MAT-203 : The Leibniz Rule

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In this note, I'll give a quick proof of the Leibniz Rule I mentioned in class (when we computed the more general Gaussian integrals), and I'll also explain the condition needed to apply it to that context (i.e. for infinite regions of integration). A few exercises are also included.

The Leibniz Rule for a finite region

Theorem 0.1. *Suppose $f(x, y)$ is a function on the rectangle $R = [a, b]\times[c, d]$ and $\frac{\partial f}{\partial y}(x, y)$ is continuous on R. Then*

$$\frac{d}{dy}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f}{\partial y}(x, y)dx.$$

Before I give the proof, I want to give you a chance to try to prove it using the following hint: consider the double integral

$$\int_c^y \int_a^b \frac{\partial f}{\partial z}(x, z)dxdz,$$

change the order of integration and differentiate both sides of the ensuing equality.

Proof. Go ahead, give it a try.

Come on...

You sure?

Ok, fine.

So we start off with the equality the hint gives

$$\frac{d}{dy}\left(\int_c^y \int_a^b \frac{\partial f}{\partial z}(x, z)dxdz\right) = \frac{d}{dy}\left(\int_a^b \int_c^y \frac{\partial f}{\partial z}(x, z)dxdz\right).$$

Then using the fundamental theorem of calculus ($d/dt\left(\int_a^t f(x)dx\right) = f(t)$), the left-hand side becomes

$$\int_a^b \frac{\partial f}{\partial y}(x, y)dx.$$

Using the other version of the fundamental theorem of calculus ($\int_a^b F'(x)dx = F(b)-F(a)$), the right-hand side becomes

$$\frac{d}{dy}\left(\int_a^b (f(x,y) - f(x,c))dx\right),$$

and the second part of the integrand ($f(x,c)$) is independent of y, so it's derivative with respect to y is 0, thus the right-hand side is

$$\frac{d}{dy}\left(\int_a^b f(x,y)dx\right),$$

as desired. □

Exercise: Using this theorem and the chain rule, prove the more general formula

$$\frac{d}{dy}\int_{g_1(y)}^{g_2(y)} f(x,y)dx = \int_{g_1(y)}^{g_2(y)} \frac{\partial f}{\partial y}(x,y)dx + g_2'(y)f(g_2(y),y) - g_1'(y)f(g_1(y),y)$$

assuming, in addition, that g_1 and g_2 are differentiable.

Exercise: Compute

$$\int_0^1 \frac{x-1}{\log x}dx.$$

Hint: Define $I(\alpha) := \int_0^1 \frac{x^\alpha - 1}{\log x}dx$ for $\alpha > 0$, and use the Leibniz rule. At some point, you'll need that $\lim_{\alpha\to 0} I(\alpha) = 0$.

The Leibniz Rule for an infinite region

I just want to give a short comment on applying the formula in the Leibniz rule when the region of integration is infinite. In this case, one can prove a similar result, for example

$$\frac{d}{dy}\int_0^\infty f(x,y)dx = \int_0^\infty \frac{\partial f}{\partial y}(x,y)dx,$$

like the one we used in class, but we need to add a condition on f. Basically, we need to make sure that $\partial f/\partial y$ is well-behaved as x goes to infinity. The condition is the following: there is a positive function $g(x,y)$ that is integrable, with respect to x, on $[0,\infty)$, for each y, and such that $|\frac{\partial f}{\partial y}(x,y)| \leq g(x,y)$ for all (x,y). (In a more general context, this theorem is a corollary of the Lebesgue Dominated Convergence Theorem).