The following problems from the textbook:

**Section 2.5:** 1 and 6, 14, 28 and 26

(1) Consider the group $\mathbb{C}^\times$ whose elements are the non-zero complex numbers and the group operation is multiplication of complex numbers (if you don’t believe me that this is a group, show it!). Recall that in class we showed that

$$G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ not both zero} \right\}$$

is a subgroup of $\text{GL}_2(\mathbb{R})$. Define a function $\varphi : \mathbb{C}^\times \to G$ by

$$\varphi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$  

Show that this is a bijective homomorphism.

(2) Let $m, n \in \mathbb{Z}_{\geq 1}$ and define $\varphi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by

$$\varphi([a]_m) = [a]_n.$$  

In the first few parts, we’ll figure when this is well-defined.

(a) Show that if $m < n$, then $\varphi$ is not well-defined.

(b) From part (a), we must have $m \geq n$. Show that if $n \nmid m$, then $\varphi$ is still not well-defined.

(c) From parts (a) and (b), we must have $n|m$. Show that in this case $\varphi$ is well-defined.

(d) Finally, show that when $n|m$ the map $\varphi$ is a homomorphism. When is it injective? When is it surjective?

(3) Consider the function $\varphi : \mathbb{Z} \to 2\mathbb{Z}$ defined by $\varphi(a) = 2a$. Is $\varphi$ a homomorphism?

(4) Consider the function $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(a) = a + 1$. Is $\varphi$ a homomorphism?

(5) Find a bijective homomorphism from $U_{10}$ to $U_5$. 
(6) Suppose $\varphi : G \to G'$ is a homomorphism. If $\varphi$ is injective, show that $G \cong \text{im}(\varphi)$.

(7) If $G$ is a cyclic group and $\varphi : G \to H$ is a surjective homomorphism, show that $H$ is cyclic.

(8) Show that:

(a) Every group of order 2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
(b) Every group of order 3 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.
(c) Every group of order 4 is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (if you prefer, you can replace the latter with $U_8$).

Hint: Write out the possible multiplication tables (a.k.a. Cayley tables). Remember that each row of the table has to contain each element exactly once. Similarly with each column. If you like Sudoku, surely you’ll like this.

(9) In class, we defined an injective homomorphism $L : G \to A(G)$ sending $g$ to $m_g$, where $m_g : G \to G$ is the bijection $m_g(h) = gh$. This is called the left regular representation of $G$.

(a) Define $m'_g : G \to G$ by $m'_g(h) = hg$ and define $r : G \to A(G)$ by $r(g) = m'_g$. Show that $r$ is not always a homomorphism!

(b) Instead, define $\mu_g : G \to G$ by $\mu_g(h) = hg^{-1}$ and define $R : G \to A(G)$ by $R(g) = \mu_g$. Show that $R$ is a homomorphism. This is called the right regular representation of $G$.

(10) Suppose $G = \langle g_0 \rangle$ is cyclic and $\varphi : G \to G'$ is a homomorphism. Show that $\varphi$ is completely determined by $\varphi(g_0)$.

(11) Find all homomorphisms $\varphi : \mathbb{Z} \to \mathbb{Z}$. For each one, determine its kernel and its image.

Extra problems for the honours students:

(X1) If $m$ is a positive odd integer, find a bijective homomorphism $U_{2m} \to U_m$.

(X2) Show that a group $G$ is cyclic if and only if there is a surjective homomorphism $\mathbb{Z} \to G$.

(X3) Show that every group of order 5 is isomorphic to $\mathbb{Z}/5\mathbb{Z}$.

(X4) Show that there is no isomorphism $\mathbb{Z} \to \mathbb{Q}$ (recall that $\mathbb{Q}$ denotes the rational numbers; the group operation on both of these is addition).