Assignment 9 – Part 1 & 2

The following problems from the textbook:

Section 4.1: 14, 15, 16

(1) The goal of this problem is to classify groups of order 8.

(a) There are three abelian groups of order 8 (up to isomorphism). What are they?

(b) From now on, assume $G$ is a non-abelian group of order 8. Show that $G$ must have an element of order 4. (Hint: exercise 16 of section 2.1 of your book, which you did on assignment 4, will be helpful).

(c) Let $a$ be an element of order 4 and let $b \notin \langle a \rangle$. Show that the 8 elements \{e, a, a^2, a^3, b, ab, a^2b, a^3b\} are distinct and hence are all the elements of $G$.

(d) Show that $bab^{-1} = a^3$. (Hint: first explain why $\langle a \rangle$ is normal, then think about the order of $bab^{-1}$).

(e) Part (c) shows that $G$ is generated by two elements $a$ and $b$. We want to show that $G \cong D_4$ or $G \cong Q$. We know that

\[ D_4 = \langle \rho, r : \rho^4 = e, r^2 = e, r\rho = \rho^{-1}r \rangle \]

and I’m telling you now that

\[ Q = \langle i, j : i^4 = e, j^2 = i^2, ji = i^{-1}j \rangle \]

(You don’t have to show this, but it’s not so hard). We already know that $a^4 = e$ and $ra = a^{-1}r$, so we need to show that either $b^2 = e$ or $b^2 = a^2$. First, show that $b^2 \in \langle a \rangle$.

(f) Finally, show that if either $b^2 = a$ or $b^2 = a^3$, then $G$ is abelian, which is a contradiction. So, there are 5 groups of order 8.

(2) Consider the set $R = \{0, 1, a, a+1\}$ with addition and multiplication tables

\[
\begin{array}{c|cccc}
+ & 0 & 1 & a & a+1 \\
\hline
0 & 0 & 1 & a & a+1 \\
1 & 1 & 0 & a+1 & a \\
a & a & a+1 & 0 & 1 \\
a+1 & a+1 & a & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & 1 & a & a+1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & a & a+1 \\
a & 0 & a & a+1 & 1 \\
a+1 & 0 & a+1 & 1 & a \\
\end{array}
\]
Taking for granted the associativity of addition and multiplication as well as the distributive laws (otherwise this would take a while!), explain why this is a commutative ring with identity. Is it a field?

(3) Consider the elements \( i \) and \( j \) of the quaternions \( \mathbb{H} \) and show that

\[
(i + j)^2 \neq i^2 + 2ij + j^2.
\]

(By the way, example 13 of section 4.1 of the book contains some details about the quaternions \( \mathbb{H} \) that I skipped in class, so you can look there to see these details). This shows that it is not always true in a general ring that \((a + b)^2 = a^2 + 2ab + b^2\). That identity requires \( a \) and \( b \) to commute. What is true is that \((a + b)^2 = a^2 + ab + ba + b^2\).

(4) Show that if \( a \in R \) has a multiplicative inverse, then it can’t be a zero-divisor. Conclude that a field is an integral domain.

(5) Find a zero-divisor in the ring of two-by-two matrices \( M_2(\mathbb{R}) \).

(6) The centre of a ring \( R \) is

\[
Z(R) := \{ z \in R : az = za \text{ for all } a \in R \}.
\]

Let’s figure out the centre of \( M_2(\mathbb{R}) \).

(a) Find all \( a, b, c, d \in \mathbb{R} \) such that

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

commutes with \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

(b) Find all \( a, b, c, d \in \mathbb{R} \) such that

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

commutes with \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

(c) Show that the matrices that are left after doing parts (a) and (b) commute with all other matrices and hence form the centre of \( M_2(\mathbb{R}) \).