Critical integers of motivic L-functions and Hodge numbers

Robert Harron

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version 1.1

Abstract

A quick write-up of how to relate the critical integers of a motivic L-function (in the sense of Deligne [D79]) to the Hodge numbers of the motive. We work out the cases of the Tate motive $\mathbf{Q}(1)$, and the symmetric powers of elliptic modular forms.

1 Definition

The integers which are critical for the motive M are determined by the poles of the gamma factors of M and $M^\vee$. The gamma factors are, in turn, defined by the Hodge structure of the motive. Suppose M is a motive over $\mathbf{Q}$ (with coefficients in $\mathbf{Q}$) whose Hodge structure is

$$M_H = \bigoplus M_H^{p,q}$$

equipped with an involution F_∞ which maps $M_H^{p,q}$ to $M_H^{q,p}$.

Let $h^{p,q} = \dim M_H^{p,q}$ and for $\epsilon \in \{0,1\}$, let $h^{p,p,\epsilon} = \dim \left(M_H^{p,p}\right)^{F_\infty = (-1)^{p+\epsilon}}$.

Definition 1.1. The gamma factor of M is

$$\Gamma_M(s) := \prod_{p<q} \Gamma_{\mathbf{C}}(s-p)^{h^{p,q}} \times \prod_{p=q} \prod_{\epsilon\in\{0,1\}} \Gamma_{\mathbf{R}}(s-p+\epsilon)^{h^{p,p,\epsilon}}$$

where

$$\Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s).$$

See [D79, §5.3].

Definition 1.2 (Definition 1.3 of [D79])**.** An integer n is critical for M if neither $\Gamma_M(s)$, nor $\Gamma_{M^\vee}(1-s)$, has a pole at $s=n$. The motive is called critical if $s=0$ is critical for M.

Note $M^\vee$ is the usual dual $\mathrm{Hom}(M, \mathbf{Q})$.

Remark 1.3. Criticality can also be characterized in terms of vanishing of the Deligne cohomology.

Department of Mathematics and Statistics, Boston University

2 Relation to Hodge numbers

Let $h^{\vee,p,q} = \dim M_H^{\vee,p,q}$ and $h^{\vee,p,p,\epsilon} = \dim\big(M_H^{\vee,p,p}\big)^{F_\infty=(-1)^{p+\epsilon}}$. Since $\Gamma_{\mathbf{C}}(s-p)$ has poles at all integers $\leq p$, the only possible critical integers are those greater than all p that satisfy $h^{p,q}\neq 0$ with $p<q$. Furthermore, since $h^{\vee,-p,-q}=h^{p,q}$ and $\Gamma_{\mathbf{C}}((1-s)-(-q))=\Gamma_{\mathbf{C}}(1+q-s)$ has poles at all integers $>q$, critical integers must also be less than or equal to all q that satisfy the same condition ($h^{p,q}\neq 0$ with $p<q$). Thus, for example, if $h^{p,p}=0$ for all p, the critical integers for M are those satisfying

$$\max_{\substack{h^{p,q}\neq 0\\ p<q}} p < n \leq \min_{\substack{h^{p,q}\neq 0\\ p<q}} q. \tag{1}$$

Thus, the presence of a non-zero $M_H^{p,q}$ with $p\neq q$ constrains the critical integers to lie within an interval (as in the case of modular forms, see below).

The consideration of nontrivial $h^{p,p}$ is slightly more complicated. We summarize the constraints imposed by the various gamma factors in the table 1. Let $\mathscr{N}$ denote the set of negative odd integers.

Factor	$\Gamma\left(\frac{s-p}{2}\right)$	$\Gamma\left(\frac{1-s+p}{2}\right)$	$\Gamma\left(\frac{s-p+1}{2}\right)$	$\Gamma\left(\frac{2-s+p}{2}\right)$
Prevents criticality at	$s=p+1+\mathscr{N}$	$s=p-\mathscr{N}$	$s=p+\mathscr{N}$	$s=p+1-\mathscr{N}$

Table 1: Location of poles for factors that show up in gamma factors of motives. Poles prevent integers from being critical.

The second row of this table is just the location of poles of the factor in the first row. The factors in the first row are just (up to factors of π^s)

$$\Gamma_{\mathbf{R}}(s-p), \Gamma_{\mathbf{R}}((1-s)-(-p)), \Gamma_{\mathbf{R}}(s-p+1), \text{ and } \Gamma_{\mathbf{R}}((1-s)-(-p)+1).$$

Note that $h^{\vee,-p,-p,\epsilon}=h^{p,p,\epsilon}$. If $h^{p,p,0}\neq 0$, the first and second factors appear in $\Gamma_M(s)$ and $\Gamma_{M^\vee}(1-s)$, respectively. Similarly, if $h^{p,p,1}\neq 0$, the third and fourth factors appear in $\Gamma_M(s)$ and $\Gamma_{M^\vee}(1-s)$, respectively. We can already state the following:

$$\text{if } h^{p,p,0}\neq 0 \text{ and } h^{p,p,1}\neq 0 \text{ for some } p, \text{ then no integers are critical for } M. \tag{2}$$

Indeed, the table shows that all factors combined prevent all integers from being critical. Thus, F_∞ must act as a scalar on each $M_H^{p,p}$ for motives with critical integers. If M is pure, there is at most one p for which $h^{p,p}\neq 0$. The above table then tells us the constraints on the critical integers coming from $M_H^{p,p}$ (see figure 1). These must be added to those coming from the $p\neq q$ factors.

For mixed motives, the situation becomes messier.

(a) Case $\epsilon = 0$ (b) Case $\epsilon = 1$

Figure 1: Integers excluded by $h^{p,p,\epsilon} \neq 0$. An $\times$ marks integers prevented from being critical by $\Gamma_M(s)$, while $\times$ marks those prevented by $\Gamma_{M^\vee}(1-s)$.

3 Examples

3.1 The Tate motive $\mathbf{Q}(1)$

The Tate motive is $\mathbf{Q}(1) = H^2(\mathbb{P}^1)^\vee$. It can be shown (via some classical cohomology computations) that its Hodge structure is

$$H^{-1.-1} \text{ one-dimensional with } F_\infty = -1 = (-1)^{-1+0}$$

so $h^{-1,-1,0} = 1$ is the only non-zero Hodge number. From figure 1(a), it's critical at the negative even integers and the positive odd integers. Since $L(s, \mathbf{Q}(1)) = \zeta(s+1)$, this is the expected answer. Note that its gamma factor is accordingly

$$\Gamma_{\mathbf{Q}(1)}(s) = \Gamma_{\mathbf{R}}(s+1) = \pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right).$$

3.2 Elliptic modular forms

Suppose f is a weight $k \geq 2$ holomorphic modular form over $\mathbf{Q}$. The latter condition is simply to ensure that the associated motive M_f has coefficients in $\mathbf{Q}$ (removing this condition would not affect the following statements in any meaningful way). The motive attached to f is $M_f = H^{k-1}(\mathcal{E}^{k-2})$, where $\mathcal{E}^{k-2}$ is some canonical resolution of the (k-2)-fold self-product of the universal generalized elliptic curve over the appropriate modular curve. Its Hodge structure is

$$H^{k-1,0} \oplus H^{0,k-1} \text{ with each factor one-dimensional.}$$

By (1), its critical integers are thus

$$0 < n \leq k-1. \tag{3}$$

Its gamma factor is

$$\Gamma_{M_f}(s) = \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

3.3 Symmetric powers of elliptic modular forms

Again, suppose f is a weight $k \geq 2$ holomorphic modular form over $\mathbf{Q}$. For a positive integer m, denote $M_m = \mathrm{Sym}^m M_f$, and denote its Hodge structure by

$$H_m = H_m^{m(k-1),0} \oplus H_m^{(m-1)(k-1),k-1} \oplus H_m^{(m-2)(k-1),2(k-1)} \oplus \cdots \oplus H_m^{0,m(k-1)},$$

where each factor is one-dimensional. If $m = 2r+1$ is odd, then the "left" middle term is

$$H_m^{r(k-1),(r+1)(k-1)},$$

so, by (1), the critical integers are

$$r(k-1) < n \leq (r+1)(k-1). \tag{4}$$

When m is even, the situation is complicated by the presence of $H_m^{m(k-1)/2,m(k-1)/2}$. Since the parity of $m(k-1)/2$ matters in determining ϵ, we must split up into two cases corresponding to the combined parities of $m/2$ and k. Of course, in both cases, the $p \neq q$ contributions imply that all critical integers satisfy

$$\left(\frac{m}{2}-1\right)(k-1) < n \leq \left(\frac{m}{2}+1\right)(k-1). \tag{5}$$

Let $m = 2r$.

• (r odd, k even): in this case, $r(k-1)$ is odd. Since F_∞ acts trivially, we must have $\epsilon = 1$. Referring to figure 1(b), this case therefore allows the integers

$$\ldots, r(k-1)-4, r(k-1)-2, r(k-1), r(k-1)+1, r(k-1)+3, r(k-1)+5, \ldots \tag{6}$$

• (r odd, k odd) or (r even, any k): in these cases, $r(k-1)$ is even. Since F_∞ acts trivially, we must have $\epsilon = 0$. Referring to figure 1(a), this case therefore allows the integers

$$\ldots, r(k-1)-5, r(k-1)-3, r(k-1)-1, r(k-1)+2, r(k-1)+4, r(k-1)+6, \ldots \tag{7}$$

To combine (5) with (6) and (7) requires splitting up into three cases. The result is the following.

Proposition 3.1 (Lemma 3.3 of [RgS08]). *Let $m = 2r$ be even. The critical integers for $M_m = \mathrm{Sym}^m M_f$ are*

• *r odd, k even:* $(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1)-2, r(k-1)$ *and*

$$r(k-1)+1, r(k-1)+3, \ldots, (r+1)(k-1)-2, (r+1)(k-1)$$

• *k odd:* $(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1)-3, r(k-1)-1$ *and*

$$r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)-2, (r+1)(k-1)$$

• *r even, k even:* $(r-1)(k-1)+2, (r-1)(k-1)+4, \ldots, r(k-1)-3, r(k-1)-1$ *and*

$$r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)-2, (r+1)(k-1)-1.$$

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