The following problems from the textbook:

Section 2.1: 1, 8, 9, 16

Section 2.3: 1, 2, 3, 8 and 5, 6, 7, 12, 13, 14, 16

(1) Consider the four matrices $1, i, j, k$ given by

$$
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
$$

where $i$ is the imaginary number such that $i^2 = -1$. Show that the set of eight elements $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is a group under matrix multiplication. It is called the quaternion group.

(2) Suppose $G$ and $H$ are two groups and define a binary operation on $G \times H$ by

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, h_1 \cdot h_2).$$

Show that $G \times H$ is a group under this operation. This is called the direct product of $G$ and $H$.

(3) This exercise is about what we’ll call $GL_2(\mathbb{Z})$.

(a) Show that

$$H := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is not a subgroup of $GL_2(\mathbb{R})$.

(b) Show that

$$GL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \text{ and } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1 \right\}$$

is a subgroup of $GL_2(\mathbb{R})$.

(4) If $g \in G$ and $H \leq G$ with $g \in H$, show that $\langle g \rangle \leq H$ (we interpret this as saying that $\langle g \rangle$ is the smallest subgroup of $G$ containing $g$).

(5) The subgroups of $(\mathbb{Z}, +)$: this exercise tells you what all the subgroups of $\mathbb{Z}$ are.
(a) Which subgroup of $\mathbb{Z}$ is the cyclic subgroup of $\mathbb{Z}$ generated by 1?

(b) Suppose $H \leq \mathbb{Z}$. If both $2 \in H$ and $3 \in H$, show that $1 \in H$.

(c) More generally, if $m, n \in H$, show that $\gcd(m, n) \in H$.

(d) Conclude that the subgroups of $\mathbb{Z}$ are exactly the cyclic subgroups $\langle n \rangle$ where $n$ varies over all non-negative integers. I.e. the subgroups are $\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \ldots$

Hint: If $H \leq \mathbb{Z}$ is not the trivial subgroup $\{0\}$, show that $H = \langle d \rangle$ where $d$ is the least positive number in $H$.

Extra problems for the honours students:

Section 2.1: 28, 29, 30
Section 2.2: 5
Section 2.3: 18, 21, 24 and 30