Sets of equations implying congruence semidistributivity and \( n \)-permutability

Ralph Freese
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So a Maltsev term $p(x, y, z)$ is weakly independent of all of its places.

$\Sigma'$, the derivative, is the augmentation of $\Sigma$ by equations that say that $F$ is independent of its $i$th place whenever $\Sigma$ implies $F$ is weakly independent of its $i$th place.
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- If x ≈ F(w), where w is a vector of not necessarily distinct variables, then F is weakly independent of its i\textsuperscript{th} place for each i with w_i ≠ x. So a Maltsev term p(x, y, z) is weakly independent of all of its places.
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- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).

If $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But the converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies CM.

This contrasts McNulty's Theorem that there is no effective way to decide if a (nonlinear) idempotent $\Sigma$ implies CM.
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A similar theorem holds for $\forall$ satisfying some congruence identity if

"$\Sigma'$ is inconsistent"

is replaced by

"$\Sigma^{(k)}$ is inconsistent for some $k$."

The Theorems of Dent, Kearnes, Szendrei
The **order derivative**, $\Sigma^+$, augments $\Sigma$ by

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whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$. 
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If some iterated order derivative $\Sigma^{+^k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutable, for some $n$. 

Ralph Freese ()

semidistributivity and $n$-permutability

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If $\mathcal{V}$ is a congruence $n$-permutable, for some $n$, then $\mathcal{V}$ realizes some $\Sigma$ whose iterated order derivative $\Sigma^{+^k}$ is inconsistent. (The Hagemann-Mitschke terms work.)
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For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies congruence $n$-permutability, for some $n$. 
The weak derivative, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the $y$ is in the $i^{th}$ place.
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- If some iterated weak derivative $\Sigma^{*k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.
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If $\mathcal{V}$ is a congruence semidistributive then $\mathcal{V}$ realizes some $\Sigma$ whose iterated weak derivative $\Sigma^*_k$ is inconsistent. (In a joint paper with Matt, Ross and others we have a variant of the Hobby-McKenzie-Kearnes-Kiss terms work.)
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- The converse of the first statement is false, even if \( \Sigma \) is linear. Nevertheless

- For a finite linear, idempotent \( \Sigma \) one can effectively decide if \( \Sigma \) implies congruence semidistributivity.
A variety is congruence semidistributive iff there are terms $d_i(x, y, z), i = 0, \ldots, n$, such that

$$d_0(x, y, z) \approx x \quad d_n(x, y, z) \approx z$$

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and for each $i$ two of the following three hold:

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Let $\Sigma$ be these equations. Assume inductively that $\Sigma^*^k$ implies $x \approx d_i(x, y, z)$. Then, using the above equations, one can show that $\Sigma^*^{k+2}$ implies $x \approx d_{i+1}(x, y, z)$. 
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So some iterated weak derivative implies $x \approx d_n(x, y, z) \approx z$ and so is inconsistent.
Theorem

For each property \( P \) listed below, given a finite, idempotent, linear set of equations \( \Sigma \) one can effectively decide if every variety that realizes \( \Sigma \) satisfies \( P \).

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence \( n \)-permutable for some \( n \).
- Is congruence semidistributive.
- Is congruence meet-semidistributive.
- Is congruence distributive.
### Decidable properties of finite, idempotent linear $\Sigma$’s

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