PROPORTIONS OF SIDON SETS ARE I_0 SUBSETS

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Abstract. It is proved that proportions of Sidon sets are I_0 subsets of controlled degree. That is, a set E is Sidon if and only if, there are $r > 0$ and positive integer n such that, for every finite subset $F \subset E$, there is $H \subset F$ with the cardinality of H at least r times the cardinality of F and $N(H) \le n$ ($N(H)$ is a measure of the degree of being I_0). This paper leaves open David Grow's question of whether Sidon sets are finite unions of I_0 sets.

Introduction

An I_0 degree, $N(E)$, will be defined below; it is finite if and only if E is an I_0 set and allows a quantification of being I_0. The purpose of this paper is to prove the following theorem, which offers weak affirmative evidence to David Grow's question: must Sidon sets be finite unions of I_0 sets [G]?

Theorem 1. *Let Γ be a discrete abelian group. Then $E \subset \Gamma$ is Sidon if and only if, there are some real $r > 0$ and positive integer n such that, for all finite $F \subset E$, there is some $H \subset F$ for which $|H| \ge r|F|$ and $N(E) \le n$.*[1]

In what follows, Γ is a discrete, abelian group and G its compact dual. $M(G)$ is the Banach algebra of bounded Borel measures on G; $M_d(G)$ is the subalgebra of $M(G)$ consisting of discrete measures. For $E \subset \Gamma$, $B(E)$ is the Banach algebra of the restrictions to E of Fourier transforms of measures $\mu \in M(G)$; $B_d(E)$ consists of the restrictions to E of Fourier transforms of measures $\mu \in M_d(G)$. The closure of $B_d(E)$ in $\ell_\infty(E)$ is called $AP(E)$ (the almost periodic functions restricted to E). $E \subset \Gamma$ is said to be **Sidon** if and only if $B(E) = \ell_\infty(E)$ [LR]; E is called an I_0 set if and only if $AP(E) = \ell_\infty(E)$ [HR]. The following definition offers a measure of being an I_0 set.

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[1] $|H|$ denotes the cardinality of H. Unless otherwise specified, variables such as n denote positive integers.

Definition. *Let $D(N)$ denote the set of discrete measures μ on G for which*

$$\mu = \sum_{j=1}^{N} c_j \delta_{t_j},$$

where $|c_j| \leq 1$ and $t_j \in G$ for each j. For $E \subset \Gamma$ and $\delta \in \mathcal{R}^+$, let $AP(E, N, \delta)$ be the set of $f \in \ell_\infty(E)$ for which there exists $\mu \in D(N)$ such that

$$\| f - \hat{\mu} \mid_E \|_\infty \leq \delta.$$

E is said to be $I(N, \delta)$ if the unit ball in $\ell_\infty(E)$ is a subset of $AP(E, N, \delta)$. The I_0 degree of E, $N(E)$, is defined to be the first N such that E is $I(N, 1/2)$; if no such N exists, $N(E)$ is set equal to ∞.

By the following theorem, I_0 sets are exactly those for which $N(E) < \infty$. The following theorem was known to Kahane, Mèla, Ramsey and Wells much earlier, but the authors like Kalton's more recent formulation and proof ([Kl],[Kh], [M], [RW]).

Theorem 2. *For any discrete abelian group Γ and $E \subset \Gamma$, the following are equivalent:*

1. *E is an I_0-set.*
2. *There is some real $\delta \in (0, 1)$ and some N for which E is $I(N, \delta)$.*
3. *There is some real $\delta \in (0, 1)$ and some $M \in \mathcal{R}^+$ such that, for all f in the unit ball of $\ell_\infty(E)$, there are points $g_j \in G$ and complex numbers c_j with $|c_j| \leq M\delta^j$ for which*

$$f = \hat{\mu} \mid_E \quad \text{where } \mu = \sum_{j=1}^{\infty} c_j \delta_{g_j}.$$

4. *For all real $\delta \in (0, 1)$ there is some N for which E is $I(N, \delta)$.*
5. *$B_d(E) = \ell_\infty(E)$.*

Corollary 3. *For any discrete abelian group Γ and $E \subset \Gamma$, if E is $I(N, \delta)$ for some real $\delta \in (0, 1)$, then condition (3) holds with $M = 1/\delta$ and $\delta^{1/N}$ in the role of δ.*

Proof. This is implicit in Kalton's proof, and made explicit in [R]. □

One can weaken the conditions of interpolation and still attain an equivalent "degree" for I_0 sets [R].

Definition. *Let C_1 and C_2 be closed subsets of the complex plane. For $E \subset \Gamma$, E is said to be $J(N, C_1, C_2)$ if and only if, for all $F \subset E$, there is some $\mu \in D(N)$ such that $\hat{\mu}(F) \subset C_1$ and $\hat{\mu}(E \backslash F) \subset C_2$. When $C_1 = \{z \mid \Re(z) \geq \delta\}$, and $C_2 = \{z \mid \Re(z) \leq -\delta\}$, $J(N, C_1, C_2)$ is abbreviated as $J(N, \delta)$. $J(E)$ is defined to be the first N such that E is $J(N, 1/2)$; if no such N exists, $J(E)$ is set equal to ∞.*

The next theorem is proved in [R], and shows that E is I_0 if and only if $J(E) < \infty$.

Theorem 4. *The following are equivalent:*

1. *E is an I_0 set.*
2. *E is $J(N, C_1, C_2)$ for some N and disjoint subsets C_1 and C_2.*
3. *For all real $\delta \in (0, 1)$, there is some N such that E is $J(N, \delta)$.*

The next lemma relates $J(N, \delta)$ to $J(E)$.

Lemma 5. *If E is $J(N, \delta)$ for some $\delta \in (0, 1)$, then $J(E) \leq KN$ where $K = \lceil 1/(2\delta) \rceil$.*

Proof. Assume that E is $J(N, \delta)$. Then, for any $F \subset E$, there is some $\mu \in D_N$ such that

$$(\forall \gamma \in F)\, (\Re(\hat{\mu}(\gamma)) \geq \delta)\,,$$

and

$$(\forall \gamma \in E \backslash F)\, (\Re(\hat{\mu}(\gamma)) \leq -\delta)\,.$$

Because $K \geq 1/(2\delta)$, $K\delta \geq 1/2$ and thus

$$\Re\left(\widehat{K\mu}(\gamma)\right) \geq 1/2, \quad \text{for} \quad \gamma \in E,$$

while

$$\Re\left(\widehat{K\mu}(\gamma)\right) \leq -1/2, \quad \text{for} \quad \gamma \in (E \backslash F).$$

One can write $K\mu$ as a sum of KN point masses with complex coefficients bounded by 1 in absolute value. Thus E is $J(KN, 1/2)$ and $J(E) \leq KN$. □

It is readily evident that $J(E) \leq N(E)$. In [R], it is proved that there is a bounded relation between $J(E)$ and $N(E)$:

Theorem 6. *There is a function ϕ with $\phi(\mathcal{Z}^+) \subset \mathcal{Z}^+$ such that, for all discrete abelian groups Γ and all $E \subset \Gamma$,*

$$J(E) \leq N(E) \leq \phi(J(E)).$$

A key ingredient of the proof of Theorem 1 is this theorem [P]:

Theorem 7. *E is a Sidon set if and only if, there is some $\delta > 0$ with the following property: for every finite $A \subset E$, there are points $g_j \in G$, $1 \leq j \leq N$ with $N \geq 2^{\delta|A|}$, such that*

$$\sup_{\gamma \in A} |\gamma(g_i) - \gamma(g_j)| \geq \delta, \quad \textit{for all} \quad i \neq j.$$

The last ingredients of the proof are Elton's theorem about sign-embeddings of ℓ_1^n into real Banach spaces [E] and Pajor's generalization of Elton's theorem to complex Banach spaces [Pa]. The proof given in this paper does not quote their theorems verbatim; rather, parts of the their proofs are adapted to this situation.

Proof of Theorem 1

Sufficiency. Suppose that $E \subset \Gamma$ has some real $r > 0$ and positive integer N such that, for every finite subset $F \subset E$,

$$(\exists H \subset F)\,(|H| \geq r|F| \quad \text{and} \quad N(H) \leq n)\,.$$

Then H is $I(n, 1/2)$. By Corollary 3, condition (3) of Theorem 2 holds with $M = 2$ and $\delta = (1/2)^{1/n}$. It follows that, for every f in the unit ball of $\ell_\infty(E)$, there is some $\mu \in M_d(G)$ such that $\hat{\mu}\,|_H = f$ and

$$\|\mu\|_{M_d(G)} \leq L = 2\sum_{j=1}^{\infty} 2^{-j/n} < \infty.$$

For all $f \in \ell_\infty(H)$, there is a constant L which depends only on n such that

$$\|f\|_{B_d(H)} \leq L\|f\|_{\ell_\infty(H)}.$$

Since $\|f\|_{B(H)} \leq \|f\|_{B_d(H)}$, one has

$$\|f\|_{B(H)} \leq L\|f\|_{\ell_\infty(H)}.$$

Thus H is a Sidon set with Sidon constant at most L, with L independent of $F \subset E$. That suffices to make E be Sidon, by Corollary 2.3 of [P].

Necessity. Suppose that E is Sidon. Apply Theorem 7. There is some $\delta > 0$ such that, for all finite $F \subset E$, there are at least $2^{\delta|F|}$ points g_j of G such that, for $i \neq j$,

$$\sup_{\gamma \in F} |\gamma(g_j) - \gamma(g_i)| \geq \delta. \tag{1}$$

Necessarily, $\delta \leq 2$.

Let $F \subset E$ of cardinality n. Enumerate F as $\gamma_1, \ldots, \gamma_n$. Choose p so that $\tau = 2\pi/p < \delta/2$. To be specific, let $p = 1 + \lceil 4\pi/\delta \rceil$. Let T denote the unit circle in the complex plane. Partition T into disjoint arcs, T_k, $1 \leq k \leq p$, of the form

$$T_k = \{\, e^{i\theta} \mid (k-1)\tau \leq \theta < k\tau \,\}.$$

Let $Q = \lceil (1 - 2^{-\delta/2})^{-1} \rceil$. and set $\tau' = \tau/Q$. Partition each T_k into Q arcs $U_{k,m}$ of the form

$$U_{k,m} = \{\, e^{i\theta} \mid (k-1)\tau + (m-1)\tau' \leq \theta < (k-1)\tau + m\tau' \,\},$$

for $1 \leq m \leq Q$. Finally, let $\mathcal{S}_0$ denote a set of at least $2^{\delta|F|}$ points of G which satisfy inequality (1).

Define $\mathcal{S}_i$ inductively. Let

$$\mathcal{S}^i_k = \{\, g \in \mathcal{S}_{i-1} \mid \gamma_i(g) \in T_k \,\}$$

and

$$\mathcal{S}^i_{k,m} = \{\, g \in \mathcal{S}_{i-1} \mid \gamma_i(g) \in U_{k,m} \,\}.$$

Then

$$\mathcal{S}_{i-1} = \cup_{k=1}^{p} \mathcal{S}_k^i$$

and

$$\mathcal{S}_k^i = \cup_{m=1}^{Q} \mathcal{S}_{k,m}^i.$$

There is some $m(i,k)$ such that

$$|\mathcal{S}_{k,m(i,k)}^i| \leq Q^{-1}|\mathcal{S}_k^i|.$$

So,

$$\left|\cup_{k=1}^{p} \mathcal{S}_{k,m(i,k)}^i\right| \leq Q^{-1} \left|\mathcal{S}_{i-1}\right|.$$

Let

$$\mathcal{S}_i = \mathcal{S}_{i-1} \setminus \cup_{k=1}^{p} \mathcal{S}_{k,m(i,k)}^i.$$

Then

$$|\mathcal{S}_i| \geq (1 - Q^{-1}) \left|\mathcal{S}_{i-1}\right|.$$

By induction one has

$$|\mathcal{S}_n| \geq (1 - Q^{-1})^n \left|\mathcal{S}_0\right|.$$

Note that $Q \geq (1 - 2^{-\delta/2})^{-1}$; consequently,

$$(1 - Q^{-1}) \geq 2^{-\delta/2}.$$

Therefore,

$$\begin{aligned}|\mathcal{S}_n| &\geq (1 - Q^{-1})^n \left|\mathcal{S}_0\right| \\ &\geq (2^{-\delta/2})^n 2^{\delta n} \\ &= 2^{n\delta/2}.\end{aligned}$$

For $1 \leq i \leq n$ and $1 \leq k < p$, let $I_{i,k}$ be the arc between $U_{k,m(i,k)}$ and $U_{k+1,m(i,k+1)}$. For $k = p$, let $I_{i,k}$ be the arc between $U_{k,m(i,k)}$ and $U_{1,m(i,1)}$. Necessarily,

$$I_{i,k} \subset \{\, e^{i\theta} \mid (k-1)\tau + \tau' \leq \theta < (k+1)\tau - \tau' \,\}. \tag{2}$$

The length (and hence the diameter) of each of these arcs is at most $(2 - 2/Q)\tau < 2 * (\delta/2) = \delta$.

It is possible for $I_{i,k} = \emptyset$, which will happen when $k < p$, $m(i,k) = Q$ and $m(i,k+1) = 1$; it will also happen when $k = p$, $m(i,k) = Q$ and $m(i,1) = 1$. Otherwise, $e^{ik\tau}$ is in the closure of $I_{i,k}$: it is in $I_{i,k}$ when $m(i,k+1) > 1$ and when $k = p$ and $m(i,1) > 1$. When $m(i,k+1) = 1$ and $I_{i,k} \neq \emptyset$,

$$\{e^{i\theta} \mid (k-1)\tau + (Q-1)\tau' \leq \theta < k\tau\} \subset I_{i,k}.$$

Likewise, when $m(i,1) = 1$ and $I_{i,p} \neq \emptyset$,

$$\{e^{i\theta} \mid (p-1)\tau + (Q-1)\tau' \leq \theta < p\tau\} \subset I_{i,p}.$$

For all other $j \neq k$, there is an arc of length τ' between $I_{i,k}$ and $e^{ij\tau}$ (e.g., $U_{k,m(i,k)}$ or $U_{k+1,m(i,k+1)}$ when $1 \leq k < p$).

Each sequence $\{k_i\}_{i=1}^n$, with $1 \leq k_i \leq p$, defines a cylinder in $\ell_\infty(F)$ of the following form:

$$W[\{k_i\}_{i=1}^n] = \{\, f \in \ell_\infty(F) \mid f(\gamma_i) \in I_{i,k_i} \,\}.$$

For $g \in G$, let $f_g(\gamma) = \gamma(g)$ for $\gamma \in F$. Because these cylinders are disjoint, each f_g is in at most one of them. $\mathcal{S}_n$ was chosen to guarantee the f_g would be in at least one cylinder for $g \in \mathcal{S}_n$. For $g \in \mathcal{S}_n$, let $h(g) = \{k_i\}_{i=1}^n$ define the cylinder which contains f_g.

Because each cylinder has diameter less than δ, inequality (1) implies that each cylinder contains at most one f_g for $g \in \mathcal{S}_n$. Hence

$$|\mathcal{S}_n| = |h(\mathcal{S}_n)|.$$

For any subset $H \subset F$, let Π^H be this projection: for $f \in \ell_\infty(F)$,

$$\Pi^H(f) = f\mid_H .$$

By Corollary 2 of [Pa, p. 742], there is a constant $c'' > 0$ which depends only on $\delta/2$ and p (which itself depends only on δ) such that there are some $H \subset F$ and integers $a < b$ from $[1, p]$ such that

$$|H| \geq c''|F|$$

and

$$\{a, b\}^H \subset \Pi^H(h(\mathcal{S}_n)).$$

Case 1: $|(a - b) \mod p| \geq 2$**.** On the circle, there are two arcs between $e^{ia\tau}$ and $e^{ib\tau}$. Choose c so that $e^{ic\tau}$ is the center of the shorter of these two arcs, $a \leq c \leq a + p$. Necessarily $c \neq a$ and $c \neq b$. c is either a half-integer or an integer. If c is an integer, then $e^{ic\tau}$ is separated by arcs of length τ' from $I_{i,a}$ and $I_{i,b}$. If c is a half-integer, $c - 1/2$ and $c + 1/2$ are both integers which are distinct from a and b. Since there are arcs of length τ' between each of $e^{i(c-1/2)\tau}$ and $e^{i(c+1/2)\tau}$ and each of $I_{i,a}$ and $I_{i,b}$, there are arcs of length τ' between $e^{ic\tau}$ and each of $I_{i,a}$ and $I_{i,b}$.

Case 1A. Assume that $a < c < b$**.** Let $z_2 \in I_{i,b}$ and $z_1 \in I_{i,a}$. Then

$$I_{i,b} = \{\, e^{i\theta} \mid x \leq \theta \leq y \,\},$$

where x and y can be chosen to satisfy

$$x \geq b\tau - \tau + \tau' \quad \text{and} \quad y \leq b\tau + \tau - \tau'.$$

[See equation (2).] Moreover, since $e^i c\tau$ is separated from $I_{i,b}$ by an arc of length τ', and both $c\tau < b\tau$ and $x < b\tau$, we have

$$c\tau + \tau' \leq x.$$

Because $e^{ic\tau}$ is the center of the shorter of the two arcs between $e^{ia\tau}$ and $e^{ib\tau}$,

$$b\tau - c\tau \leq \pi/2.$$

Since $\delta \le 2$ and $\tau < \delta/2$ (and $\tau' > 0$), we have $z_2 = e^{i\theta}$ with

$$c\tau + \tau' \le \theta < c\tau + \pi/2 + 1.$$

Hence

$$e^{-ic\tau} z_2 = e^{i(\theta - c\tau)}, \quad \text{with} \quad \tau' \le \theta - c\tau \le \pi/2 + 1.$$

Thus $e^{-ic\tau} z_2$ is in the top half-plane, with

$$\Re(e^{-ic\tau} z_2) \ge \tau'' = \min\{\sin(\tau'), \sin(\pi/2+1)\} > 0.$$

Likewise, $e^{-ic\tau} z_1$ is in the lower half-plane, with

$$\Re(e^{-ic\tau} z_1) \le -\tau'' < 0.$$

Because $\{a,b\}^H \subset \Pi^H(h(\mathcal{S}_n))$, for any $A \subset H$ there is some $g \in \mathcal{S}_n$ such that $h(g)(\gamma) = b$ for $\gamma \in A$ and $h(g)(\gamma) = a$ for $\gamma \in H \backslash A$. Let $\mu = e^{-ic\tau}\delta_g$; $\mu \in D(1)$. Because $h(g)(\gamma_i) = b$ if and only if $\gamma_i(g) \in I_{i,b}$, for $\gamma \in A$ we have

$$\Re(\widehat{e^{-ic\tau}\delta_g}(\gamma)) = \Re(e^{-ic\tau}\gamma(g)) \ge \tau''$$

Likewise, for $\gamma_i \in H\backslash A$, $\gamma_i(g) \in I_{i,a}$ and hence

$$\Re(\widehat{e^{-ic\tau}\delta_g}(\gamma)) = \Re(e^{-ic\tau}\gamma(g)) \le -\tau''$$

This proves that H is $J(1,\tau'')$.

Case 1B. Assume that $b < c < a + p$**.** Let $z_2 \in I_{i,a}$ and $z_1 \in I_{i,b}$. Then $z_2 = e^{i\theta}$ with

$$c\tau + \tau' \le \theta < c\tau + \pi/2 + 1,$$

and

$$e^{-ic\tau} z_2 = e^{i(\theta - c\tau)}, \quad \text{with} \quad \tau' \le \theta - c\tau < \pi/2 + 1.$$

Thus $e^{-ic\tau} z_2$ is in the top half-plane, with

$$\Re(e^{-ic\tau} z_2) \ge \tau'' > 0.$$

Likewise, $e^{-ic\tau} z_1$ is in the lower half-plane, with

$$\Re(e^{-ic\tau} z_1) \le -\tau'' < 0.$$

Because $\{a,b\}^H \subset \Pi^B(h(\mathcal{S}_n))$, for any $A \subset B$ there is some $g \in \mathcal{S}_n$ such that $h(g)(\gamma) = a$ for $\gamma \in A$ and $h(g)(\gamma) = b$ for $\gamma \in H\backslash A$. Let $\mu = e^{-ic\tau}\delta_g$; again, $\mu \in D(1)$. Because $h(g)(\gamma_i) = a$ if and only if $\gamma_i(g) \in I_{i,a}$, for $\gamma \in A$ we have

$$\Re(\widehat{e^{-ic\tau}\delta_g}(\gamma)) = \Re(e^{-ic\tau}\gamma(g)) \ge \tau''.$$

Likewise, for $\gamma_i \in H\backslash A$, $\gamma_i(g) \in I_{i,b}$ and hence

$$\Re(\widehat{e^{-ic\tau}\delta_g}(\gamma)) = \Re(e^{-ic\tau}\gamma(g)) \le -\tau''.$$

This proves that H is $J(1,\tau'')$.

Case 2A: $b = a+1$. Because $\{a,b\}^H \subset \Pi^H(h(\mathcal{S}_n))$, for every $A \subset H$ there are g_1 and g_2 such that

$$(\forall \gamma \in A)\,(h(g_1)(\gamma) = b \quad \text{and} \quad h(g_2)(\gamma) = a)\,,$$

while

$$(\forall \gamma \in H\backslash A)\,(h(g_2)(\gamma) = a \quad \text{and} \quad h(g_2)(\gamma) = b)\,.$$

The arc $U_{i,m(i,b)}$ is between $I_{i,b}$ and $I_{i,a}$. Let

$$U_{i,m(i,b)} = \{e^{i\theta} \mid a' \le \theta < b'\},$$

for some a' and b' such that $a\tau \le a' < b' \le b\tau$. If $z \in I_{i,b}$, then $z = e^{i\theta}$ for some θ such that

$$b' \le \theta < b\tau + \tau - \tau'.$$

[See inclusion (2).] Likewise, if $z \in I_{i,a}$, then $z = e^{i\theta}$ for some θ such that

$$a\tau - \tau + \tau' \le \theta < a'.$$

Thus, when $\gamma_i(g_1) \in I_{i,b}$ and $\gamma_i(g_2) \in I_{i,a}$,

$$\gamma_i(g_1 - g_2) = \gamma_i(g_1)/\gamma_i(g_2) = e^{i\theta}$$

with

$$\tau' \le b' - a' < \theta < (b-a)\tau + 2\tau - 2\tau' = (3 - 2/Q)\tau < 3.$$

Thus, when $\gamma \in A$, $\gamma(g_1 - g_2)$ is in the upper half-plane and

$$\Re(\gamma(g_1 - g_2)) \ge \tau''' = \min\{\sin(\tau'), \sin(3)\}.$$

When $\gamma_i(g_1) \in I_{i,a}$ and $\gamma_i(g_2) \in I_{i,b}$, then

$$\gamma_i(g_1 - g_2) = \gamma_i(g_1)/\gamma_i(g_2) = e^{i\theta}$$

with

$$-3 < (-3 + 2/Q)\tau < \theta < a' - b' = -\tau'.$$

Thus, when $\gamma \in H\backslash A$, This puts $\gamma_i(g_1 - g_2)$ in the lower half plane with

$$\Re(\gamma(g_1 - g_2)) \le -\tau'''.$$

This makes H a $J(1,\tau''')$ set.

Case 2B: $a = 1$ and $b = p$. This is just like Case 2A, if one treats a as $p+1$ and switch the roles of a and b.

Set $\tau'''' = \min\{\tau'', \tau'''\}$. Every finite $F \subset E$ has $H \subset F$ such that

$$|H| \ge c''|F|$$

and H is $J(1,\tau'''')$. By Lemma 5, $J(H) \le \lceil 2/\tau'''' \rceil$. By Theorem 5, $N(H) \le \phi(J(H))$ for a function ϕ which is independent of E. Note that $J(H)$ depends only on τ'''' which in turn depends only on δ (and is independent of $F \subset E$). Also, c'' depends only on δ, and is independent of $F \subset E$. $\square$

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