Assignment 10 – Part 1 – Math 541

The following problems from the textbook:

Section 4.3: 4, 5, 6, 15

(1) Suppose \( \varphi : R \to R' \) is a homomorphism of rings. Show that \( \ker(\varphi) \leq R \) and \( \text{im}(\varphi) \leq R' \). (This was stated in class, but not proved).

(2) Suppose \( R \) is a ring and \( S \) is a set with two binary operations denoted + and \( \cdot \). Suppose \( \varphi : R \to S \) is a surjective function such that \( \varphi(a + b) = \varphi(a) + \varphi(b) \) and \( \varphi(ab) = \varphi(a) \cdot \varphi(b) \) for all \( a, b \in R \). Show that \( S \) is a ring. Show that if \( R \) has an identity, then so does \( S \). Show that if \( R \) is commutative, then so is \( S \).

(3) Suppose \( \varphi : R \to R' \) is a homomorphism of rings and suppose \( S' \) is a subring of \( R' \). Show that \( \varphi^{-1}(S') \) is a subring of \( R \). If \( S' \) is an ideal, show that \( \varphi^{-1}(S') \) is, too.

(4) Suppose \( I \) and \( J \) are ideals in \( R \) and \( I \subseteq J \). Show that \( I \) is an ideal in \( J \).

(5) Let \( F \) be the field \( \mathbb{Z}/p\mathbb{Z} \) where \( p \) is a prime and consider the polynomials with coefficients in \( F \), denoted \( F[x] \). An element \( f(x) \in F[x] \) can be thought of as a function from \( F \) to itself \( F \); its value on \( a \in F \) is obtained by plugging \( a \) in to \( f(x) \), i.e. it is the function \( a \mapsto f(a) \). Show that the polynomial \( f(x) = x^p - x \) gives the zero function (i.e. the function that sends every \( a \in F \) to 0).