1) Recall from assignment 6 question (4) that, for \( h_1, h_2 \in G \), we say that \( h_2 \) is conjugate to \( h_1 \) if there is \( g \in G \) such that

\[
  h_2 = g^{-1}h_1g.
\]

(The map from \( G \) to itself sending \( h \) to \( g^{-1}hg \) is called “conjugation by \( g \”).)

(a) Show that “conjugacy” is an equivalence relation.
(b) For \( h \in G \), we use the term “conjugacy class of \( h \)” to refer to the equivalence class under the above equivalence relation. We denote the conjugacy class of \( h \) by \( C_h \). Show that \( \#C_h = 1 \) if and only if \( h \in Z(G) \).
(c) In question 4(b) of assignment 6, we found that for \( D_n \) when \( n \) is odd, the reflections are all in the same conjugacy class \( C_r = \{ \rho^i r : 0 \leq i < n \} \). Show that, for \( n \) odd, all the other non-identity conjugacy classes in \( D_n \) have size 2.
(d) When \( n \) is even, the answer is slightly different. What are the conjugacy classes of \( D_n \) in this case?

2) Determine the conjugacy classes of the quaternion group \( Q \).

3) (a) Suppose \( N_1 \) and \( N_2 \) are two normal subgroups of \( G \) such that \( N_1 \cap N_2 = \{ e \} \). If \( a_1 \in N_1 \) and \( a_2 \in N_2 \), show that \( a_1 \) and \( a_2 \) commute. (Hint: consider \( a_1^{-1}a_2^{-1}a_1a_2 \).)
(b) Suppose \( N_1 \) and \( N_2 \) are two normal subgroups of \( G \) such that every \( g \in G \) can be uniquely factored as \( g = a_1a_2 \) with \( a_1 \in N_1 \) and \( a_2 \in N_2 \). Show that \( N_1 \cap N_2 = \{ e \} \).
(c) Recall the definition of the direct product of two groups from Assignment 4 question (2). Suppose again that \( N_1 \) and \( N_2 \) are two normal subgroups of \( G \) such that every \( g \in G \) can be uniquely factored as \( g = a_1a_2 \) with \( a_1 \in N_1 \) and \( a_2 \in N_2 \). Show that the function \( \varphi : N_1 \times N_2 \to G \) defined by

\[
  \varphi((a_1, a_2)) = a_1a_2
\]

is an isomorphism. The group \( G \) is then said to be the internal direct product of \( N_1 \) and \( N_2 \).
(d) For two subsets $S, T \subseteq G$, define $ST := \{st : s \in S, t \in T\}$. Suppose $N_1$ and $N_2$ are two normal subgroups of $G$ such that $G = N_1N_2$ and $N_1 \cap N_2 = \{e\}$. Show that $G$ is the internal direct product of $N_1$ and $N_2$.

(4) Let $p$ be a prime and suppose $G$ is an abelian group of order $p^2$. It is possible that there is an element $a \in G$ such that $#a = p^2$, in which case $G \cong \mathbb{Z}/p^2\mathbb{Z}$ is cyclic. Suppose there is no such $a$ in $G$.

(a) Suppose $a \in G$ with $a \neq e$. Show that $#a = p$.

(b) Suppose $a_1, a_2 \in G$ both not the identity. Let $N_i = \langle a_i \rangle$. Show that if $a_1^i = a_2^j$ for some integers $i$ and $j$ not divisible by $p$, then $a_2 \in N_1$. (Hint: Bézout’s identity will likely pop up in your proof).

(c) Conclude from part (b) that if $a_2 \not\in N_1$, then $N_1 \cap N_2 = \{e\}$.

(d) Show that $G = N_1N_2$. (Hint: think about the cosets of $N_1$).

(e) Conclude that $G$ is the internal direct product of $N_1$ and $N_2$. Then conclude that $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Extra problems for the honours students:

(X1) Let $p > q$ be primes with $p \equiv 1 \pmod{q}$. Define

$$G_{pq} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}/p\mathbb{Z} \text{ with } a^q \equiv 1 \pmod{p} \right\}.$$

(i) Show that $G_{pq}$ is a group.

(ii) Show that $G_{pq}$ is nonabelian.

(iii) What is the order of $G_{pq}$?

(iv) If $p \not\equiv 1 \pmod{q}$, what is the order of $G_{pq}$?