(1) Show that every $A \in O_2(\mathbb{R})$ has determinant $\pm 1$. (Hint: look at $\det(A^T A)$).

(2) The goal of this exercise is to determine the centre of $D_n$ ($n \geq 3$). It will be useful to use the fact that every element of $D_n$ is of the form $\rho^i r^j$ with $0 \leq i \leq n$ and $0 \leq j \leq 1$, as well as the presentation of $D_n = \langle \rho, r : \rho^n = r^2 = 1, r\rho = \rho^{-1}r \rangle$.

(a) When $n$ is even, show that $\rho^{n/2}$ commutes with everything.

(b) Determine when $\rho^i$ commutes with $r$.

(c) Determine when $\rho^i r$ commutes with $r$.

(d) What is the centre $Z(D_n)$? (Hint: the answer depends on whether $n$ is even or odd).

(3) We showed in class that if $H$ is a subgroup of $G$ and $g \in G$, then $g^{-1} H g$ is some other subgroup of $G$. For $G = D_n$ ($n \geq 3$) and the cyclic subgroup $H = \langle \rho \rangle$, show that $g^{-1} H g = H$ for all $g \in G$. Is this true for $H = \langle r \rangle$?

(4) Two elements $h_1$ and $h_2$ in $G$ are called “conjugate” if there is a $g \in G$ such that $g^{-1} h_1 g = h_2$ (we will soon see that this is an equivalence relation on $G$).

(a) Consider the reflection $\rho^a r \in D_n$. Show that for all $g \in D_n$, we have that $g^{-1} (\rho^a r) g = \rho^c r$ with $c \equiv a \pmod{2}$.

(b) With a bit more work, conclude first that if $n$ is even, then $\rho^a r$ is conjugate to $\rho^b r$ if and only if $a \equiv b \pmod{2}$. Then, when $n$ is odd, conclude that $\rho^a r$ is conjugate to $\rho^b r$ for all $a$ and $b$. (This is a mathematical expression of the fact that there are two types of axes of reflection for a regular polygon with an even number of sides: those through a pair of opposite vertices and those through a pair of opposite edges; in the odd case, every reflection is along an axis that goes through a vertex and an edge).

Extra problems for the honours students:

(X1) Show that the quaternion group $Q$ (Assignment 4, problem (1)) has only one subgroup of order 2. Find more than one subgroup of order 2 in $D_4$ (there are in fact 5). Conclude that $Q$ and $D_4$ are not isomorphic (though they both have order 8).