L-invariants of low symmetric powers of modular forms and Hida deformations

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Abstract

We obtain formulae for Greenberg's L-invariant of symmetric square and symmetric sixth power motives attached to p-ordinary modular forms in the vein of theorem 3.18 of [GS93]. For the symmetric square of f, the formula obtained relates the L-invariant to the derivative of the p-adic analytic function interpolating the pth Fourier coefficient (equivalently, the unit root of Frobenius) in the Hida family attached to f. We present a different proof than Hida's, [Hi04], with slightly different assumptions. The symmetric sixth power of f requires a bigger p-adic family. We take advantage of a result of Ramakrishnan–Shahidi ([RS07]) on the symmetric cube lifting to $\mathrm{GSp}(4)_{/\mathbf{Q}}$, Hida families on the latter ([TU99] and [Hi02]), as well as results of several authors on the Galois representations attached to automorphic representations of $\mathrm{GSp}(4)_{/\mathbf{Q}}$, to compute the L-invariant of the symmetric sixth power of f in terms of the derivatives of the p-adic analytic functions interpolating the eigenvalues of Frobenius in a Hida family on $\mathrm{GSp}(4)_{/\mathbf{Q}}$. We must however impose stricter conditions on f in this case. Here, Hida's work (e.g. [Hi07]) does not provide answers as specific as ours.

In both cases, the method consists in using the big Galois deformations and some multilinear algebra to construct global Galois cohomology classes in a fashion reminiscent of [Ri76]. The method employs explicit matrix computations.

Contents

Introduction

In the mid 1980s, searching for a p-adic analogue of the Birch and Swinnerton-Dyer conjecture, Mazur, Tate, and Teitelbaum ([MTT86]) discovered some exceptional behaviour occurring in the p-adic interpolation of L-functions of elliptic curves. Specifically, the p-adic L-function can vanish at $s = 1$ even when the archimedean L-function does not. This phenomenon is called an "exceptional zero" (or "trivial zero", as it trivially arises from the vanishing of the interpolation factor). Based on numerical computations, they conjectured that a certain p-adic quantity, the *L-invariant*, should recuperate a non-trivial interpolation property relating the *derivative* of the p-adic L-function to the archimedean L-function. This quantity was defined in terms of the multiplicative p-adic period of the elliptic curve (i.e. the Tate period, q). They further formulated a conjecture for higher weight modular forms that such an L-invariant should depend only *locally* on E.

In the early 1990s, Greenberg and Stevens proved Mazur, Tate and Teitelbaum's conjecture ([GS93]) for modular elliptic curves and weight two modular forms. Their clever method consists in adding a variable to the problem via the use of Hida families (that is, p-adic analytic families of p-ordinary modular forms of varying weight). The usual p-adic L-function can then be replaced by a two-variable p-adic L-function that additionally interpolates values as the weight varies in the Hida family. The problem is then broken into two pieces: the first is to relate the L-invariant to the derivative of the two-variable p-adic L-function in the weight direction, the second, to use the

p-adic functional equation to transfer this result to the usual "cyclotomic" direction. By reinterpreting the L-invariant in terms of Galois cohomology and infinitesimal deformations, Greenberg and Stevens obtain a key result in their proof, namely theorem 3.18 of [GS93] stating that the L-invariant of a weight two modular form, f, that has an exceptional zero, is given by

$$\mathcal{L}(f) = -2a_p' \tag{1}$$

where a_p' is the derivative of the p-adic analytic function that interpolates the pth Fourier coefficient, a_p, (equivalently, the unit root of Frobenius) as the weight varies in the Hida family. In this thesis, we prove analogues of this result. Note that several more definitions of the L-invariant and proofs of the Mazur–Tate–Teitelbaum conjecture have been given by other mathematicians; see Colmez's article [Cz05] for an overview.

Shortly thereafter, in [G94], Greenberg proposed a definition for the L-invariant of rather general p-ordinary p-adic Galois representations to go along with the conjectures of Coates and Perrin-Riou concerning p-adic L-functions ([CoPR89, Co89]). It is this version of the L-invariant that we treat in this thesis. In [G94, p. 170], Greenberg treats the case of arbitrary symmetric powers of an elliptic curve with split multiplicative reduction at p, and even symmetric powers of CM elliptic curves with good, ordinary reduction at p. In both cases, he shows that the L-invariant is independent of the power.

Subsequent work on Greenberg's L-invariant of symmetric powers of ordinary modular forms has mostly been done by Hida in a series of papers starting with [Hi04] (see [Hi07] for the most up-to-date version of his work). His results rely on conjectures concerning the local structure of the versal nearly ordinary Galois deformation ring of the symmetric power Galois representations. Furthermore, for powers greater than two, his formulae say very little about the case where the modular form has level prime to p.

This thesis establishes formulae for the L-invariants of the symmetric square and some symmetric sixth power ordinary modular forms. For the symmetric square, Hida's results are mostly complete. We include our work on the subject as the approach differs from Hida's and does not require knowledge of the versal nearly ordinary Galois deformation ring. The theorem we obtain is the following:

Theorem A. *Let f be a p-ordinary, weight $k_0 \geq 2$, new, holomorphic eigenform of character ψ (of conductor prime to p) and arbitrary level. Assume the balanced Selmer group vanishes:*

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}\big(\mathrm{Sym}^2\rho_f(1-k_0)\big) = 0.$$

Then,

$$\mathcal{L}\big(\mathrm{Sym}^2\rho_f(1-k_0)\big) = -\frac{2a_p'}{a_p}. \tag{2}$$

For the symmetric sixth power, the Hida deformation of f on $\mathrm{GL}(2)$ is insufficient for our purposes. We take advantage of the symmetric cube lift to $\mathrm{GSp}(4)_{/\mathbf{Q}}$ of Ramakrishnan–Shahidi in [RS07] and the work of Tilouine–Urban on Hida deformations on $\mathrm{GSp}(4)$ ([TU99]). This requires extra conditions on the modular form f, but provides us with the following result of chapter 3:

Theorem B. *Let $f \in S_{k_0}(\Gamma_1(N))$ be a p-ordinary, non-CM, even weight $k_0 \geq 4$ new eigenform of level prime to p and trivial character. Suppose $\mathrm{Sym}^3\rho_f$ is residually absolutely irreducible. Assume the balanced Selmer group vanishes:*

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}\big(\mathrm{Sym}^6\rho_f(3(1-k_0))\big) = 0,$$

and that the $\mathrm{GSp}(4)$ universal p-adic Hecke algebra is unramified over the Iwasawa *algebra at the prime corresponding to* $\mathrm{Sym}^3 f$*. Then, for all but finitely many* p

$$\mathcal{L}(\rho_6) = \mathcal{L}\left(\mathrm{Sym}^6 \rho_f(3(1-k_0))\right) = 3a_p a_p^{(2,1)} - a_p^3 a_p^{(1,1)}, \tag{3}$$

where $a_p^{(i,1)}$ *are derivatives of specific* p*-adic analytic functions interpolating eigenvalues of Frobenius in a certain Hida family on* $\mathrm{GSp}(4)$*.*

In particular, we obtain a formula for the symmetric sixth power L-invariant when f has level prime to p. We view such a result as encouraging evidence that further instances of functoriality can be combined with Hida deformations on higher rank groups to yield results on higher symmetric power L-invariants.

The symmetric sixth power L-invariant computation also yields the symmetric square L-invariant. This allows us to compare the two in an attempt to verify if they are equal — as is the case for CM elliptic curves. We fall slightly short of a conclusive result, but at least obtain the following:

Theorem B′. *In the situation of theorem B, we have that*

$$\mathcal{L}(\rho_6) = -10a_p^3 a_p^{(1,1)} + 6\mathcal{L}(\rho_2). \tag{4}$$

The results of this thesis lay groundwork for proving generalizations of the Mazur–Tate–Teitelbaum conjecture. Furthermore, they suggest that higher symmetric powers could be addressed via new instances of functoriality — such as the very recent potential automorphy results of [BLGHT09]. In [Hi07, p. 6], Hida mentions that alternate methods of computing L-invariants could lead to a proof of [Hi07, Conjecture 0.1]. That is, our results may be useful in studying the local structure of nearly ordinary Galois deformation rings.

Notation

Let us record some common notation and nomenclature that we will use throughout this thesis.

We fix throughout a rational prime p. Let K be a finite extension of $\mathbf{Q}_p$. Let $G_{\mathbf{Q}}$ denote the absolute Galois group of $\mathbf{Q}$ (relative to a fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$). Fix an embedding, ι_p, of $\overline{\mathbf{Q}}$ into a fixed algebraic closure $\overline{K} = \overline{\mathbf{Q}}_p$. This identifies the absolute Galois group $G_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ with the decomposition group $G_p \subseteq G_{\mathbf{Q}}$ of the prime above p corresponding to ι_p. Let $I_p \subseteq G_p$ denote the inertia subgroup. Let χ_p denote the p-adic cyclotomic character, and let $K(1)$ denote the vector space K with the Galois action given by χ_p. We denote the adeles of a global field F by $\mathbb{A}_F$, dropping the subscript for $F = \mathbf{Q}$. The finite adeles will be denoted $\mathbb{A}_{F,f}$ and the infinite adeles, $\mathbb{A}_{F,\infty}$.

For compatibility with [G94], we use Frob_p to denote an *arithmetic* Frobenius at p, and we normalize the local reciprocity map

$$\mathrm{rec} : \mathbf{Q}_p^\times \longrightarrow G_{\mathbf{Q}_p}^{ab}$$

so that Frob_p corresponds to p. We remark that under this normalization

$$\chi_p(\mathrm{rec}(u)) = u^{-1}$$

for any principal unit u in $\mathbf{Z}_p^\times$.

All representations will be finite-dimensional vector spaces (or finite free modules). Given a Galois representation on a K-vector space V, we will denote by V^* the *Tate dual* of V defined by

$$V^* := \mathrm{Hom}_K\left(V, K(1)\right).$$

Given a field F with algebraic closure $\overline{F}$, we denote its absolute Galois group by G_F. If M is a topological G_F-module M, we will denote the continuous Galois cohomology of G_F with coefficients in M by

$$H^i(F, M) := H^i(G_F, M).$$

Given a G_F-submodule M' of M, and a class $c \in H^i(F, M)$, we let $c \bmod M'$ denote the image of c under the natural map

$$H^i(F, M) \longrightarrow H^i(F, M/M').$$

Given a representation V of $G_{\mathbf{Q}}$ and a place v of $\mathbf{Q}$, we will denote res_v the natural restriction map

$$\mathrm{res}_v : H^1(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_v, V).$$

Furthermore, given a class $c \in H^1(\mathbf{Q}, V)$, we will often denote its image under res_v by c_v.

Given an algebraic number field[1] F, a reductive algebraic group G over F, and an automorphic representation π of $G(\mathbb{A}_F)$, we recall that π can be decomposed as a restricted tensor product

$$\pi = {\bigotimes_v}' \pi_v$$

over the places v of F, where π_v denotes an admissible representation of $G(F_v)$, for v finite, and an admissible $(\mathfrak{g}_v, K_v)$-module, for v infinite, where $\mathfrak{g}_v$ is the real Lie

[1] An algebraic number field will always refer to a *finite* extension of $\mathbf{Q}$.

algebra of $G(F_v)$ and K_v is a maximal compact subgroup of $G(F_v)$. We will denote the finite part of π by π_f and the infinite part by π_∞.

Chapter 1

Greenberg's L-invariant

In this chapter, we recall Greenberg's theory of L-invariants associated to the phenomenon of trivial zeroes of p-adic L-functions attached to p-ordinary motives (as developed in [G94]). We present the theory "with coefficients"[1] so that it is suited to our use in later chapters.

Greenberg's theory grows out of the Galois cohomological reinterpretation of the Mazur–Tate–Teitelbaum L-invariant presented in section 3 of [GS93]. In that original case, the L-invariant has a manifestly "local" definition[2] that makes it easier to deal with than the more general cases, two of which we attack in subsequent chapters. The basic premise of our approach is introduced in section 1.5. The contents of the other sections are as follows. Section 1.1 introduces the most basic notions of interest, while section 1.2 provides some refinements related to the exceptional nature of the p-adic Galois representations we are studying. Section 1.3 defines the ordinary and balanced Selmer groups and describes the fundamental properties that we will need. Then, section 1.4 gives Greenberg's definition of the L-invariant.

[1] i.e. we consider Galois representations on a vector space over a finite extension K of $\mathbf{Q}_p$.

[2] It only depends on the action of the decomposition group at p. This is true for any case of type M (see section 1.2 for the definition).

1.1 Basic setup

Our basic object of study in this chapter will be a continuous representation

$$\rho : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_K(V)$$

on a finite-dimensional vector space V over the p-adic field K (endowed with the p-adic topology) which we will refer to simply as a p-adic representation. A fundamental condition we impose on this representation is that of *ordinarity* (see definition 1.1.1 below).[3] In this chapter, we will also impose three conditions Greenberg calls (S), (T), and (U) in [G94]. These three conditions are easily seen to be satisfied for the situations we deal with in subsequent chapters. The goal of Greenberg's theory is to study a certain exceptional behaviour in the theory of p-adic interpolation of values of L-functions. In particular, we will be interested in the value of the archimedean L-function of V at the point $s = 1$, so we will assume that V is *critical* at $s = 1$ (in the sense of Deligne, [D79]) and is *exceptional* (in the sense of [G94], see definition 1.2.3 below). Later, in order to define the L-invariant, we will require that a certain Selmer group of V — the *balanced* Selmer group — vanishes. According to Greenberg's conjectures in [G94], this latter condition should be equivalent to the non-vanishing of $L(1, V)$.

Let us now define the conditions we impose on V.

Definition 1.1.1. *A p-adic representation (ρ, V) is called* ordinary *if there exists an exhaustive, separated, descending filtration $\{F^iV\}_{i\in\mathbf{Z}}$ of $G_{\mathbf{Q}_p}$-stable K-subspaces of V such that the inertia subgroup, I_p, acts on the ith graded piece $gr^iV = F^iV/F^{i+1}V$ via multiplication by χ_p^i.*

[3] A recent preprint of Denis Benois ([Be09]) uses the theory of (φ, Γ)-modules to suggest a generalization of Greenberg's L-invariant with ordinarity replaced by semistability. We do not address this notion here.

Remark 1.1.2. a) In other words, we have

$$\cdots \supseteq F^{i-1}V \supseteq F^iV \supseteq F^{i+1}V \supseteq \cdots$$

with

$$\begin{aligned} &\text{(exhaustive):} \quad F^iV = V \text{ for } i \ll 0 \\ &\text{(separated):} \quad F^iV = 0 \text{ for } i \gg 0 \end{aligned}$$

and each F^iV is sent to itself under the action of $G_{\mathbf{Q}_p}$. Furthermore, for any $\sigma \in I_p$ and $v \in F^iV$

$$\rho(\sigma)v \equiv \chi_p^i(\sigma)v \bmod F^{i+1}V.$$

b) It is common to denote

$$F^+V := F^1V \text{ and } F^-V := F^0V.$$

The first two conditions that Greenberg introduces are the following:

(S) for all $i \in \mathbf{Z}$, $G_{\mathbf{Q}_p}$ acts semisimply on gr^iV;

(U) V has no $G_{\mathbf{Q}_p}$-subquotient isomorphic to a cristalline extension of K by $K(1)$.[4]

Before defining the condition (T), we require a slightly finer understanding of the structure of V as a $G_{\mathbf{Q}_p}$-representation which we explain in the following section.

1.2 The exceptional subquotient

In [G94], Greenberg isolates a certain subquotient W of V that is supposed to control the behaviour of V with respect to the exceptional zeros of its p-adic L-function. For

[4] Such an extension is cristalline if, and only if, under the Kummer map, it comes from a unit (equivalently, if it is in the Bloch–Kato $H^1_f(\mathbf{Q}_p, K)$). Morally, this case is excluded as the L-invariant should be infinite (see, for example, Proposition 4.5.5(2) of [Em06] for the case of two-dimensional V over $\mathbf{Q}_p$).

this reason, we refer to W as the *exceptional subquotient* of V. In this short section, we describe its basic structure and state the condition (T) that we impose on V.

The definition of W basically relies on a slight refinement of the filtration on V between steps 0 and 2. Specifically, we introduce $F^{00}V$ and $F^{11}V$ with

$$F^0V \supseteq F^{00}V \supseteq F^1V \supseteq F^{11}V \supseteq F^2V.$$

Definition 1.2.1.

a) Let $F^{00}V$ be the maximal $G_{\mathbf{Q}_p}$-stable subspace of F^0V such that $G_{\mathbf{Q}_p}$ acts trivially on the quotient $F^{00}V/F^1V$.

b) Let $F^{11}V$ be the minimal $G_{\mathbf{Q}_p}$-stable subspace of F^1V such that $G_{\mathbf{Q}_p}$ acts by multiplication by χ_p on the quotient $F^1/F^{11}V$.

c) Define the exceptional subquotient *W of V as*

$$W := F^{00}V/F^{11}V.$$

We denote

$$m_0 := \dim_K F^{00}V/F^1V \text{ and } m_1 := \dim_K F^1V/F^{11}V$$

so that we have the equality $\dim_K W = m_0 + m_1$.

Remark 1.2.2. Since the action of $G_{\mathbf{Q}_p}$ on $gr^iV(-i)$ is unramified, we may consider the action of Frob_p on it. Packaging this information together for all i, we define a polynomial

$$H(x) := \prod_{i \in \mathbf{Z}} \det\left(I - \mathrm{Frob}_p\, p^i x | gr^i(V)(-i)\right).$$

Then, under hypothesis (S), for $i = 0, 1$, m_i is the multiplicity of p^i as an inverse root of $H(x)$. The inverse roots of $H(x)$ enter into the conjectural p-adic interpolation properties of the critical values of $L(s,V)$ as stated in the work of Coates and Perrin-Riou ([CoPR89, Co89]). This is the link that motivates the definition of W; we refer to the introduction of [G94] for the details.

The exceptional subquotient is an ordinary $G_{\mathbf{Q}_p}$-representation; indeed, it has a two step filtration given by

$$\begin{aligned} F^0W &= W = F^{00}V/F^{11}V \\ F^1W &= F^1V/F^{11}V \\ F^2W &= 0. \end{aligned}$$

Let $t_0 := \dim_K W^{G_{\mathbf{Q}_p}}$ and $t_1 := \dim_K (W^*)^{G_{\mathbf{Q}_p}}$.[5] We obtain two $G_{\mathbf{Q}_p}$-linear maps: an injection $\varphi_0 : K^{t_0} \hookrightarrow W$ and a surjection $\varphi_1 : W \twoheadrightarrow K(1)^{t_1}$. Letting

$$M := \ker(\varphi_1)/\operatorname{im}(\varphi_0),$$

we obtain an isomorphism of $G_{\mathbf{Q}_p}$-representations

$$W \cong M \oplus K^{t_0} \oplus K(1)^{t_1}.$$

The subspace M is a non-split extension of K^t by $K(1)^t$ for $2t = \dim_K W - t_0 - t_1$. A representation V with $W \cong M$ is said to be *of type* M. This is the case in the original Greenberg–Stevens work in [GS93]; indeed, in their situation $V = W$ is a non-split extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$. One of our goals in this work is to investigate the case where V is not of this type. This requires an understanding of the global Galois cohomology of V.[6]

[5] Recall that we use $-^*$ to denote the Tate dual $\operatorname{Hom}_K(-, K(1))$.

[6] See the discussion on pages 169–170 of [G94] for more details.

We may now state condition (T) as follows:

(T) At least one of t_0 or t_1 is zero.

Definition 1.2.3. *An ordinary p-adic representation satisfying conditions (S), (T), and (U) is called* exceptional *if its exceptional subquotient is non-zero. We define*

$$e := t + t_0 + t_1.$$

Then V is exceptional if, and only if, $e \neq 0$.

Greenberg conjectures that the order of vanishing of the p-adic L-function of V at $s = 1$ is more than that of the archimedean L-function by e.

1.3 Selmer groups

Selmer groups for arbitrary ordinary p-adic Galois representations were introduced by Greenberg in [G89] for the purpose of generalizing the main conjecture of Iwasawa theory (and Mazur's generalization of it to good, ordinary, modular elliptic curves, [Mz72]). The Selmer groups he deals with in that paper are over the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty/\mathbf{Q}$ and are conjecturally related to the p-adic L-function of V. In the last section of [G89], Greenberg suggests a possible definition for a Selmer group over $\mathbf{Q}$ that should be related to the archimedean L-function of V. Trivial zero considerations led Greenberg in [G94] to define a new Selmer group over $\mathbf{Q}$ (which we follow Hida in calling the *balanced* Selmer group). In this section, we introduce these Selmer groups and prove some basic properties that they satisfy. We expect that everything here is known to the experts.

Recall that a Selmer group is defined by a collection of *local conditions*, i.e a subspace $L_v(V) \subseteq H^1(\mathbf{Q}_v, V)$ for each place v of $\mathbf{Q}$. For the Selmer groups we consider, the conditions at $v \neq p$ will always be

$$L_v(V) = H^1_{\mathrm{nr}}(\mathbf{Q}_v, V)$$

where

$$\begin{aligned} H^1_{\mathrm{nr}}(\mathbf{Q}_v, V) &:= \ker\left(H^1(\mathbf{Q}_v, V) \longrightarrow H^1(I_v, V)\right) \\ &= \operatorname{im}\left(H^1\left(G_v/I_v, V^{I_v}\right) \longrightarrow H^1(\mathbf{Q}_v, V)\right) \end{aligned}$$

is the *unramified part* of $H^1(\mathbf{Q}_v, V)$ (also called the space of unramified classes). At p, several interesting conditions may be used. Given any such condition $L_p^?(V)$, the associated Selmer group is defined as

$$\mathrm{Sel}^?_{\mathbf{Q}}(V) := \ker\left(H^1(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_p, V)/L_p^?(V) \times \prod_{v \nmid p} H^1(\mathbf{Q}_v, V)/L_v(V)\right).$$

Remark 1.3.1. Given a $G_{\mathbf{Q}}$-stable lattice T in V and defining $A := V/T$, we may define Selmer groups for T and A by the same formula, but using the local conditions given by the inverse images (resp. the images) of those for V in T (resp. A).

The Selmer groups we are interested in are the following:

a) $\mathrm{Sel}_{\mathbf{Q}}(V)$, the ordinary Selmer group,

b) $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$, the balanced Selmer group.

The ordinary Selmer group (first introduced by Greenberg in [G89]) is defined with respect to the local condition at p given by

$$L_p(V) := \ker\left(H^1(\mathbf{Q}_p, V) \longrightarrow H^1\left(I_p, V/F^+V\right)\right).$$

Remark 1.3.2. In [G94], Greenberg modified his definition over number fields slightly to account for trivial zeroes; however, we follow Hida (e.g. [Hi-HMI]) and refer to the modified Selmer group as the "balanced Selmer group".

The local condition at p for the balanced Selmer group is defined as follows: it consists of $c \in H^1(\mathbf{Q}_p, V)$ such that

(Bal1) $c \in \ker\left(H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V/F^{00}V)\right)$, and

(Bal2) $c \bmod F^{11}V$ is in the image of

$$H^1_{\mathrm{unit}}\big(\mathbf{Q}_p, F^+V/F^{11}V\big) \oplus H^1_{\mathrm{nr}}\big(\mathbf{Q}_p, (F^{00}V/F^{11}V)^{G_p}\big)$$

in $H^1(\mathbf{Q}_p, V/F^{11}V)$,

where $H^1_{\mathrm{unit}}(\mathbf{Q}_p, F^+V/F^{11}V) \cong H^1_{\mathrm{unit}}(\mathbf{Q}_p, K(1)^{t+t_1}) \cong H^1_{\mathrm{unit}}(\mathbf{Q}_p, K(1)) \otimes_K K^{t+t_1}$ and $H^1_{\mathrm{unit}}(\mathbf{Q}_p, K(1))$ classifies cristalline extensions of K by $K(1)$.[7]

The basic relation between these two Selmer groups is given by the following lemma.

Lemma 1.3.3. *At p,*

$$\overline{L}_p(V) \subset L_p(V)$$

so

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) \subseteq \mathrm{Sel}_{\mathbf{Q}}(V).$$

Proof. Suppose $c \in \overline{L}_p(V)$. Condition (Bal1) implies that there is a $c' \in H^1(\mathbf{Q}_p, F^{00}V)$ mapping to c. We will show that $c \in L_p(V)$ by using the commutative diagram

$$\begin{tikzcd}
H^1(\mathbf{Q}_p, V) \arrow[r] & H^1(I_p, V/F^+V) \\
H^1(\mathbf{Q}_p, F^{00}V) \arrow[r] \arrow[u] & H^1(I_p, W/F^+W) \arrow[u]
\end{tikzcd}$$

[7] We could also write this as the Bloch–Kato Selmer group $H^1_f\big(\mathbf{Q}_p, F^+V/F^{11}V\big) \cong H^1_{\mathrm{unit}}\big(\mathbf{Q}_p, F^+V/F^{11}V\big)$

and by showing that c' maps to 0 under the map in the bottom row. Let $\overline{c}$ denote the image of c' in $H^1(\mathbf{Q}_p, W)$. Then, by (Bal2),

$$\overline{c} = \overline{c}_1 + \overline{c}_2$$

with $\overline{c}_1$ coming from $H^1(\mathbf{Q}_p, F^+W)$ and $\overline{c}_2$ coming from $H^1\big(G_p/I_p, W^{G_p}\big)$. In the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & & H^1(\mathbf{Q}_p, W/F^+W) \arrow[dr] & \\
H^1(\mathbf{Q}_p, F^{00}V) \arrow[r] & H^1(\mathbf{Q}_p, W) \arrow[ur] \arrow[dr] & & H^1(I_p, W/F^+W) \\
 & & H^1(I_p, W) \arrow[ur] &
\end{tikzcd}$$

$\overline{c}_1$ goes to zero along the upward path and $\overline{c}_2$ goes to zero along the downward path, thus c' is sent to zero on the right.

Since the local conditions away from p are the same in both cases, the second conclusion follows immediately from the first. $\square$

In the case we will be dealing with in chapters 2 and 3, the ordinary and balanced Selmer groups are equal.

Lemma 1.3.4. *Suppose* $F^{11}V = F^+V$, *i.e.* $t = t_1 = 0$. *Then*

$$\overline{L}_p(V) = L_p(V).$$

Thus,

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) = \mathrm{Sel}_{\mathbf{Q}}(V).$$

Proof. We know the inclusion $\overline{L}_p(V) \subseteq L_p(V)$ from the previous lemma. Now, pick an element $c \in L_p(V)$ and consider the following commutative diagram

$$\begin{tikzcd} H^1\Big(G_p/I_p, (V/F^{00}V)^{I_p}\Big) \arrow[r, "f_1"] & H^1(\mathbf{Q}_p, V/F^{00}V) \arrow[r, "f_2"] & H^1(I_p, V/F^{00}V) \\ & c \in H^1(\mathbf{Q}_p, V) \arrow[u, "f_3"] \arrow[r, "f_5"] & H^1(I_p, V/F^+V) \arrow[u, "f_4"] \end{tikzcd}$$

whose top row is exact (by inflation–restriction). Since $c \in L_p(V)$, we know that $f_5(c) = 0$, and thus $f_2 \circ f_3(c) = 0$ by commutativity. Showing that $\ker f_2 = \operatorname{im} f_1 = 0$ would therefore imply that $c \in \ker f_3$ and thus satisfies condition (Bal1). In fact,

$$H^1\Big(G_p/I_p, \big(V/F^{00}V\big)^{I_p}\Big) = 0;$$

indeed, $(V/F^{00}V)^{I_p} = F^0V/F^{00}V$ and $G_p/I_p \cong \hat{\mathbf{Z}}$ with (topological) generator Frob_p, so by the cohomology of a pro-cyclic group

$$H^1\Big(G_p/I_p, \big(V/F^{00}V\big)^{I_p}\Big) = \big(F^0V/F^{00}V\big) \,/\, \big((\mathrm{Frob}_p - 1)\, \big(F^0V/F^{00}V\big)\big) = 0.$$

The latter equality stems from the fact that $F^{00}V$ is the part of F^0V on which Frob_p acts trivially.

It remains to check (Bal2). Since $F^{11}V = F^+V$, we have that $W^{G_p} = W \cong K^{t_0}$. The condition (Bal2) then reads that $c \mod F^+V$ must be in the image of $H^1_{\mathrm{nr}}(\mathbf{Q}_p, F^{00}V/F^+V)$ in $H^1(\mathbf{Q}_p, V/F^+V)$. Consider the commutative diagram

$$\begin{tikzcd} c \in H^1(\mathbf{Q}_p, V) \arrow[r] & H^1(\mathbf{Q}_p, V/F^+V) \arrow[r] & H^1(I_p, V/F^+V) \\ H^1(\mathbf{Q}_p, F^{00}V) \arrow[r] \arrow[u, hook] & H^1(\mathbf{Q}_p, F^{00}V/F^+V) \arrow[r] \arrow[u, hook] & H^1(I_p, F^{00}V/F^+V)\,. \arrow[u, hook] \end{tikzcd}$$

The first two vertical arrows are injective since $\big(V/F^{00}V\big)^{G_p} = 0$. The injectivity of the third arrow follows from the exactness (on the right) of

$$0 \longrightarrow \left(F^{00}V/F^{+}V\right)^{I_p} \longrightarrow \left(V/F^{+}V\right)^{I_p} \longrightarrow \left(V/F^{00}V\right)^{I_p} \longrightarrow 0.$$

By the first part of this proof, there is a $c' \in H^1(\mathbf{Q}_p, F^{00}V)$ mapping to c. Let $\overline{c}'$ denote the image of c' in $H^1(\mathbf{Q}_p, F^{00}V/F^{+}V)$. Then, the image of $\overline{c}'$ under the middle vertical map is $c \bmod F^{+}V$. Thus, it remains to show that

$$\overline{c}' \in H^1_{\mathrm{nr}}\big(\mathbf{Q}_p, F^{00}V/F^{+}V\big) = \ker\big(H^1\big(\mathbf{Q}_p, F^{00}V/F^{+}V\big) \longrightarrow H^1\big(I_p, F^{00}V/F^{+}V\big)\big)\,.$$

By the definition of $L_p(V)$, c is sent to zero in the top-right entry of the commutative diagram above. By commutativity, so is $\overline{c}'$. Since the third vertical map is injective, this implies $\overline{c}'$ is sent to zero in the bottom-right entry of the above diagram, as desired. $\square$

Remark 1.3.5. In fact, we can relate these Selmer groups to Bloch–Kato Selmer groups. Indeed, recall that if V is ordinary, then it is semi-stable in the sense of p-adic Hodge theory ([PR94, Théorème 1.5], due to Fontaine). A lemma of Flach ([Fl90, Lemma 2]) shows that

$$\mathrm{Sel}_{\mathbf{Q}}(V) = H^1_g(\mathbf{Q}, V)\,.$$

Under the hypotheses of the above lemma, namely that $F^1V = F^{11}V$,

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) = \mathrm{Sel}_{\mathbf{Q}}(V) = H^1_g(\mathbf{Q}, V) = H^1_f(\mathbf{Q}, V)\,, \tag{1.1}$$

where the last equality follows from [BK90, Corollary 3.8.4].

Let us mention that if one were interested in replacing $\mathbf{Q}$ by a finite extension F, Pottharst has recently shown that

$$\mathrm{Sel}_F(V) = H^1_g(F, V)$$

(see [Po09] Theorem 3.5(2), Proposition 3.11(1), and example 3.13) so that the equations in (1.1) hold with $\mathbf{Q}$ replaced with F.

An important property of the balanced Selmer group — which gives it its name — is the following result due to Greenberg. The proof relies on the criticality of $L(s, V)$ at $s = 1$ and, for this reason, we make this condition explicit in our statement.

Proposition 1.3.6 (Proposition 2 of [G94]). *Assume V is critical at $s = 1$. Then $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ and $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V^*)$ have the same dimension.*

1.4 Definition of the L-invariant

To define the L-invariant, Greenberg begins by isolating an e-dimensional subspace, $H^{\mathrm{exc}}_{\mathrm{glob}}(V)$, of a global Galois cohomology group. Taking its image, $H^{\mathrm{exc}}_{\mathrm{loc}}(V)$, in the local cohomology with coefficients in W/F^+W, he defines the L-invariant, $\mathcal{L}(V)$, as the "slope" of $H^{\mathrm{exc}}_{\mathrm{loc}}(V)$ in certain natural coordinates. In this section, we give the definitions of these objects, and in the next section, we outline a general idea for computing the L-invariant in the case $e = 1$.

Recall that we are assuming that V is an ordinary p-adic Galois representation satisfying conditions (S), (T), and (U), that is critical at $s = 1$ and exceptional. We will now also be assuming that $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) = 0$.

Let us introduce a little bit of notation before continuing. For a group G acting on V, we denote

$$\begin{aligned} F^i H^1(G,V) &:= \ker\big(H^1(G,V) \longrightarrow H^1\big(G, V/F^iV\big)\big) \\ &= \operatorname{im}\big(H^1\big(G, F^iV\big) \longrightarrow H^1(G,V)\big) \end{aligned}$$

with similar notations for F^{ii} and W.

Let Σ be the set of primes ramified for V together with p and ∞, and let G_Σ denote $\mathrm{Gal}(\mathbf{Q}_\Sigma/\mathbf{Q})$, where $\mathbf{Q}_\Sigma$ is the maximal extension of $\mathbf{Q}$ unramified outside of Σ. From the local conditions that we have imposed away from p, it is clear that $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) \subseteq H^1(G_\Sigma, V)$. Consider the beginning of the Poitou–Tate exact sequence with local conditions $\overline{L}_v(V)$:

$$0 \longrightarrow \overline{\mathrm{Sel}}_{\mathbf{Q}}(V) \longrightarrow H^1(G_\Sigma, V) \longrightarrow \bigoplus_{v\in\Sigma} H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V) \longrightarrow \overline{\mathrm{Sel}}_{\mathbf{Q}}(V^*)$$

(see, for example, [PR00, Proposition A.3.3]). The vanishing of $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$, together with proposition 1.3.6, thus gives an isomorphism

$$H^1(G_\Sigma, V) \xrightarrow{\sim} \bigoplus_{v\in\Sigma} H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V). \tag{1.2}$$

We may exploit this isomorphism by defining an e-dimensional subspace of the right-hand side, and transferring it over to the left. We use the following lemma.

Lemma 1.4.1. *The subspace $\overline{L}_p(V)$ has codimension e in $F^{00}H^1(\mathbf{Q}_p, V)$.*

For the proof, see [G94, p. 161].

Definition 1.4.2. *Define $H^{\mathrm{exc}}_{\mathrm{glob}}(V)$ to be the e-dimensional subspace of $H^1(G_\Sigma, V)$ corresponding to the subspace $F^{00}H^1(\mathbf{Q}_p, V)\,/\overline{L}_p(V)$ of $\bigoplus_{v\in\Sigma} H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V)$ under the isomorphism in (1.2).*

Remark 1.4.3. In [G94], Greenberg uses the notation $\widetilde{\mathbf{T}}$ for this subspace, whereas Hida uses the notation $\mathbf{H}$ (see, for example, [Hi07]). Our notation attempts to convey the sense that this subspace is a space of global cohomology classes attached to the exceptional nature of V. Hopefully it is not too cumbersome.

Since we are assuming that V satisfies hypothesis (T), we will assume that in fact $t_1 = 0$ (as will be the case in the following chapters); otherwise, one can replace V with V^*. Consider the image of $H^{\mathrm{exc}}_{\mathrm{glob}}(V)$ under the composition

$$\lambda : H^1(G_\Sigma, V) \longrightarrow H^1(\mathbf{Q}_p, V) \longrightarrow H^1\big(\mathbf{Q}_p, V/F^+V\big)$$

and note that the natural map

$$H^1\big(\mathbf{Q}_p, W/F^+W\big) \hookrightarrow H^1\big(\mathbf{Q}_p, V/F^+V\big)$$

is an injection (indeed, $H^0(\mathbf{Q}_p, V/F^{00}V) = 0$). We identify $H^1(\mathbf{Q}_p, W/F^+W)$ with its image inside $H^1(\mathbf{Q}_p, V/F^+V)$.

Proposition-Definition 1.4.4. *The image of $H^{\mathrm{exc}}_{\mathrm{glob}}(V)$ under λ satisfies the following three properties:*

a) $\lambda(H^{\mathrm{exc}}_{\mathrm{glob}}(V)) \subseteq H^1(\mathbf{Q}_p, W/F^+W)$,

b) $\dim \lambda(H^{\mathrm{exc}}_{\mathrm{glob}}(V)) = e$, *and*

c) $\lambda(H^{\mathrm{exc}}_{\mathrm{glob}}(V)) \cap H^1_{\mathrm{nr}}(\mathbf{Q}_p, W/F^+W) = 0$.

We define $H^{\mathrm{exc}}_{\mathrm{loc}}(V)$ to be $\lambda(H^{\mathrm{exc}}_{\mathrm{glob}}(V))$ considered as a subspace of $H^1(\mathbf{Q}_p, W/F^+W)$.

We refer to section 3 of [G94] for these statements.

One last step before defining the L-invariant is to discuss the natural coordinates mentioned at the beginning of this section. The space W/F^+W is a trivial $G_{\mathbf{Q}_p}$-module, so

$$H^1\big(\mathbf{Q}_p, W/F^+W\big) \cong \mathrm{Hom}\left(G_{\mathbf{Q}_p}, W/F^+W\right).$$

Any such homomorphism factors through the maximal pro-p quotient of $G_{\mathbf{Q}_p}^{ab}$ given by $\mathrm{Gal}(\mathbf{F}_\infty/\mathbf{Q}_p)$, where $\mathbf{F}_\infty$ is the compositum of two $\mathbf{Z}_p$-extensions of $\mathbf{Q}_p$: the cyclotomic one, $\mathbf{Q}_{\infty,p}$, and the maximal unramified abelian extension $\mathbf{Q}_p^{\mathrm{nr}}$. Writing

$$\Gamma_\infty := \mathrm{Gal}\left(\mathbf{Q}_{\infty,p}/\mathbf{Q}_p\right) \cong \mathrm{Gal}\left(\mathbf{F}_\infty/\mathbf{Q}_p^{\mathrm{nr}}\right)$$

and

$$\Gamma_{\mathrm{nr}} := \mathrm{Gal}\left(\mathbf{Q}_p^{\mathrm{nr}}/\mathbf{Q}_p\right) \cong \mathrm{Gal}\left(\mathbf{F}_\infty/\mathbf{Q}_{\infty,p}\right),$$

we have that

$$\mathrm{Gal}\left(\mathbf{F}_\infty/\mathbf{Q}_p\right) = \Gamma_\infty \times \Gamma_{\mathrm{nr}}.$$

Thus,

$$H^1\big(\mathbf{Q}_p, W/F^+W\big) = \mathrm{Hom}\left(\Gamma_\infty, W/F^+W\right) \times \mathrm{Hom}\left(\Gamma_{\mathrm{nr}}, W/F^+W\right).$$

Let pr_∞ and $\mathrm{pr}_{\mathrm{nr}}$ denote the associated projections, let pr'_∞ and $\mathrm{pr}'_{\mathrm{nr}}$ denote their restrictions to $H^{\mathrm{exc}}_{\mathrm{loc}}(V)$, and note that, by part c) of the above proposition-definition, pr'_∞ is invertible. The one-dimensional $\mathbf{Q}_p$-vector spaces $\mathrm{Hom}(\Gamma_\infty, \mathbf{Q}_p)$ and $\mathrm{Hom}(\Gamma_{\mathrm{nr}}, \mathbf{Q}_p)$ each have a natural basis given by the p-adic logarithm of the cyclotomic character, $\log \chi_p$, and the ord function (sending Frob_p to 1), respectively. Since

$$\mathrm{Hom}(\Gamma_\infty, W/F^+W) = \mathrm{Hom}(\Gamma_\infty, \mathbf{Q}_p) \otimes W/F^+W$$

and

$$\mathrm{Hom}(\Gamma_{\mathrm{nr}}, W/F^+W) = \mathrm{Hom}(\Gamma_{\mathrm{nr}}, \mathbf{Q}_p) \otimes W/F^+W,$$

$\log \chi_p$ and ord define isomorphisms

$$\begin{aligned} \iota_\infty : \mathrm{Hom}(\Gamma_\infty, W/F^+W) &\to W/F^+W \\ \log \chi_p \otimes w &\mapsto w \end{aligned}$$

and

$$\begin{aligned} \iota_{\mathrm{nr}} : \mathrm{Hom}(\Gamma_{\mathrm{nr}}, W/F^+W) &\to W/F^+W \\ \mathrm{ord} \otimes w &\mapsto w. \end{aligned}$$

We thus have the following commutative diagram

$$\begin{tikzcd}
 & H^1(\mathbf{Q}_p, W/F^+W) \arrow[ddl, bend right, "\mathrm{pr}_\infty"'] \arrow[ddr, bend left, "\mathrm{pr}_{\mathrm{nr}}"] & \\
 & H^{\mathrm{exc}}_{\mathrm{loc}}(V) \arrow[u, hook] \arrow[dl, "\sim", "\mathrm{pr}'_\infty"'] \arrow[dr, "\mathrm{pr}'_{\mathrm{nr}}"'] & \\
\mathrm{Hom}\left(\Gamma_\infty, W/F^+W\right) \arrow[d, "\iota_\infty", "\wr"'] & & \mathrm{Hom}\left(\Gamma_{\mathrm{nr}}, W/F^+W\right) \arrow[d, "\iota_{\mathrm{nr}}", "\wr"'] \\
W/F^+W \arrow[rr, dashed, "\underline{\mathcal{L}}"] & & W/F^+W
\end{tikzcd}$$

which defines the linear map

$$\underline{\mathcal{L}} = \iota_{\mathrm{nr}}\, \mathrm{pr}'_{\mathrm{nr}}\, \mathrm{pr}'^{-1}_\infty\, \iota^{-1}_\infty.$$

Definition 1.4.5. *The L-invariant of V is*

$$\mathcal{L}(V) := \det \underline{\mathcal{L}}.$$

The L-invariant is basically measuring how skewed $H^{\mathrm{exc}}_{\mathrm{loc}}$ sits as a subspace of $H^1(\mathbf{Q}_p, W/F^+W)$ with respect to the natural coordinates.

1.5 Using a global Galois cohomology class

This section is basically an extended remark discussing how one can compute the L-invariant of a p-adic Galois representation with $t_0 = 1$ and $t = t_1 = 0$ starting with a global Galois cohomology class.

Suppose that $t_0 = 1$ and $t = t_1 = 0$, and that we have somehow obtained $[c] \in H^1(\mathbf{Q}, V)$. Suppose further that $[c]$ satisfies

a) $[c_v] \in H^1_{\mathrm{nr}}(\mathbf{Q}_v, V)$, for all $v \in \Sigma \setminus \{p\}$,

b) $[c_p] \in F^{00}H^1(\mathbf{Q}_p, V)$,

c) $[c_p] \not\equiv 0 \bmod F^+V$.

Then, $[c]$ generates $H^{\mathrm{exc}}_{\mathrm{glob}}$. Thus, $[\bar{c}_p] := ([c_p] \bmod F^+V)$ generates $H^{\mathrm{exc}}_{\mathrm{loc}}$. Since $W/F^+W \cong K$, we may view the maps $\iota_\infty \operatorname{pr}_\infty$ and $\iota_{\mathrm{nr}} \operatorname{pr}_{\mathrm{nr}}$ as giving K-coordinates on $H^1(\mathbf{Q}_p, W/F^+W)$. If we have a cocycle representative $\bar{c}_p$ of $[\bar{c}_p]$ whose image lies in $F^{00}V/F^+V$ — such a representative exists by condition b) — then we can explicitly realize these coordinates as evaluation maps as follows. Let $\varphi \in \operatorname{Hom}(G_{\mathbf{Q}_p}, W/F^+W)$. Suppose $\operatorname{pr}_{\mathrm{nr}} \varphi = y \operatorname{ord}$, then

$$\iota_{\mathrm{nr}} \operatorname{pr}_{\mathrm{nr}} \varphi = y = y \operatorname{ord}(\mathrm{Frob}_p) = \varphi(\mathrm{Frob}_p),$$

so the second coordinate is obtained by evaluating at Frob_p. The first coordinate is only slightly more complicated. Suppose $\operatorname{pr}_\infty \varphi = x \log \chi_p$. Let u be any principal unit, so that $\chi_p(\mathrm{rec}(u)) = u^{-1}$. Then

$$\iota_\infty \operatorname{pr}_\infty \varphi = x = -\frac{x}{\log u} \log \chi_p(\mathrm{rec}(u)) = -\frac{1}{\log u} \varphi(\mathrm{rec}(u)).$$

Therefore, the coordinates of $[\overline{c}_p]$ are

$$\left(-\frac{1}{\log u}\overline{c}_p(\mathrm{rec}(u)), \overline{c}_p(\mathrm{Frob}_p)\right) \tag{1.3}$$

and the linear map $\underline{\mathcal{L}}$ acts by scalar multiplication by the ratio of the second coordinate to the first. In other words, the L-invariant is simply the slope of the line generated by $[\overline{c}_p]$:

$$\mathcal{L}(V) = \frac{-\overline{c}_p(\mathrm{Frob}_p)}{\frac{1}{\log u}\overline{c}_p(\mathrm{rec}(u))} \tag{1.4}$$

independent of the choice of principal unit u.

Chapter 2

L-invariant of the symmetric square of a modular form

In this chapter, we prove a formula for Greenberg's L-invariant of the symmetric square of a p-ordinary elliptic modular form, f, only assuming that its balanced Selmer group vanishes (corollary 2.3.5, also known as theorem A). The first results on these L-invariants were those of Greenberg concerning Tate curves, and good ordinary CM elliptic curves ([G94, p. 170]). Since then Hida has provided in [Hi04] a conditional proof (known to be true in most cases) of the formula we prove in this chapter relating the L-invariant to the p-adic analytic function interpolating the eigenvalues of Frobenius in the Hida family attached to f. Our approach differs from Hida's in two important ways. Firstly, it does not require the existence of the versal nearly p-ordinary Galois deformation ring, nor any knowledge of its local structure. We also do not require the module of differentials he uses to carry out his computations. For these reasons, we believe our proof to be of interest.

The difficult case of our theorem occurs when p does not divide the level of f (the simpler case being the result of Greenberg on Tate curves). In this case, the L-invariant is of a different nature than both the Tate curve case and the Greenberg–

Stevens result ([GS93]). Specifically, in those cases, the L-invariant is defined in terms of local Galois cohomology groups, and techniques of Kummer theory and local Tate duality suffice to arrive at an answer. In the case of a p-unramified modular form, a *global* Galois cohomology class is required to compute the L-invariant. It is here that we take inspiration from Ribet [Ri76] and seek out an irreducible Galois deformation which becomes reducible at our point of interest. Specifically, we tensor the Hida deformation with the constant deformation and study the family obtained.

In view of Greenberg's result for Tate curves, we may restrict ourselves to the case where p does not divide the level. In section 2.1, we describe the symmetric square Galois representation and its exceptional subquotient. What little information we require regarding the Hida deformation is contained in section 2.2. In section 2.3, we present our computation of the L-invariant by first describing how we obtain a global Galois cohomology class, then computing its local coordinates as in section 1.5. We finish up this chapter with a short discussion of the questions not addressed in this work, followed by other concluding remarks.

2.1 The symmetric square Galois representation

Let $f \in S_{k_0}(\Gamma_0(N), \psi)$ be a p-ordinary, weight $k_0 \geq 2$ new eigenform of level N (prime to p) and character ψ (of conductor prime to p). To ensure the vanishing of the balanced Selmer group, one may make additional assumptions (see theorem 2.1.1 and the remark following it). Write α_p for the unit root of the Hecke polynomial of f at p and β_p for the other root. Let $E := \mathbf{Q}(f)$ be the number field generated by the Fourier coefficients of f. Let $\mathfrak{p}|p$ be the prime of E determined by the embedding ι_p and let

$$\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}(V_f)$$

be the contragredient[1] of the $\mathfrak{p}$-adic Galois representation attached to f by Deligne ([D71]) on a two-dimensional $E_{\mathfrak{p}}$-vector space V_f.

Wiles showed ([W88, Theorem 2.1.4]) that

$$\rho_f|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \chi_p^{k_0-1}\delta^{-1} & \varphi \\ 0 & \delta \end{pmatrix} \tag{2.1}$$

where χ_p is the p-adic cyclotomic character and δ is the unramified character sending Frob_p to α_p. The symmetric square of $\rho_f|_{G_{\mathbf{Q}_p}}$ is then equivalent to

$$\begin{pmatrix} \chi_p^{2(k_0-1)}\delta^{-2} & 2\chi_p^{k_0-1}\varphi\delta^{-1} & \varphi^2 \\ & \chi_p^{k_0-1} & \varphi\delta \\ & & \delta^2 \end{pmatrix}.$$

Let $\rho := \big(\mathrm{Sym}^2\rho_f\big)(1-k_0)$ and denote its representation space by V. Then

$$\rho|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \chi_p^{k_0-1}\delta^{-2} & 2\varphi\delta^{-1} & \chi_p^{1-k_0}\varphi^2 \\ & 1 & \chi_p^{1-k_0}\varphi\delta \\ & & \chi_p^{1-k_0}\delta^2 \end{pmatrix}. \tag{2.2}$$

[1]For example, we take the Tate module of an elliptic curve, not its first étale cohomology group.

Thus, ρ is an ordinary representation with filtration given by

$$\begin{array}{ccc} V = F^{1-k_0}V & & \\ \cup\!\! \mid & & \\ F^{00}V = F^{2-k_0}V = \cdots = F^0 V & \sim & \begin{pmatrix} \chi_p^{k_0-1}\delta^{-2} & 2\varphi\delta^{-1} \\ & 1 \end{pmatrix} \\ \cup\!\! \mid & & \\ F^{11}V = F^1 V = \cdots = F^{k_0-1}V & \sim & \left(\chi_p^{k_0-1}\delta^{-2}\right) \\ \cup\!\! \mid & & \\ F^{k_0}V = 0. & & \end{array}$$

Since the graded pieces are all one-dimensional, ρ clearly satisfies condition (S) of chapter 1. The only chance condition (U) has of being violated occurs when $k_0 = 2$, in which case δ must be trivial; this cannot happen since N is prime to p and $\psi(p) = 1$. The exceptional subquotient of V is $W \cong \mathbf{1}$ appearing in the middle of the matrix in (2.2). Clearly, $t_1 = 0$, so ρ satisfies Greenberg's condition (T). As explained in chapter 1, Greenberg's theory then says that the p-adic L-function attached to ρ should have a trivial zero at $s = 1$ (of order one).

Recall that in order to define Greenberg's L-invariant, the balanced Selmer group $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ must vanish.[2] By equation (1.1) in remark 1.3.5 relating the balanced Selmer group to Bloch–Kato Selmer groups, we may use the results of either Weston ([Wes04, Theorem 5.5]) or Kisin ([Ki04, Corollary 7.12]), in addition to those of Hida ([Hi04, Lemma 2.1]). These results spring from different approaches to the subject. Kisin's approach is modeled on that of Wiles ([W95]), i.e. it is in the spirit of modular Galois deformations and requires the type of assumptions common in that field. Hida's approach is in the same family in that it relies on knowledge of the local structure

[2] Let T be a $G_{\mathbf{Q}}$-stable lattice in V and let $A := V/T$. The vanishing of $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ is equivalent to the finiteness of $\overline{\mathrm{Sel}}_{\mathbf{Q}}(A)$.

of the universal p-nearly ordinary Galois deformation ring that can be obtained from modularity theorems. Weston's approach is via geometric Euler systems and, with some work, should apply to large classes of adjoint motives (such as those dealt with in the next chapter). Let us combine these results.

Theorem 2.1.1 ([Hi04, Ki04, Wes04])**.** *Let $\overline{\rho}_f$ denote a reduction of ρ_f modulo $\mathfrak{p}$. Suppose at least one of the following holds*

a) for any quadratic extension $L/\mathbf{Q}$ contained in $\mathbf{Q}(\zeta_{p^3})$, $\overline{\rho}_f|_{G_L}$ is absolutely irreducible;

b) $\overline{\rho}_f$ is not absolutely irreducible, and the diagonal characters of $\overline{\rho}_f \otimes \overline{\mathbf{F}}_p$ are distinct when restricted to $G_{\mathbf{Q}(\zeta_{p^\infty})}$;

c) f is special or supercuspidal at all primes dividing N, non-CM, (and $p \nmid N$);

d) f satisfies the assumptions in theorem 1 of [Hi04].

Then, $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ vanishes.

Remark 2.1.2. Conditions a) and b) are due to Kisin, condition c) is due to Weston, and the last condition is due to Hida. Weston remarks that one should be able to remove the assumptions on the ramification of f away from p via a more careful analysis of the bad reduction of the self-product of the Kuga–Sato variety. We entertain hopes of addressing this question in future work.

2.2 The Hida deformation of ρ_f

To put f in a p-adic analytic family, we must replace it with its p-stabilization f_p given by

$$f_p(z) = f(z) - \beta_p f(pz);$$

in some sense, this “removes” the Euler factor corresponding to β_p.[3] Writing the q-expansion of f_p as

$$f_p(q) = \sum_{n \geq 1} a_n q^n$$

one has

$$a_p = \alpha_p.$$

By Hida theory, there exists an open disk U around $k_0 \in \mathbf{Z}_p$ and a formal q-expansion that we denote

$$\mathcal{F} = \sum_{n \geq 1} \alpha_n(k) q^n \in \mathcal{A}_U[\![q]\!]$$

(where $\mathcal{A}_U$ denotes the ring of p-adic analytic functions in the variable $k \in U$) such that when $k \in U$ is an integer greater than 1, the specialization $\mathcal{F}_k$ is the q-expansion of a (non-zero) weight k, p-stabilized, p-ordinary newform of tame conductor N. Furthermore, at k_0, we have

$$\mathcal{F}_{k_0} = f_p.$$

The latter justifies the notation for $\alpha_n(k)$ since $\alpha_p(k_0) = a_p = \alpha_p$.

Hida showed ([Hi86], [GS93, pp. 418–419]) that $\mathcal{F}$ gives rise to a “big” Galois representation

$$\rho_{\mathcal{F}} : G_{\mathbf{Q}} \to \mathrm{GL}(2, \mathcal{A}_U)$$

whose specialization at a given integer $k \geq 2$ in U is equivalent to the contragredient of that attached by Deligne to the modular form $\mathcal{F}_k$. Moreover, Wiles [W88, Theorem 2.2.2] (building on work of Mazur–Wiles [MzW86]) showed that the local Galois representation

$$\rho_{\mathcal{F}}|_{G_{\mathbf{Q}_p}} : G_{\mathbf{Q}_p} \to \mathrm{GL}(2, \mathcal{A}_U)$$

[3] For an introduction to this operation, also known as a refinement, see [Mz00].

is equivalent to

$$\begin{pmatrix} \theta\mu^{-1} & \xi \\ 0 & \mu \end{pmatrix} \tag{2.3}$$

where, for any $k \in U$,

$$\theta(k) = \chi_p \langle \chi_p \rangle^{k-2}$$

(where $\langle \cdot \rangle$ is the projection to the principal units), and $\mu(k)$ is the unramified character sending a Frobenius element to $\alpha_p(k)$. It is for this reason that we refer to $\alpha_p(k)$ as the p-adic analytic function interpolating the eigenvalues of Frobenius in the Hida family.

2.3 Computing the L-invariant

In this section, we compute the L-invariant, $\mathcal{L}(\rho)$, using a global Galois cohomology class as explained in section 1.5. We obtain the class via a method inspired by Ribet ([Ri76, Prop. 2.1]) using the Hida deformation $\rho_{\mathcal{F}}$ and some multilinear algebra. Basically, we produce an irreducible family of Galois representations whose specialization at k_0 decomposes as $\rho \oplus \mathbf{1}$. This type of behaviour leads to a non-trivial element in $H^1(\mathbf{Q}, \rho)$.

2.3.1 Obtaining a global Galois cohomology class

Consider the representation over $\mathcal{A}_U$ given by

$$r^\circ := \rho_{\mathcal{F}} \otimes_{\mathcal{A}_U} (\rho_f(1-k_0) \otimes \mathcal{A}_U)$$

on a free $\mathcal{A}_U$-module of rank 4, and let

$$r := r^\circ \otimes_{\mathcal{A}_U} \mathrm{Frac}(\mathcal{A}_U).$$

Since $\rho_f^\vee \cong \rho_f(1-k_0)$, specializing at k_0 gives

$$r_{k_0}^\circ \cong \rho_f \otimes \rho_f^\vee \cong \mathrm{End}(\rho_f) \cong \rho \oplus \mathbf{1}$$

So we see that r has reducible reduction. It will in fact be a corollary of our method that the representation r is irreducible (though one can easily prove this directly in the non-CM case).

From the reducibility above, we construct a non-split extension of $\mathbf{1}$ by ρ, i.e. a non-zero element of $H^1(\mathbf{Q}, \rho)$. The method basically consists in scaling part of the lattice by $(s-k_0)$. The explicit calculations are as follows.

Using the bases giving equations (2.1) and (2.3), one obtains

$$r^\circ|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \frac{\theta}{\mu\delta} & \frac{\theta\varphi}{\chi_p^{k_0-1}\mu} & \frac{\xi}{\delta} & \frac{\xi\varphi}{\chi_p^{k_0-1}} \\ & \frac{\theta\delta}{\chi_p^{k_0-1}\mu} & 0 & \frac{\xi\delta}{\chi_p^{k_0-1}} \\ & & \frac{\mu}{\delta} & \frac{\mu\delta}{\chi_p^{k_0-1}} \\ & & & \frac{\mu\delta}{\chi_p^{k_0-1}} \end{pmatrix}.$$

Conjugating by the change of basis matrix

$$U = \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ & 1 & & -1 \\ & & 1 & \end{pmatrix}$$

which corresponds to the direct sum decomposition

$$\rho_f \otimes \rho_f(1-k_0) = \left(\mathrm{Sym}^2\rho_f \oplus \wedge^2\rho_f\right)(1-k_0) \tag{2.4}$$

gives

$$r^\circ|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \frac{\theta}{\mu\delta} & \frac{\theta\varphi}{\chi_p^{k_0-1}\mu} + \frac{\xi}{\delta} & \frac{\xi\varphi}{\chi_p^{k_0-1}} & \frac{\theta\varphi}{\chi_p^{k_0-1}\mu} - \frac{\xi}{\delta} \\ 0 & \frac{1}{2}\left(\frac{\theta\delta}{\chi_p^{k_0-1}\mu} + \frac{\mu}{\delta}\right) & \frac{1}{2\chi_p^{k_0-1}}(\xi\delta + \mu\varphi) & \frac{1}{2}\left(\frac{\theta\delta}{\chi_p^{k_0-1}\mu} - \frac{\mu}{\delta}\right) \\ 0 & 0 & \frac{\mu\delta}{\chi_p^{k_0-1}} & 0 \\ 0 & \frac{1}{2}\left(\frac{\theta\delta}{\chi_p^{k_0-1}\mu} - \frac{\mu}{\delta}\right) & \frac{1}{2\chi_p^{k_0-1}}(\xi\delta - \mu\varphi) & \frac{1}{2}\left(\frac{\theta\delta}{\chi_p^{k_0-1}\mu} + \frac{\mu}{\delta}\right) \end{pmatrix}. \tag{2.5}$$

The latter indeed specializes to

$$\rho|_{G_{\mathbf{Q}_p}} \oplus \mathbf{1} \sim \left(\begin{array}{ccc|c} \frac{\chi_p^{k_0-1}}{\delta^2} & \frac{2\varphi}{\delta} & \frac{\varphi^2}{\chi_p^{k_0-1}} & \\ & 1 & \frac{\varphi\delta}{\chi_p^{k_0-1}} & \\ & & \frac{\delta^2}{\chi_p^{k_0-1}} & \\ \hline & & & 1 \end{array}\right)$$

at k_0. Since the decomposition in (2.4) depends only on the representation theory of GL(2), it is global, so that in fact we have picked a basis that gives a global direct sum decomposition

$$r_{k_0}^\circ = \rho \oplus \mathbf{1}.$$

Expanding some of the entries to first order in $\epsilon := (s - k_0)$ using the formulae

$$\begin{array}{rcl} \theta & \approx & \chi_p^{k_0-1} + \epsilon\theta' \\ \xi & \approx & \varphi + \epsilon\xi' \end{array} \quad \text{and} \quad \begin{array}{rcl} \mu & \approx & \delta + \epsilon\mu' \\ \frac{1}{\mu} & \approx & \frac{1}{\delta} - \epsilon\frac{\mu'}{\delta^2} \end{array}$$

(where the prime denotes a derivative at $s = k_0$) yields

$$r^\circ|_{G_p} \sim \left(\begin{array}{ccc|c} & & & \frac{\epsilon}{\delta}\left(\varphi\left(\frac{\theta'}{\chi_p^{k_0-1}} - \frac{\mu'}{\delta}\right) + \xi'\right) \\ & A & & \epsilon\left(\frac{\theta'}{2\chi_p^{k_0-1}} - \frac{\mu'}{\delta}\right) \\ & & & 0 \\ \hline 0 & \epsilon\left(\frac{\theta'}{2\chi_p^{k_0-1}} - \frac{\mu'}{\delta}\right) & \frac{\epsilon}{2\chi_p^{k_0-1}}\left(\xi'\delta - \mu'\varphi\right) & D \end{array}\right) \tag{2.6}$$

where A is a deformation of ρ and D is a deformation of $\mathbf{1}$. Note that $\epsilon = (s - k_0)$ is a factor of both the top-right and the bottom-left, as it should be. Suppose the basis of r° giving the matrix in equation (2.5) is (e_1, e_2, e_3, e_4).

Lemma 2.3.1. *If we replace e_4 with*

$$e_4' := \frac{1}{\epsilon} e_4,$$

then we obtain a new $G_{\mathbf{Q}}$-stable lattice r_1° in r whose specialization at k_0 is

$$\mathcal{E} := r_{1,k_0}^\circ \sim \begin{pmatrix} \rho & c \\ 0 & 1 \end{pmatrix}$$

with

$$c|_{G_{\mathbf{Q}_p}} = \begin{pmatrix} \frac{1}{\delta}\left(\varphi\left(\frac{\theta'}{\chi_p^{k_0-1}} - \frac{\mu'}{\delta}\right) + \xi'\right) \\ \frac{\theta'}{2\chi_p^{k_0-1}} - \frac{\mu'}{\delta} \\ 0 \end{pmatrix}. \tag{2.7}$$

Remark 2.3.2. Note that we are not (yet) claiming that $\mathcal{E}$ is a non-split extension; we will however see this below in theorem 2.3.4.

Proof. Since r° is residually reducible, we know that in the basis $(e_1, \ldots, e_4)$ it is given by a matrix of the form

$$\begin{pmatrix} A & \epsilon B \\ \epsilon C & D \end{pmatrix}$$

where A, B, C, and D have coefficients in $\mathcal{A}_U$ (and A is 3×3). Replacing e_4 with e_4' has the effect of conjugating this matrix by

$$P = \begin{pmatrix} I_3 & \\ & \epsilon \end{pmatrix}$$

where I_3 is the 3×3 identity matrix. A simple calculation yields

$$P \begin{pmatrix} A & \epsilon B \\ \epsilon C & D \end{pmatrix} P^{-1} = \begin{pmatrix} A & B \\ \epsilon^2 C & D \end{pmatrix}. \tag{2.8}$$

This matrix still has coefficients in $\mathcal{A}_U$ and thus the lattice spanned by (e_1, e_2, e_3, e_4') is $G_{\mathbf{Q}}$-stable.

The statement regarding $c|_{G_{\mathbf{Q}_p}}$ is immediate from (2.8) and the expression (2.6). □

2.3.2 The L-invariant

We now find ourselves in the situation described at the beginning of section 1.5. Our first step is to verify that the cocycle c defines a cohomology class that satisfies conditions a) – c) listed there. Then, a computation of the coordinates of $\overline{c}_p$ will yield the L-invariant.

Let us begin with condition a) regarding ramification away from p. Our proof is adapted from lemmas 1.2 and 1.3 of [Hi07].

Lemma 2.3.3. *The class $[c] \in H^1(\mathbf{Q}, V)$ is unramified away from p.*

Proof. Let $q \neq p$ be a prime where ρ is ramified. The proof breaks into two parts according to whether ρ_f is potentially unramified or potentially multiplicative at q.

Case 1): ρ_f potentially unramified at q. The representation ρ is then also potentially unramified. Let $L/\mathbf{Q}_q$ be a finite Galois extension such that the inertia group of L, I_L, acts trivially on ρ. Since I_q/I_L is a finite group and ρ is an $E_{\mathfrak{p}}$-vector space

$$H^1(I_q/I_L, \rho) = 0.$$

The inflation–restriction exact sequence of G_q/I_q-modules yields

$$0 \longrightarrow H^1(I_q, \rho) \longrightarrow H^1(I_L, \rho)^{I_q/I_L}.$$

Therefore, if we can show that c becomes zero in $H^1(I_L, \rho)$, then c is unramified at q. By definition, ρ is a trivial I_L-module, so

$$H^1(I_L, \rho)^{I_q/I_L} = \mathrm{Hom}_{I_q/I_L}(I_L, \rho) = \mathrm{Hom}_{I_q/I_L}(\mathbf{Z}_p(1), \rho).$$

The second equality holds because every homomorphism must factor through the maximal pro-p quotient of I_L. Note that since c passes through $H^1(G_q, \rho)$ on its way to $\mathrm{Hom}_{I_q/I_L}(\mathbf{Z}_p(1), \rho)$, its image in the latter actually lies in $\mathrm{Hom}_{G_q/I_L}(\mathbf{Z}_p(1), \rho)$, i.e. it gives a Frob_q-equivariant map. Recall that Frob_q acts by multiplication by q on $\mathbf{Z}_p(1)$. Its action on ρ is determined by its action on ρ_f and a Tate twist by $(1 - k_0)$. Specifically, the eigenvalues of Frob_q on ρ_f, denoted λ_1, λ_2, are Weil numbers with

$$|\lambda_i| = q^{\frac{k_0-1}{2}},$$

so the eigenvalues of Frob_q on ρ are

$$\lambda_1^{2-j}\lambda_2^j q^{1-k_0}, \text{ for } j = 0, 1, 2$$

all of which have absolute value 1. Equivariance implies that, for each $z \in \mathbf{Z}_p(1)$,

$$qc(z) = c(qz) = c(\mathrm{Frob}_q z) = \mathrm{Frob}_q c(z) = \lambda c(z)$$

where $|\lambda| = 1 \neq |q|$. This is a contradiction. Hence, c must be zero in $H^1(I_q, V)$.

Case 2): ρ_f potentially multiplicative at q. Its restriction to the decomposition group at q is

$$\rho_f|_{G_q} \sim \begin{pmatrix} \eta\chi_p^{k_0-1} & \eta\sigma \\ 0 & \eta \end{pmatrix}$$

and

$$\rho_{\mathcal{F}}|_{G_q} \sim \begin{pmatrix} \nu\theta & \nu\tau \\ 0 & \nu \end{pmatrix}.$$

Going through the matrix computations that give c shows that

$$c_q = \begin{pmatrix} \eta^2\left(\sigma\theta' - \chi_p^{k_0-1}\tau'\right) \\ \frac{\eta^2\theta'}{2} \\ 0 \end{pmatrix}. \tag{2.9}$$

Since $\rho_{\mathcal{F}}$ is a Hida deformation, it satisfies conditions ($\mathrm{K}_1$1–4) on page 4 of [Hi07]. The proof of lemma 1.3 of [Hi07] then shows that $\rho_{\mathcal{F}}|_{I_q}$ is constant. This shows that $c_q = 0$ since every term in equation (2.9) has a derivative in it. □

Condition b) of section 1.5 is simply that the third component of c_p in (2.7) is zero. Condition c) that $\overline{c}_p$ is non-zero in $H^1\big(G_{\mathbf{Q}_p}, W\big) \cong \mathrm{Hom}\left(G_{\mathbf{Q}_p}, \mathbf{Q}_p\right)$ follows from the following theorem.

Theorem 2.3.4. *The coordinates of $\overline{c}_p$, discussed in section 1.5, are*

$$\left(\frac{1}{2}, -\frac{\mu'(k_0)(\mathrm{Frob}_p)}{a_p}\right). \tag{2.10}$$

Since the first coordinate is non-zero,

- $[c]$ *is a non-zero cohomology class, and*
- $\mathcal{E}$ *is a non-split extension.*

Since $a_p = \mu(k_0)(\mathrm{Frob}_p)$, we suggestively denote $a'_p := \mu'(k_0)(\mathrm{Frob}_p)$. From the formula (1.4) for the L-invariant, the main theorem of this chapter is an immediate corollary of the above theorem.

Corollary 2.3.5 (Theorem A)**.** *Let f be a p-ordinary, weight $k_0 \geq 2$, new, holomorphic eigenform of character ψ (of conductor prime to p) and arbitrary level. Assume $\overline{\mathrm{Sel}}_{\mathbf{Q}}\big(\mathrm{Sym}^2\rho_f(1-k_0)\big) = 0$. Then,*

$$\mathcal{L}(\rho) = \mathcal{L}\big(\mathrm{Sym}^2\rho_f(1-k_0)\big) = -\frac{2a'_p}{a_p}. \tag{2.11}$$

Recall that, by theorem 2.1.1 and remark 2.1.2, the assumption on the vanishing of the balanced Selmer group is known in many cases, and is expected in all cases. Also, note that by the ordinarity assumption the order to which p divides the level is less than or equal to one. In the case where p divides the level, the weight must be two and this result is due to Greenberg [G94, p. 170].

It remains to prove theorem 2.3.4. We begin with a lemma.

Lemma 2.3.6. *Let $k \in U$ be an integer greater than or equal to 2, then, for any principal unit u,*

$$1) \qquad \theta'(k)(\mathrm{Frob}_p) = 0 \tag{2.12}$$

$$2) \qquad \frac{\theta'(k)(\mathrm{rec}(u))}{\chi^{k-1}(\mathrm{rec}(u))} = -\log u. \tag{2.13}$$

Proof. Recall that

$$\theta(k) = \chi_p^{k-1}.$$

The first conclusion follows from the fact that $\chi_p(\mathrm{Frob}_p) = 1$. The second conclusion is a formula for the logarithmic derivative of θ at k. Recall that $\chi_p(\mathrm{rec}(u)) = u^{-1}$, so that

$$\theta(k)(\mathrm{rec}(u)) = u^{1-k}.$$

Thus, its logarithmic derivative at k is indeed $-\log u$. $\square$

Proof of theorem 2.3.4. By equation (2.7),

$$\bar{c}_p = \frac{\theta'(k_0)}{2\chi_p^{k_0-1}} - \frac{\mu'(k_0)}{\delta}.$$

The first coordinate is obtained by evaluating $\bar{c}_p$ at $\mathrm{rec}(u)$ and dividing by $-\log(u)$. Since μ is unramified,

$$\mu'(\mathrm{rec}(u)) = 0.$$

Combining this with the previous lemma yields

$$\frac{1}{\log u}\bar{c}_p(\mathrm{rec}(u)) = \frac{-\log u}{-2\log u} = \frac{1}{2}.$$

The second coordinate is obtained by evaluating $\bar{c}_p$ at Frob_p. The formula given follows immediately from part 1) of the previous lemma and the fact that $\delta(\mathrm{Frob}_p) = a_p$. $\square$

2.4 Concluding remarks

We begin our remarks by pointing out what we have not addressed in the above sections.

As mentioned in the introduction, the computation of the L-invariant of the symmetric square of a p-unramified modular form f requires a global Galois cohomology class. One may ask if this L-invariant is in fact global in nature. In other words, can one modify the action of Galois on V away from p and change the value of the L-invariant?

Another important question we have not addressed is the non-vanishing of the L-invariant. Greenberg conjectures this non-vanishing and in fact shows that it is equivalent to the order of vanishing of the arithmetic p-adic L-function being $e = 1$ as it should be (see [G94, Proposition 3]). We should mention that in Hida's situation, he is capable of proving non-vanishing in most cases (see p. 3180 of [Hi04]). This requires in a crucial way the assumptions of his paper as well as his work in [Hi00].

A final point in this vein is in regards to the relation of the L-invariant with the analytic p-adic L-function. The point $s = 1$ is not central and hence the functional equation approach of [GS93] cannot hope to succeed. A new idea is needed. Citro has shown in [Ci08] that if

$$L_p\left(s, \mathrm{Ad}\,\rho_f\right) = \zeta_p(s) L_p\left(s, \mathrm{Ad}^0 \rho_f\right)$$

near $s = 1$, then Greenberg's L-invariant is indeed the fudge factor required to relate the derivative of the p-adic L-function to the archimedean L-function. Let us also mention that both Greenberg ([G94, Proposition 4]) and Hida ([Hi07, Corollary 3.2]) have conditional results relating Greenberg's L-invariant to the *arithmetic* p-adic L-function.

Additionally, we would like to mention that though our approach is through multilinear algebra, we could arrive at the same conclusion in essentially the same fashion by considering ρ as $\mathrm{Ad}^0\rho_f$. From such a point of view, the Hida deformation $\rho_{\mathcal{F}}$ provides an infinitesimal deformation of ρ_f and hence a global class in $H^1\big(\mathbf{Q}, \mathrm{Ad}^0\rho_f\big)$.

The methods of this chapter suggest that more general cases can be addressed. In the next chapter, we take a first step in this direction by studying the symmetric sixth power L-invariants in a way that indeed generalizes what we have done here.

In fact, The computations of the next chapter also provide another proof of theorem A, albeit with some additional assumptions on f.

Chapter 3

L-invariant of the symmetric sixth power of a modular form

The aim of this chapter is to show that instances of Langlands functoriality together with Hida families of automorphic forms on higher rank groups (admitting PEL Shimura varieties) can be exploited to compute L-invariants of symmetric powers of holomorphic modular forms, f, on GL(2). In fact, for the critical[1] even symmetric powers strictly greater than two, the Hida family of f on $\mathrm{GL}(2)_{/\mathbf{Q}}$ appears inadequate for computing the L-invariant (at least using our method), so that we *must* move to higher rank groups — hence larger Hida families — to obtain results. In this chapter, we partially carry out this program for the symmetric sixth power, exploiting the symmetric cube lift to GSp(4) of Ramakrishnan–Shahidi ([RS07, Theorem A′]) and the Hida families on this group ([TU99],[Hi02]). As in the previous chapter, we obtain a formula relating the L-invariant to p-adic analytic functions interpolating eigenvalues of Frobenius in a Hida family.

Many of the comments made in the introduction to chapter 2 apply here as well with a few modifications. Greenberg's computations in [G94, p. 170] are for arbitrary

[1] The symmetric $2m$-th power (twisted by $m(1-k)$) of a modular form of weight k is critical at $s=1$ if, and only if, m is odd. We refer to these as the *critical* even symmetric powers.

symmetric powers so we may consider the cases of Tate curves and good ordinary CM elliptic curves taken care of. This is convenient as the methods in this chapter require that f not be CM. Hida's work on the subject in [Hi07] leaves the case that we deal with here mostly unaddressed, that is the case where f has level prime to p. Thus, our work provides the first formulae for the L-invariants of symmetric sixth powers of such forms. The method of computation we use here generalizes that of the previous chapter.[2]

Our general strategy for computing L-invariants of even symmetric powers is as follows. For the even symmetric power $\mathrm{Sym}^n V_f$ of a Galois representation attached to a modular form f of weight k ($n = 2m$ with m odd, for criticality), we compute a Hida deformation for $\mathrm{Sym}^m V_f$ and tensor it with V_f (in the previous chapter, $n = 2$, so $m = 1$, and $\mathrm{Sym}^m V_f = V_f$). Taking advantage of the decomposition of tensor products of symmetric powers in the theory of finite-dimensional representations of GL(2), we use this family to construct a global extension class. From this class, one can obtain a global Galois cohomology class and (hope to) extract enough information from it to compute Greenberg's L-invariant.

The contents of this chapter are distributed as follows. The first part of section 3.1 outlines the basic facts concerning symmetric powers of Galois representations attached to ordinary modular forms. Then it very briefly goes over what is needed from the theory of finite-dimensional representations of GL(2). Section 3.2 reviews the statement of the symmetric cube lifting of Ramakrishnan–Shahidi. In section 3.3, we present what we need of Tilouine and Urban's work ([TU99]) on Hida deformations on GSp(4). We go on to explain how one may view the symmetric cube of the GL(2) deformation $\rho_{\mathcal{F}}$ as a specialization of the GSp(4) deformation. Then, we begin section 3.4 with a description of the general method we use to compute even symmetric power L-invariants. We follow this with the computation in earnest, and wind up the

[2] Hida's work in [Hi07] similarly generalizes his approach in [Hi04].

section with a discussion on relating the symmetric sixth power L-invariant to the symmetric square L-invariant. We finish the chapter by briefly reviewing questions left unanswered and suggesting future developments for work in the subject.

For automorphic representations on $\mathrm{GSp}(4)_{/\mathbf{Q}}$ and their associated Galois representations, we refer to [Wei05], [Wei09], and [Fli-GSp].

3.1 The symmetric power Galois representations

To place our problem in a slightly larger context, in this section we give a general description of the symmetric powers of Galois representations attached to weight $k_0 \geq 2$, p-ordinary holomorphic modular forms on $\mathrm{GL}(2)_{/\mathbf{Q}}$. In particular, we discuss criticality and trivial zeroes. We also briefly review the decomposition of tensor products of symmetric powers of the standard representation of $\mathrm{GL}(2)$ for our later purposes.

Recall from equation (2.1) of the previous chapter that the p-adic Galois representation attached to f has the form

$$\rho_f|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \chi_p^{k_0-1}\delta^{-1} & \varphi \\ 0 & \delta \end{pmatrix}$$

when restricted to the decomposition group at p. So as to consider trivial zeroes at $s = 1$, we will be interested in a specific Tate twist of the symmetric nth power of ρ_f:

$$\rho_n' := \mathrm{Sym}^n \rho_f \left((1 - k_0) \left\lfloor \frac{n}{2} \right\rfloor \right).$$

Restricting this representation to the decomposition group at p, one obtains

$$\begin{pmatrix} \chi_p^{(k_0-1)\lceil n/2\rceil}\delta^{-n} & & & & & \\ & \chi_p^{(k_0-1)(\lceil n/2\rceil-1)}\delta^{-n+2} & & & & * \\ & & \ddots & & & \\ & & & \chi_p^{(k_0-1)(\lceil n/2\rceil-i)}\delta^{-n+2i} & & \\ & & 0 & & \ddots & \\ & & & & & \chi_p^{-(k_0-1)\lfloor n/2\rfloor}\delta^{n} \end{pmatrix}. \tag{3.1}$$

The questions of criticality and trivial zeroes split up into two cases: n even or odd.

Case n even: when $n = 2m$ is even, ρ'_n is critical at $s = 1$ if, and only if, m is odd. In this case, the middle entry of the matrix in (3.1) is 1. Thus, Greenberg's theory tells us that ρ'_n should have a trivial zero at $s = 1$. The middle part of the matrix looks like

$$\begin{pmatrix} \ddots & & & & \\ & \chi_p^{k_0-1}\delta^{-2} & * & * & \\ & & 1 & * & \\ & & & \chi_p^{1-k_0}\delta^{2} & \\ & & & & \ddots \end{pmatrix}.$$

In particular, if $k_0 > 2$, its exceptional subquotient is isomorphic to the trivial representation K. If $k_0 = 2$, the same holds as long as δ is not trivial, i.e. as long as f is not attached to a split, multiplicative elliptic curve. In the latter case, the exceptional subquotient is two-dimensional of type M and the L-invariant of ρ'_n equals that of ρ_f ([G94, p. 170]).

Case n odd: when n is odd, ρ'_n is critical. It can only be exceptional if δ is trivial. Again, this case is treated by Greenberg.

Thus, we are left to consider even symmetric powers congruent to two modulo four. In this chapter, we concentrate on the case $n = 6$.

3.1.1 Just enough on finite-dimensional representations of GL(2)

Recall that the irreducible finite-dimensional characteristic zero representations of GL(2) are

$$\mathrm{Sym}^a M \otimes \det^b$$

for $a \in \mathbf{Z}_{\geq 0}$, $b \in \mathbf{Z}$, where M denotes the standard representation. We will use the following decomposition of the tensor product of two symmetric power representations: for $a \geq b \in \mathbf{Z}_{\geq 0}$

$$\mathrm{Sym}^a M \otimes \mathrm{Sym}^b M \cong \bigoplus_{i=0}^{b} \left(\mathrm{Sym}^{a+b-2i} M \otimes \det^i\right). \tag{3.2}$$

Additionally, the following self-duality (up to a twist) will be useful:

$$(\mathrm{Sym}^a M)^\vee \cong \mathrm{Sym}^a M \otimes \det^{-a}. \tag{3.3}$$

3.2 The symmetric cube lifting to $\mathrm{GSp}(4)_{/\mathbf{Q}}$

Let f be a p-ordinary, holomorphic, non-CM, cuspidal eigenform, new of level N (prime to p), of even weight $k_0 \geq 2$, and trivial character. The added hypotheses on the weight, the character, and non-CM are present so that we may use the symmetric cube lifting to $\mathrm{GSp}(4)_{/\mathbf{Q}}$ of Ramakrishnan–Shahidi ([RS07, Theorem A′]). We briefly review the statement of this theorem.

As in chapter 2, we write ρ_f for the p-adic Galois representation attached to f. Let $\rho_3 := \mathrm{Sym}^3 \rho_f$ and let π denote the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ defined by f. We have the following automorphy result for the symmetric cube of π.

Theorem 3.2.1 (Ramakrishnan–Shahidi, [RS07]). *There exists a cuspidal automorphic representation* Π *of* $\mathrm{GSp}(4, \mathbb{A})$ *of trivial character, unramified away from* N*, such that*

a) Π_∞ *is in the holomorphic discrete series, with its* L*-parameter being the symmetric cube of that of* π_∞*;*

b) $L(s, \Pi) = L\big(s, \pi, \mathrm{Sym}^3\big)$.

Furthermore,

- $\Pi^K \neq 0$ *for some compact open subgroup* K *of* $\mathrm{GSp}(4, \mathbb{A}_f)$ *of level equal to the conductor of* ρ_3*;*
- Π *is of cohomological weight* $(a_0, b_0) = (2(k_0 - 2), k_0 - 2)$.[3]

3.3 The Hida deformation of ρ_3

In this section and the following, we add a simplifying assumption:

(RAI) ρ_3 is residually absolutely irreducible.

Without this assumption, one may still obtain a formula for the L-invariant (see remarks 3.4.5a) and 3.4.8c) below). We will also assume $k_0 \geq 4$ and condition (Unr) below to be able to apply the results of [TU99] without modification.

For the explicit matrices with which we will be dealing, we choose the Borel subgroup, B_4, of GSp(4) as in [RoSch07, Chapter 2] so that it consists of upper-

[3]Here, by cohomological weight, we mean $(k_1 - 3, k_2 - 3)$ (as in [TU99]), where (k_1, k_2) gives the highest weight of the lowest K_∞-type of Π_∞, as described in [Wei05] or [Wei09].

triangular matrices of the form

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix}.$$

3.3.1 The Hida deformation

Since Π is cuspidal, with Π_∞ in the discrete series, of regular non-parallel cohomological weight, B_4-ordinary at p (in the sense of [TU99]), and ρ_3 is residually absolutely irreducible, for all but finitely many[4] p we may combine [Wei05], [U05, Théorème 1 and Corollaire 1], and [TU99, Corollary 6.7] with [TU99, Theorem 7.1] and [Ny96][5] to obtain a Galois deformation of ρ_3,

$$\widetilde{\rho}_3 : G_{\mathbf{Q}} \longrightarrow \mathrm{GSp}(4, \mathcal{A}),$$

where $\mathcal{A}$ is a finite flat extension of $\Lambda[\![T_1, T_2]\!]$.[6] We make the following assumption:

(Unr) the GSp(4) universal p-adic Hecke algebra is unramified over the Iwasawa algebra at the height one prime corresponding to Π.

Under this assumption, we may proceed as in [GS93, pp. 418–419] and consider the elements of $\mathcal{A}$ as p-adic analytic functions on some neighbourhood of $(a_0, b_0) = (2(k_0 - 2), k_0 - 2)$ in two variables (s_1, s_2). The deformation $\widetilde{\rho}_3$ has the property that its restriction to the decomposition group at p has image contained in B_4. Thus we

[4]In corollary 6.7 of [TU99], they exclude a finite set of primes larger than the set of ramified primes.

[5]See the comment on page 567 of [TU99].

[6]In [TU99], a three-variable deformation is obtained. One of these variables is cyclotomic and we disregard it.

may write

$$\widetilde{\rho}_3|_{G_p} \sim \begin{pmatrix} \theta_1\theta_2\mu_1 & \xi_{12} & \xi_{13} & \xi_{14} \\ & \theta_2\mu_2 & \xi_{23} & \xi_{24} \\ & & \theta_1\mu_2^{-1} & \xi_{34} \\ & & & \mu_1^{-1} \end{pmatrix} \tag{3.4}$$

with μ_i unramified characters, and

$$\mu_1(a_0, b_0) = \delta^{-3} \tag{3.5}$$

$$\mu_2(a_0, b_0) = \delta^{-1} \tag{3.6}$$

$$\theta_1(s_1, s_2) = \chi_p^{s_2+1} \tag{3.7}$$

$$\theta_1(a_0, b_0) = \chi_p^{k_0-1} \tag{3.8}$$

$$\theta_2(s_1, s_2) = \chi_p^{s_1+2} \tag{3.9}$$

$$\theta_2(a_0, b_0) = \chi_p^{2(k_0-1)}. \tag{3.10}$$

3.3.2 The GL(2) Hida family within

Recall that in the previous chapter we used the Hida family $\mathcal{F}$ containing f_p to obtain a Galois deformation $\rho_{\mathcal{F}}$ of ρ_f. By [GhVa04, Remark 9], we know that every arithmetic specialization of $\mathcal{F}$ is non-CM. Thus, for the specializations of even weight greater than or equal to two, we may apply the symmetric cube lifting of theorem 3.2.1 to conclude that $\mathrm{Sym}^3\rho_{\mathcal{F}}$ is an ordinary modular deformation of ρ_3. By (Unr), there is a unique family containing ρ_3, and $\mathrm{Sym}^3\rho_{\mathcal{F}}$ must therefore be a specialization of $\widetilde{\rho}_3$. Thus, we may identify $\mathrm{Sym}^3\rho_{\mathcal{F}}$ as a one-dimensional "sub-family" of $\widetilde{\rho}_3$ by looking at how the weights vary. Specifically, since the weights of the symmetric cube of a modular form of weight k are given by theorem 3.2.1 as $(2(k-2), k-2)$, we conclude that $\mathrm{Sym}^3 \rho_{\mathcal{F}}$ corresponds to the sub-family where

$$s_1 = 2s_2.$$

From identifying $\mathrm{Sym}^3 \rho_{\mathcal{F}}$ with the specialization of $\widetilde{\rho}_3$ along this family, we obtain the equations

$$\begin{aligned} \mu_1(2s, s) &= \mu^{-3}(s+2) \\ \mu_2(2s, s) &= \mu^{-1}(s+2). \end{aligned}$$

By applying the multivariable chain rule to these equations, we get that

$$2\partial_1 \mu_1(a_0, b_0) + \partial_2 \mu_1(a_0, b_0) = -\frac{3\mu'(k_0)}{\delta^4} \tag{3.11}$$

$$2\partial_1 \mu_2(a_0, b_0) + \partial_2 \mu_2(a_0, b_0) = -\frac{\mu'(k_0)}{\delta^2}. \tag{3.12}$$

These two equations will be required for the computation of the L-invariant in the next section.

3.4 Computing the L-invariant

3.4.1 The multilinear algebra involved

For clarity's sake, in this section we outline the core structure of the computation, i.e. that coming from the multilinear algebra of the symmetric powers of the standard representation of GL(2), as briefly recalled in section 3.1.1. The full details of the computation will make up the next section.

Our computation will begin with an ordinary deformation, $\widetilde{\rho}_3$, of the symmetric cube of V_f whose diagonal elements we know explicitly. We denote the prime specializing to ρ_3 by $\mathfrak{P}$. Tensoring $\widetilde{\rho}_3$ with V_f, we obtain a big Galois representation r°

whose specialization at $\mathfrak{P}$, $r^\circ_{\mathfrak{P}}$, is reducible, given by equation (3.2) as

$$r^\circ_{\mathfrak{P}} \cong \mathrm{Sym}^4 V_f \oplus \mathrm{Sym}^2 V_f \otimes \det .$$

At this point, we use this reducibility to obtain a (possibly split) extension, $\mathcal{E}$, of $\mathrm{Sym}^2 V_f \otimes \det$ by $\mathrm{Sym}^4 V_f$. We have thus obtained an element of

$$\mathrm{Ext}^1\big(\mathrm{Sym}^2 V_f \otimes \det, \mathrm{Sym}^4 V_f\big) .$$

We use the standard adjunction of Hom and $\otimes$, together with equations (3.2) and (3.3) above, to obtain isomorphisms

$$\begin{aligned} \mathrm{Ext}^1\big(\mathrm{Sym}^2 V_f \otimes \det, \mathrm{Sym}^4 V_f\big) &\cong \mathrm{Ext}^1\Big(\mathbf{1}, \big(\mathrm{Sym}^2 V_f \otimes \det\big)^\vee \otimes \mathrm{Sym}^4 V_f\Big) \\ &\cong \mathrm{Ext}^1\left(\mathbf{1}, \bigoplus_{i=0}^{2} \mathrm{Sym}^{2(3-i)}\, V_f \otimes \det{}^{i-3}\right) \\ &\cong \bigoplus_{i=0}^{2} \mathrm{Ext}^1\Big(\mathbf{1}, \mathrm{Sym}^{2(3-i)}\, V_f \otimes \det{}^{i-3}\Big) \\ &\cong \bigoplus_{i=0}^{2} H^1\Big(\mathbf{Q}, \mathrm{Sym}^{2(3-i)}\, V_f \otimes \det{}^{i-3}\Big) . \end{aligned} \tag{3.13}$$

In particular, projecting onto the $i = 0$ factor, $\mathcal{E}$ gives rise to an element c_6 in $H^1(\mathbf{Q}, \rho_6)$ (since $\det = \chi_p^{k_0-1}$). Explicitly, this class can be obtained as follows. The extension $\mathcal{E}$ is given as a short exact sequence

$$0 \to \mathrm{Sym}^4 V_f \to \mathcal{E} \to \mathrm{Sym}^2 V_f \otimes \det \to 0.$$

Tensoring this sequence by $\big(\mathrm{Sym}^2 V_f \otimes \det\big)^\vee$ gives (using equations (3.2) and (3.3) once again)

$$0 \to \bigoplus_{i=0}^{2} \mathrm{Sym}^{2(3-i)} V_f \otimes \det^{i-3} \to \mathcal{E}' \to \bigoplus_{j=0}^{2} \mathrm{Sym}^{2(2-j)} V_f \otimes \det^{j-2} \to 0.$$

The $j = 2$ term of the quotient is simply the trivial representation, and we may project onto it giving

$$0 \to \bigoplus_{i=0}^{2} \mathrm{Sym}^{2(3-i)} V_f \otimes \det^{i-3} \to \mathcal{E}'' \to \mathbf{1} \to 0$$

as desired.

Once we have an explicit formula for c_6, we proceed as in section 2.3.2 to obtain a formula for the L-invariant in terms of derivatives of Frobenius eigenvalues varying in the deformation $\widetilde{\rho}_3$ with which we began.

Remark 3.4.1. a) As the isomorphism in (3.13) indicates, $\mathcal{E}$ also gives rise to an element c_2 in $H^1(\mathbf{Q}, \rho_2)$ (where $\rho_2 := \mathrm{Sym}^2 V_f(1-k_0)$) which can be used to compute $\mathcal{L}(\rho_2)$ again. We exploit the equality of this computation with that of chapter 2 in section 3.4.3 below in our attempt to relate $\mathcal{L}(\rho_6)$ and $\mathcal{L}(\rho_2)$.

b) The structure of this argument can be applied with three replaced by an arbitrary positive odd integer m in an attempt to compute the L-invariant of the symmetric $2m$-th power of V_f. We may now explain a comment we made in the introduction to this chapter. Some computations for small $m > 1$ suggest that if we begin with $\widetilde{\rho}_m = \mathrm{Sym}^m \rho_{\mathcal{F}}$ — the symmetric mth power of the GL(2) Hida deformation of ρ_f — then the class c_{2m} obtained as above is zero, and thus cannot be used to compute the L-invariant. The name of the game is thus to obtain a "rich enough" deformation of $\rho_m = \mathrm{Sym}^m \rho_f$ so that the c_{2m} obtained is non-zero. This chapter makes good on this idea for a large class of examples in the $m = 3$ case. In future work, we plan to investigate whether the very recent work of [BLGHT09] on the potential automorphicity of the symmetric powers of higher weight modular forms can be paired succesfully with Hida's ordinary families on unitary groups ([Hi02]) to yield L-invariants of higher symmetric powers.

3.4.2 The computation

The deformation we take for $\widetilde{\rho}_3$ is the one described in section 3.3. Thus,

$$\widetilde{\rho}_3|_{G_p} \sim \begin{pmatrix} \theta_1\theta_2\mu_1 & \xi_{12} & \xi_{13} & \xi_{14} \\ & \theta_2\mu_2 & \xi_{23} & \xi_{24} \\ & & \theta_1\mu_2^{-1} & \xi_{34} \\ & & & \mu_1^{-1} \end{pmatrix}.$$

Tensoring this with ρ_f gives r°, which after conjugating by the change of basis matrix

$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & -3 & & \\ & 1 & & & & 1 & & \\ & & 2 & & & & -2 & \\ & & 2 & & & & 2 & \\ & & & 6 & & & & -2 \\ & & & 6 & & & & 6 \\ & & & & 24 & & & \end{pmatrix}$$

gives

$$r^\circ|_{G_p} \sim \begin{pmatrix} A^\circ & B^\circ \\ C^\circ & D^\circ \end{pmatrix}.$$

Here A° is a deformation of $\mathrm{Sym}^4\rho_f$, D° is a deformation of $\mathrm{Sym}^2\rho_f \otimes \det$, B° and C° are zero mod $\mathfrak{P}$, and B° is given by

$$\begin{pmatrix} -3\theta_1\theta_2\mu_1\varphi + \frac{\xi_{12}\chi_p^{k_0-1}}{\delta} & 2\left(-\xi_{12}\varphi + \frac{\xi_{13}\chi_p^{k_0-1}}{\delta}\right) & 2\left(-\xi_{13}\varphi + \frac{3\xi_{14}\chi_p^{k_0-1}}{\delta}\right) \\ \frac{3\theta_2}{4}\left(-\theta_1\delta\mu_1 + \frac{\chi_p^{k_0-1}\mu_2}{\delta}\right) & -\frac{\xi_{12}\delta+3\theta_2\mu_2\varphi}{2} + \frac{3\xi_{23}\chi_p^{k_0-1}}{2\delta} & -\frac{\xi_{13}\delta+3\xi_{23}\varphi}{2} + \frac{9\xi_{24}\chi_p^{k_0-1}}{2\delta} \\ 0 & \frac{1}{2}\left(-\theta_2\mu_2\delta + \frac{\theta_1\chi_p^{k_0-1}}{\mu_2\delta}\right) & -\frac{\xi_{23}\delta}{2} - \frac{\theta_1\varphi}{2\mu_2} + \frac{3\xi_{34}\chi_p^{k_0-1}}{2\delta} \\ 0 & 0 & \frac{1}{4}\left(-\frac{\theta_1\delta}{\mu_2} + \frac{\chi_p^{k_0-1}}{\mu_1\delta}\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

We introduce a parameter α that describes the direction in which we will be taking derivatives at the point $(s_1, s_2) = (a_0, b_0)$. In other words, considering the entries of r° as analytic functions in (s_1, s_2) near (a_0, b_0), we first write

$$\epsilon_1 := s_1 - a_0 \text{ and } \epsilon_2 := s_2 - b_0.$$

Then, we make the substitution

$$\epsilon_1 = \epsilon(1-\alpha) \text{ and } \epsilon_2 = \epsilon\alpha$$

and view r° as a one-paramater family of one-dimensional deformations of ρ_3, which for each value of the parameter α should give an extension of $\mathrm{Sym}^2\rho_f \otimes \det$ by $\mathrm{Sym}^4\rho_f$. Expanding the entries of r° to first order in ϵ, using among others

$$\begin{aligned} \mu_1 &\approx \delta^{-3} + \epsilon\left((1-\alpha)\partial_1\mu_1(a_0,b_0) + \alpha\partial_2\mu_1(a_0,b_0)\right) \\ \mu_2 &\approx \delta^{-1} + \epsilon\left((1-\alpha)\partial_1\mu_2(a_0,b_0) + \alpha\partial_2\mu_2(a_0,b_0)\right) \\ \theta_i &\approx \chi_p^{i(k_0-1)} + \epsilon\left((1-\alpha)\partial_1\theta_i(a_0,b_0) + \alpha\partial_2\theta_i(a_0,b_0)\right), \end{aligned}$$

gives that B° and C° are both divisible by ϵ, as they should be. Then, scaling the basis vectors corresponding to the $\mathrm{Sym}^2\rho_f \otimes \det$ part by $1/\epsilon$ gives a new $G_{\mathbf{Q}}$-stable lattice r_α° such that

$$\mathcal{E}_\alpha := r^\circ_{\alpha,\mathfrak{P}} \sim \begin{pmatrix} \mathrm{Sym}^4\rho_f & B_\alpha \\ 0 & \mathrm{Sym}^2\rho_f \otimes \det \end{pmatrix}$$

where the subscript $\mathfrak{P}$ denotes specialization modulo $\mathfrak{P}$. As described in the previous section, this leads to an extension $\mathcal{E}''_\alpha$ such that

$$\mathcal{E}''_\alpha \sim \begin{pmatrix} \rho_6 & 0 & 0 & c_{6,\alpha} \\ 0 & \rho_4 & 0 & * \\ 0 & 0 & \rho_2 & c_{2,\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.14}$$

with

$$c_{6,\alpha}|_{G_p} = \begin{pmatrix} * \\ (1-\alpha)\left(\frac{2\partial_1\theta_2}{\chi_p^{2(k_0-1)}} - \frac{4\partial_1\theta_1}{\chi_p^{k_0-1}} - 2\delta^3\partial_1\mu_1 + 6\delta\partial_1\mu_2\right) \\ +\alpha\left(\frac{2\partial_2\theta_2}{\chi_p^{2(k_0-1)}} - \frac{4\partial_2\theta_1}{\chi_p^{k_0-1}} - 2\delta^3\partial_2\mu_1 + 6\delta\partial_2\mu_2\right) \\ 0 \end{pmatrix}.$$

Here the $*$ and the 0 are both 3×1 and all derivatives are evaluated at (a_0, b_0). This shows that $c_{6,\alpha}$ is indeed in $F^{00}H^1(\mathbf{Q}, \rho_6)$. The proof that $c_{6,\alpha}$ is unramified outside of p is along the same lines as lemma 2.3.3. Let

$$\bar{c}_{6,\alpha} = \text{ image of } c_{6,\alpha} \text{ in } H^1\big(\mathbf{Q}_p, W_6/F^+W_6\big)$$

and

$$a_p^{(i,j)} := \partial_j\mu_i(a_0, b_0)(\mathrm{Frob}_p).$$

Theorem 3.4.2. *The coordinates of $\overline{c}_{6,\alpha}|_{G_p}$, discussed in section 1.5, are*

$$\left(2-6\alpha,(1-\alpha)\left(-2a_p^3a_p^{(1,1)}+6a_pa_p^{(2,1)}\right)+\alpha\left(-2a_p^3a_p^{(1,2)}+6a_pa_p^{(2,2)}\right)\right). \tag{3.15}$$

In particular, if $\alpha \neq 1/3$, then $[c_{6,\alpha}]$ is non-zero.

Proof. The proof follows that of theorem 2.3.4, e.g. we use the formula (1.3) for the coordinates, combined with lemma 2.3.6. □

Remark 3.4.3. Note that for $\alpha = 1/3$, the vanishing of the first coordinate of $\overline{c}_{6,1/3}$ forces $\overline{c}_{6,1/3} = 0$ by part c) of proposition-definition 1.4.4. This implies that $c_{6,1/3} \in \overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_6)$. The assumption that the latter vanishes implies that $c_{6,1/3} = 0$. For this value of α, we have that

$$\frac{\epsilon_1}{\epsilon_2} = 2,$$

so that this direction is the one corresponding to the symmetric cube of the GL(2) Hida deformation of ρ_f. This vanishing is an instance of the behaviour described in remark 3.4.1b) that forces us to seek a richer deformation of ρ_3.

From the formula (1.4) for the L-invariant, the main theorem of this chapter is an immediate corollary of the above theorem.

Corollary 3.4.4 (Theorem B). *Let $f \in S_{k_0}(\Gamma_1(N))$ be a p-ordinary, non-CM, even weight $k_0 \geq 4$ new eigenform of level prime to p and trivial character. Assume conditions (RAI) and (Unr) hold, and that $\overline{\mathrm{Sel}}_{\mathbf{Q}}\left(\mathrm{Sym}^6\rho_f(3(1-k_0))\right) = 0$. Then, for all but finitely many p*

$$\mathcal{L}\left(\rho_6\right) = \mathcal{L}\left(\mathrm{Sym}^6\rho_f(3(1-k_0))\right) = 3a_pa_p^{(2,1)} - a_p^3a_p^{(1,1)}. \tag{3.16}$$

Proof. This corollary is simply obtained by solving the system of linear equations in $\mathcal{L}(\rho_6)$ and the $a_p^{(i,j)}$ given by equation (3.15) of theorem 3.4.2 and equations (3.11) and (3.12). It is quite nice that solving this system causes α to drop out (as it should). □

Note that the assumption on the vanishing of the balanced Selmer group should hopefully follow from a study of Weston's geometric Euler systems in this context, and is a result that is expected to be true. That finitely many p are thrown out comes from the use of corollary 6.7 of [TU99]. In principal, all p should work.

Remark 3.4.5. a) Recall that the residual absolute irreducibility assumption is only required to know that $\widetilde{\rho}_3$ takes values in $\mathrm{GSp}(4, \mathcal{A})$. Without this assumption, we must have unramified characters μ_3 and μ_4 replacing μ_2^{-2} and μ_1^{-1}, respectively, in equation (3.4). We simply record the result obtained:

$$\mathcal{L}(\rho_6) = -\frac{1}{2}\left(a_p^3 a_p^{(1,1)} - 3a_p a_p^{(2,1)} + 3\frac{a_p^{(3,1)}}{a_p} - \frac{a_p^{(4,1)}}{a_p^3}\right). \tag{3.17}$$

b) Condition (Unr) is critical as it allows us to view the elements of $\mathcal{A}$ as two-variable p-adic analytic functions near (a_0, b_0), thus providing us with the first-order expansions in ϵ.

3.4.3 Relation to the symmetric square L-invariant

This section serves two purposes. The first concerns a guess of Greenberg's that the L-invariant of a symmetric power should be independent of the power. This is true in the cases Greenberg himself treated on p. 170 of [G94], i.e. the Tate curves and good ordinary CM elliptic curves. In this section, we fall one relation short of comparing the symmetric sixth power and symmetric square L-invariants. Secondly, even without Greenberg's guess, one might be interested in replacing (at least) one of the derivatives $a_p^{(i,j)}$ present in equation (3.16) with the slightly more familiar derivative a_p'.

As already mentioned in remark 3.4.1a), we can use the element $c_{2,\alpha}$ occurring in $\mathcal{E}_\alpha''$ in equation (3.14) to recompute $\mathcal{L}(\rho_2)$. Restricting $c_{2,\alpha}$ to the decomposition group at p gives

$$c_{2,\alpha}|_{G_p} = \begin{pmatrix} * \\ (1-\alpha)\left(-\frac{2\partial_1\theta_2}{\chi_p^{2(k_0-1)}} - \frac{\partial_1\theta_1}{\chi_p^{k_0-1}} - 3\delta^3\partial_1\mu_1 - \delta\partial_1\mu_2\right) \\ +\alpha\left(-\frac{2\partial_2\theta_2}{\chi_p^{2(k_0-1)}} - \frac{\partial_2\theta_1}{\chi_p^{k_0-1}} - 3\delta^3\partial_2\mu_1 - \delta\partial_2\mu_2\right) \\ 0 \end{pmatrix}$$

where all derivatives are evaluated at (a_0, b_0). This yields the following equation for $\mathcal{L}(\rho_2)$.

Lemma 3.4.6. *In the situation of theorem B, we have that*

$$(\alpha-2)\mathcal{L}(\rho_2) = (1-\alpha)\left(-a_p a_p^{(2,1)} - 3a_p^3 a_p^{(1,1)}\right) + \alpha\left(-a_p a_p^{(2,2)} - 3a_p^3 a_p^{(1,2)}\right). \quad (3.18)$$

From chapter 2, we know that $\mathcal{L}(\rho_2) = -2a_p'/a_p$.[7] Combining this and the above lemma with Theorem B gives the following relation between $\mathcal{L}(\rho_6)$ and $\mathcal{L}(\rho_2)$.

Corollary 3.4.7 (Theorem B$'$)**.** *In the situation of theorem B, we have that*

$$\mathcal{L}(\rho_6) = -10a_p^3 a_p^{(1,1)} + 6\mathcal{L}(\rho_2). \quad (3.19)$$

Remark 3.4.8. a) Additionally, we record the following relations

$$\begin{aligned} a_p^{(2,1)} &= -3a_p^2 a_p^{(1,1)} - 4\frac{a_p'}{a_p^2} \\ a_p^{(2,2)} &= 6a_p^2 a_p^{(1,1)} + 7\frac{a_p'}{a_p^2}. \end{aligned}$$

[7]This can also be obtained by solving the system of linear equations in $\mathcal{L}(\rho_2)$ and the $a_p^{(i,j)}$ given by equations (3.11), (3.12), and (3.18).

b) To show that $\mathcal{L}(\rho_6) = \mathcal{L}(\rho_2)$, we require the relation

$$a_p^{(1,1)} = -\frac{a_p'}{a_p^4}. \tag{3.20}$$

Equivalently,

$$\begin{aligned} a_p^{(1,2)} &= -\frac{a_p'}{a_p^4} && (3.21)\\ a_p^{(2,1)} &= -\frac{a_p'}{a_p^2} && (3.22)\\ a_p^{(2,2)} &= \frac{a_p'}{a_p^2}. && (3.23) \end{aligned}$$

We do not know how to prove any of these relations. The most naïve guess that would lead to these equations would be that

$$\begin{aligned} \mu_1(s_1,s_2) &= \mu^{-2}\left(\frac{s_1}{2}+2\right)\mu^{-1}\left(s_2+2\right) && (3.24)\\ \mu_2(s_1,s_2) &= \mu^{-2}\left(\frac{s_1}{2}+2\right)\mu\left(s_2+2\right). && (3.25) \end{aligned}$$

c) As in remark 3.4.5a), we may record the result obtained without the assumption (RAI):

$$\mathcal{L}(\rho_6) = \frac{5}{3}\left(a_p a_p^{(2,1)} - \frac{a_p^{(3,1)}}{a_p}\right) - \frac{2}{3}\mathcal{L}(\rho_2) \tag{3.26}$$

with the additional relations

$$\begin{aligned} a_p^{(4,1)} &= a_p^6 a_p^{(1,1)} + \frac{1}{3}a_p^4 a_p^{(2,1)} - \frac{1}{3}a_p^2 a_p^{(3,1)} + \frac{8}{3}a_p^2 a_p'\\ a_p^{(4,2)} &= -2a_p^6 a_p^{(1,1)} - \frac{2}{3}a_p^4 a_p^{(2,1)} + \frac{2}{3}a_p^2 a_p^{(3,1)} - \frac{7}{3}a_p^2 a_p'. \end{aligned}$$

3.5 Concluding remarks

Many of the remarks made at the end of the previous chapter can be made here as well. Indeed, we do not deal with the question of globalness, the non-vanishing, nor the relation to the p-adic L-function. The only work we are aware of on any of these questions is again the work of Greenberg and Hida relating the L-invariant to the arithmetic p-adic L-function.

In Hida's work (e.g. [Hi07]), he uses $\mathrm{Ad}^0\rho_m$ to study ρ_{2m}. Though we have not verified the computations, we suspect that, as in the previous chapter we could use $\mathrm{Ad}^0\rho_f$, in this chapter we could use $\mathrm{Ad}^0\rho_3$.

The results of this chapter suggest that higher symmetric powers can be addressed via new cases of functoriality. A particular instance we have in mind is the very recent work of [BLGHT09] on the potential automorphicity of symmetric powers of higher weight modular forms as automorphic forms on unitary groups. We would hope that, together with Hida theory on unitary groups ([Hi02]), this would yield further results in this subject. The work of Harris–Li–Skinner on several variable p-adic L-functions attached to Hida families on unitary groups in [HLS06] additionally provides a basis for attempting to relate Greenberg's L-invariant of symmetric powers of modular forms to analytic p-adic L-functions. All factors indicate that there is much work ahead.

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