2500 years of very small numbers

(The myth of infinitesimals?)

David A. Ross
Department of Mathematics
University of Hawai'i

Definition:

An *infinitesimal* is a quantity which is smaller than any finite magnitude, but not zero.

Modern Terminology:

An *infinitesimal* is a number $x \neq 0$ such that $|x|$ is less than *every* positive number.

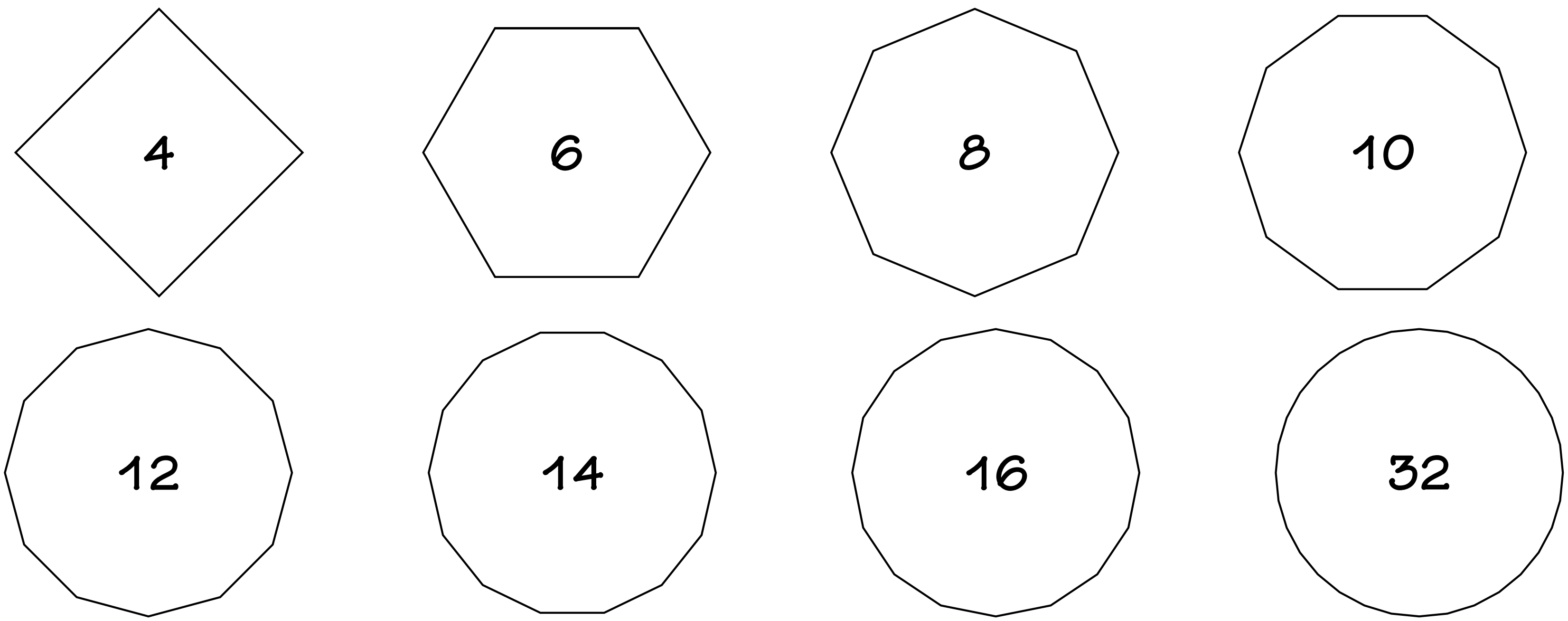
Question:

Is $|x| < |x|$?

What is the area of a circle?

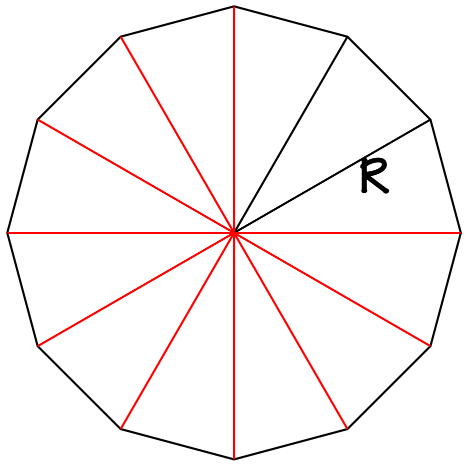
Antiphon, approx 450 B.C.:

A circle is a polygon with a very large number (infinite) of vanishingly small (infinitesimal) sides.



Area computation:

$$
\begin{aligned}
\text{Area of circle} &= (\text{area of triangle}) \times (\text{number of triangles}) \\
&= \frac{1}{2}(\text{base of triangle}) \times (\text{height of triangle}) \times (\text{number of triangles}) \\
&= \frac{1}{2}(\text{base of triangle}) \times (\text{number of triangles}) \times (\text{height of triangle}) \\
&= \frac{1}{2}(\text{circumference of circle}) \times R \\
&= \frac{1}{2}(2 \times \pi \times R) \times R \\
&= \pi R^2
\end{aligned}
$$



Centuries of indecision on infinitesimals

Eudoxus (408-355 B.C.)

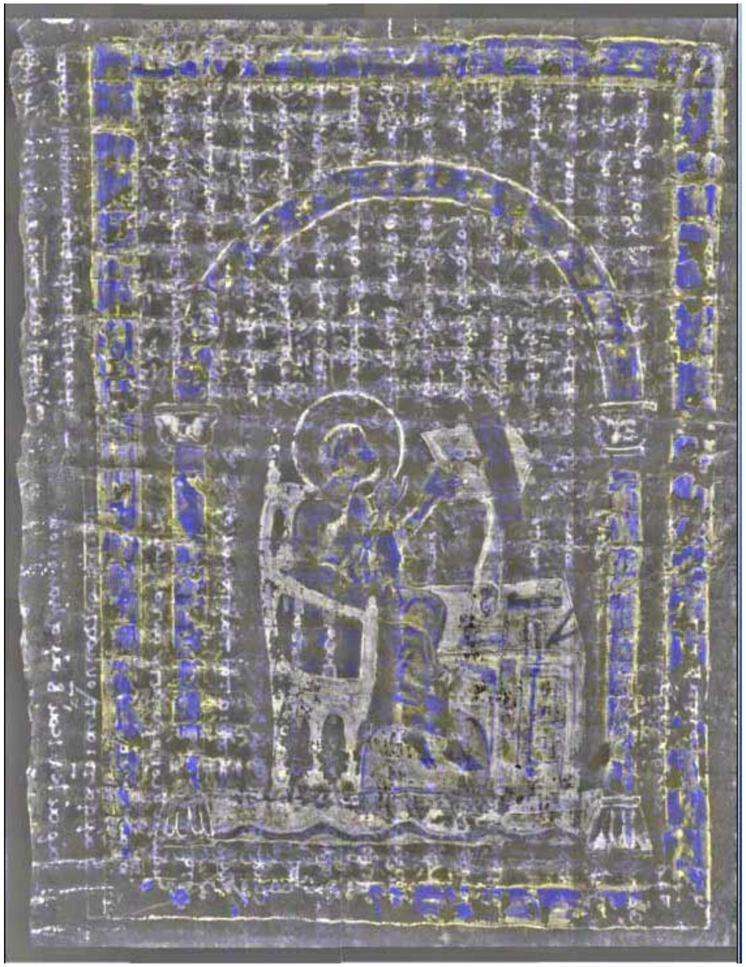
Method of exhaustion

Replaces infinitesimals with small, non-infinitesimal quantities, and a rigorous proof technique

Archimedes (~250 B.C.)

Most published proofs use exhaustion or compression

In *The Method of Mechanical Theorems*, explains how infinitesimal techniques can be used for discovery



Archimedes Palimpsest

10th century copy of several Archimedes works, including only known text of *The Method*

12th century: badly erased and rewritten as a liturgical text

1906: Inspected/photographed in Constantinople; in Baltimore (under anonymous private ownership) since 1998

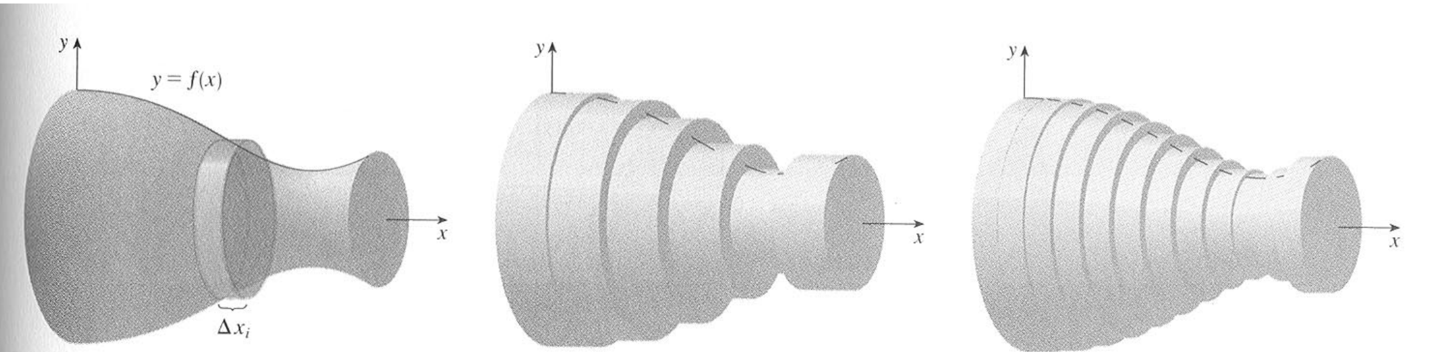
Multispectral imaging, digital image enhancement, and (more recently) X-ray flourescence used to reveal new text.

Many important new findings; for example (2002) equinumerosity of infinite sets.

Johannes Kepler (1615) *Nova stereometria dolorium vinariorium*

Still a part of the standard Calculus curriculum (under the heading "volumes of rotation"), these arguments are awkward without infinitesimals; textbooks try, but most instructors still just use infinitesimal arguments.

Many results duplicate those in *The Method*

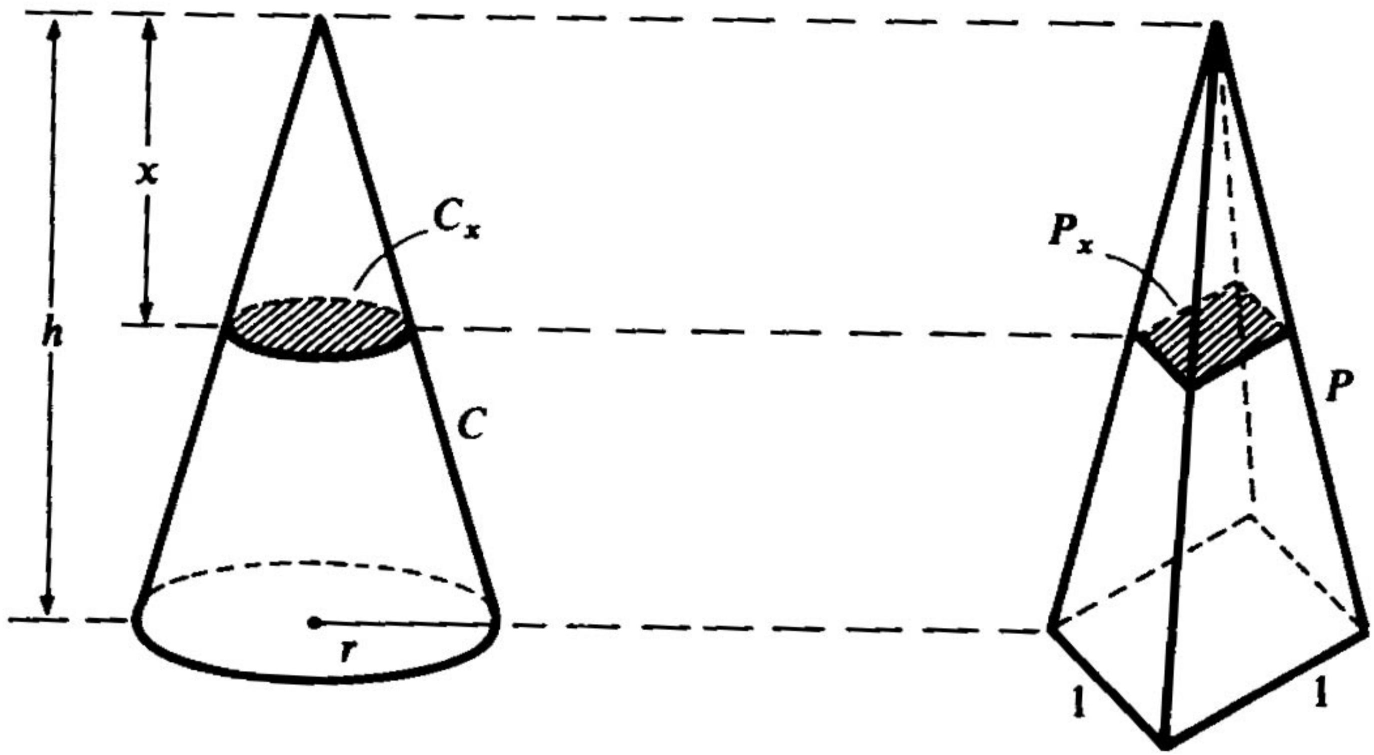


(a) ith approximating cylinder (b) $n = 6$ (c) $n = 12$

FIGURE 6

Bonaventura Cavalieri (1635), *Geometria indivisibilus*

Replaced explicit infinitesimal arguments by axiomatic statements.



(1637), *Excercitationes geometricae*:

"Rigor is the affair of philosophy rather than mathematics."

Isaac Barrow

Tangents to curves using infinitesimal methods.

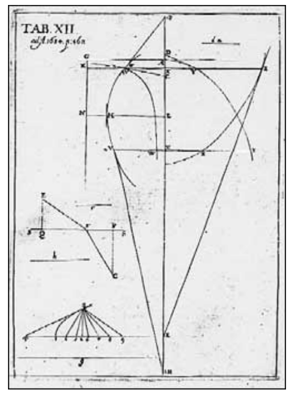
(1664-7; published 1683) Several works speculating on the methodology used by Archimedes.

Recent question: Barrow visited Constantinople at a time when the Archimedes Palimpsest was there; did he have access to it?

Gottfried Wilhelm von Leibniz

(1684): Nova methodus pro maximis et minimis..., *Acta Eruditorum*

(1686): De geometria recondita et analysi indivisibilium et infinitorum, *Acta Eruditorum*



Free use of infinitesimals. Viewed curves much as Antiphon, as composed of infinitely many small segments.

(1700+) The two questions:

a) Are infinitesimals real?

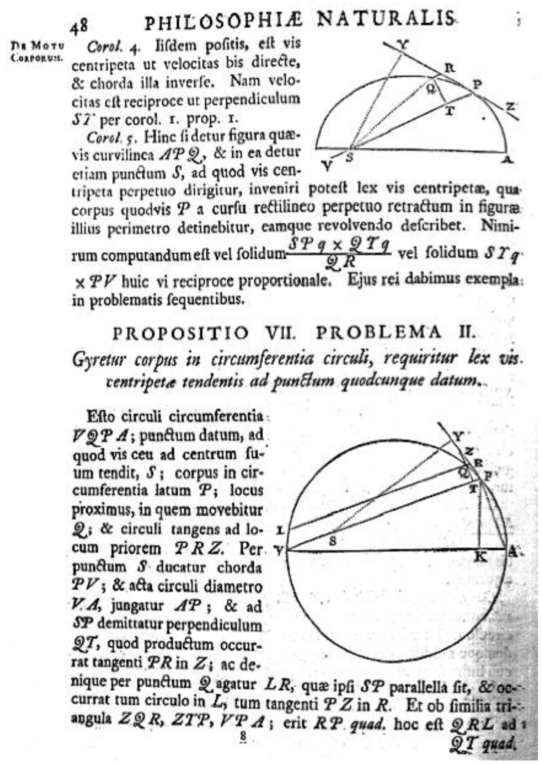
b) Do infinitesimals lead to correct theorems?

are independent.

Isaac Newton

(1687): *Principia Mathematica*

Uses no infinitesimals in the proofs (not even fluxions).



48 PHILOSOPHIÆ NATURALIS

DE MOTU CORPORUM.

Corol. 4. Iisdem positis, est vis centripeta ut velocitas bis directe, & chorda illa inverse. Nam velocitas est reciproce ut perpendiculum *ST* per corol. 1. prop. 1.

Corol. 5. Hinc si detur figura quævis curvilinea *APQ*, & in ea detur etiam punctum *S*, ad quod vis centripeta perpetuo dirigitur, inveniri potest lex vis centripetæ, qua corpus quodvis *P* a cursu rectilineo perpetuo retractum in figuræ illius perimetro detinebitur, eamque revolvendo describet. Nimirum computandum est vel solidum $\frac{SPq \times QTq}{QR}$ vel solidum $STq \times PV$ huic vi reciproce proportionale. Ejus rei dabimus exempla in problematis sequentibus.

PROPOSITIO VII. PROBLEMA II.

Gyretur corpus in circumferentia circuli, requiritur lex vis centripetæ tendentis ad punctum quodcunque datum.

Esto circuli circumferentia *VQPA*; punctum datum, ad quod vis ceu ad centrum sua tendit, *S*; corpus in circumferentia latum *P*; locus proximus, in quem movebitur *Q*; & circuli tangens ad locum priorem *PRZ*. Per punctum *S* ducatur chorda *PV*; & acta circuli diametro *VA*, jungatur *AP*; & ad *SP* demittatur perpendiculum *QT*, quod productum occurrat tangenti *PR* in *Z*; ac denique per punctum *Q* agatur *LR*, quæ ipsi *SP* parallella sit, & occurrat tum circulo in *L*, tum tangenti *PZ* in *R*. Et ob similia triangula *ZQR*, *ZTP*, *VPA*; erit *RP quad.* hoc est *QRL* ad *QT quad.*

8

(1691): *De Quadratura Curvarum*

Disavows infinitesimals.

George (Bishop) Berkeley

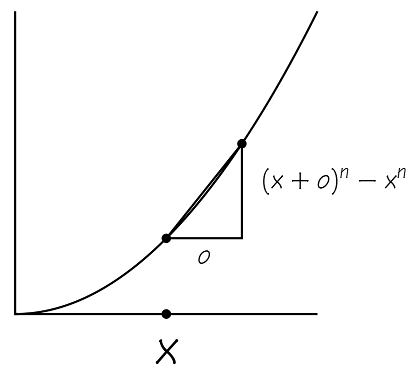
(1734): *The Analyst*

> "The foreign Mathematicians...suppose finite Quantities to consist of Parts infinitely little, and Curves to be Polygons, whereof the Sides are infinitely little, which by the Angles they make one with another determine the Curvity of the Line. Now to conceive a Quantity infinitely small, that is, infinitely less than any sensible or imaginable Quantity, or any the least finite Magnitude, is, I confess, above my Capacity. But to conceive a Part of such infinitely small Quantity, that shall be still infinitely less than it, and consequently though multiply'd infinitely shall never equal the minutest finite Quantity, is, I suspect, an infinite Difficulty to any Man whatsoever; and will be allowed such by those who candidly say what they think; provided they really think and reflect, and do not take things upon trust."

Berkeley was reacting in part to criticisms of religion from mathematicians:

> "But he who can digest a second or third Fluxion, a second or third Difference, need not, methinks, be squeamish about any Point in Divinity."

Berkeley considers the following derivation of the slope of the curve $y = x^n$:



Let o be an infinitesimal, then the slope is the change in y over the change in x:

$$\frac{(x+o)^n - x^n}{o} = \frac{(x^n + nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \cdots) - x^n}{o}$$

$$= \frac{nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \cdots}{o}$$

$$= nx^{n-1} + \frac{n(n-1)}{2}ox^{n-1} + \cdots$$

Now, the second and subsequent terms are negligible since o is infinitesimal, so we drop them, and the slope is nx^{n-1}

Berkeley wrote in response:

> "Hitherto I have supposed that x flows, that x hath a real Increment, that o is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as *one* single Step. From that Supposition it is that I get at the Increment of x^n, that I am able to compare it with the Increment of x, and that I find the Proportion between the two Increments. I now beg leave to make a new Supposition contrary to the first, i. *e.* I will suppose that there is no Increment of x, or that o is nothing; which second Supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain nx^{n-1}, which is an Expression obtained in virtue of my first Supposition, which necessarily presupposeth such Supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity."

And more:

> "Leibnitz and his followers in their calculus differentialis making no manner of scruple, first to suppose, and secondly to reject Quantities infinitely small...I shall now only observe as to the method of getting rid of such Quantities, that it is done without the least Ceremony."

As for the Leibnizian idea that correct theorems may be obtained even assuming infinitesimals are not real:

> "But then it must be remembred, that in such Case although you may pass for an Artist, Computist, or Analyst, yet you may not be justly esteemed a Man of Science and Demonstration."

Concludes with several questions, among them:

> "Whether it be necessary to consider Velocities of nascent or evanescent Quantities, or Moments, or Infinitesimals? And whether the introducing of Things so inconceivable be not a reproach to Mathematics?"

In the face of this and other criticism, mathematicians continued to use infinitesimals, because they worked. Notably:

Marquis de l'Hospital (1696) *Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes* (first Calculus textbook)

> "A curved line may be regarded as being made up of infinitely small straight line segments"
>
> "One can take as equal two quantities differing by an infinitely small quantity"

Leonhard Euler (1748): *Introductio in analysin infinitorum* (and many other works)

Pretty much every other working mathematician of the day.

In the 19th century, matematicians sought for and found a way to give Calculus a sound footing without the need for infinitesimals.

Contributors include Bernard Bolzano (∼1817), Jean Le Rond d'Alembert, *Encyclopédie méthodique* (177?) Augustin-Louis Cauchy, *Cours d'analyse* (1821), and finally Karl Weierstrass (late 1870s) gave the 'modern' definition of limit.

Once infinitesimals were seen as unnecessary, mathematicians abandoned them hastily, *eg*

Georg Cantor (1888/9): Claims to prove impossibility of infinitely small numbers, labels infinitesimals a "cholera-bacillis" infecting mathematics.

Bertrand Russell (1901) calls them "unnecessary, erroneous, and self-contradictory"

Resurrection of infinitesimals

Abraham Robinson (1960): Seminars in Princeton and at the ASL annual meeting

(1961): Nonstandard Analysis, *Proc. Royal Acad. Sciences Amsterdam*

Extended work of Skolem (1938), Los (1955), Schmieden and Laugwitz (1958), *et al*, in which methods from Model Theory (a newish branch of mathematical logic) *were used to produce extensions of* $\mathbb{R}$ *containing* infinitesimals.

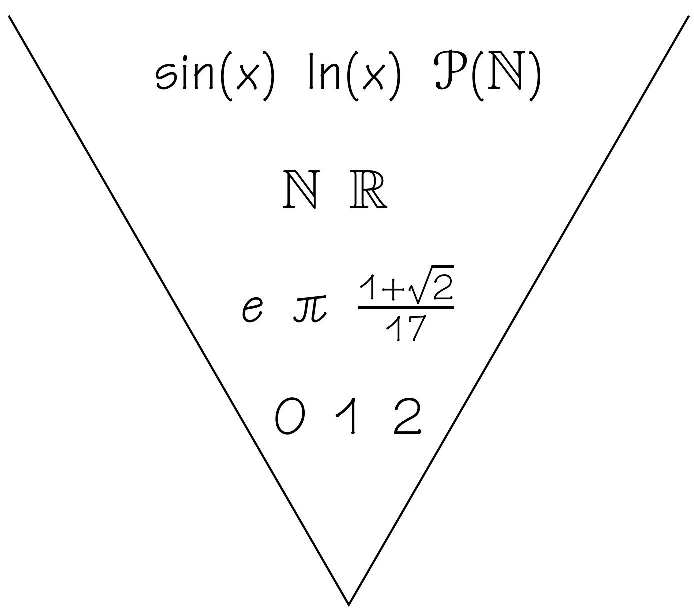
The problem with *these earlier extensions* is that *they did* not include *enough of* the *structure of* mathematical *universe to be* useful.

Robinson's idea: *extend* all of mathematics *at once* *to a* larger model, *and* use predicate logic *to* keep track *of* what's true in the bigger model.

Construction of the nonstandard model

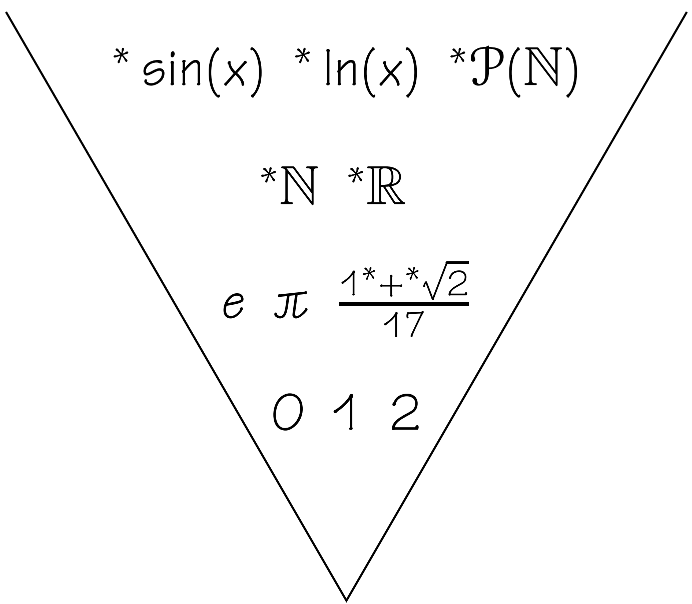
Start with a mathematical universe (*superstructure*) V, containing:

- All natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, \pi, e, \phi, \ldots$; *etc.*
- The *set* $\mathbb{N}$ of natural numbers *as an object*; the set $\mathbb{R}$ of real numbers; *etc.*
- Every function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions
- Your favorite groups, Banach spaces, etc
- Every other mathematical object we might want to talk about
- We call the elements of this mathematical universe *standard*.



Extend to a *nonstandard* mathematical universe $^*V$:

- For *every object* A in V, there is a corresponding object $^*A$ in $^*V$
- EG, $^*V$ has objects $^*\mathbb{N}$, $^*\mathbb{R}$, $^*\sin(x)$, *etc.*
- (For simplicitry, *we drop the stars* from simple objects like numbers: 12 instead of ^{*12} *etc*)
- There may (generally will) *be many more objects* in $^*V$ than in V
- An *element of* $^*V$ that is **not** in V is called *nonstandard.*



The extension should satisfy two important properties:

Transfer If S is a statement[1] about objects in V, then S is true in V if and only if it true in ${}^*V$

For example, since the following is true in V:

$$\text{For any } x \text{ and } y \text{ in } \mathbb{R},\ x + y = y + x$$

then in ${}^*V$ it follows:

$$\text{For any } x \text{ and } y \text{ in } {}^*\mathbb{R},\ x {}^*\!+ y = y {}^*\!+ x$$

[1](Technically, a "statement" is a first-order bounded-quantifier formula in a language with constants and function/predicate symbols from V.)

In particular, since (for example) 12 is an element of $\mathbb{N}$, ^{*12} is an element of $^*\mathbb{N}$.

Since we can think of the basic elements (like ^{*12}) of $^*V$ as just being the same as their counterparts (like 12) in V, $^*\mathbb{N}$ is a superset of $\mathbb{N}$.

Similarly, for *any* standard set A which is an object of V, the set $^*A$ in $^*V$ extends the set A.

Remark: We might imagine a standard mathematician living in universe V, and a nonstandard mathematician living in $^*V$. The transfer principle says that both these mathematicians experience exactly the same true statements. The reason this is possible is that they both speak the same language - the language of V. In particular, the 'nonstandard' mathematician will not be able to refer to any particular element of $^*V$ that is not the $*$−image of an element of V.

Saturation Suppose that $\mathcal{S}$ is a collection of statements about an *object* X, and that for *every* finite subcollection of $\mathcal{S}$ there is an *object* in $^*V$ for which they hold; then there is an *object* in $^*V$ for which **all** the statements in $\mathcal{S}$ hold **at the same time**.

Roughly means: Anything that can happen in $^*V$, *does* happen.

I have not *explained* how such a nonstandard model is *created*. *We* now know many ways to *construct* such models; all *employ* straightforward *techniques* from mathematical logic, and none is *especially difficult*.

Example: Consider the statements:

$$x \text{ is a real number}$$

$$x > 0$$

$$x < 1$$

$$x < 1/2$$

$$x < 1/3$$

$$x < 1/4$$

$$\vdots$$

Any finite set of these statements refers to a smallest fraction $1/N$; but then, $x = \frac{1}{N+1}$ satisfies this finite set of statements.

It follows that there is a an element of ${}^*\mathbb{R}$, call it ϵ, such that

$$\epsilon > 0$$

and, for every (standard) natural number N,

$$\epsilon < 1/N$$

We have proved that ${}^*\mathbb{R}$ contains nonzero infinitesimals, where

Definition: An *infinitesimal* is an element ϵ of ${}^*\mathbb{R}$ such that

$$|\epsilon| < 1/N$$

for *every* natural number N in $\mathbb{N}$

How does this avoid internal contradictions?

For example, the following statement, called the Archimedean Property, is true for the usual real numbers:

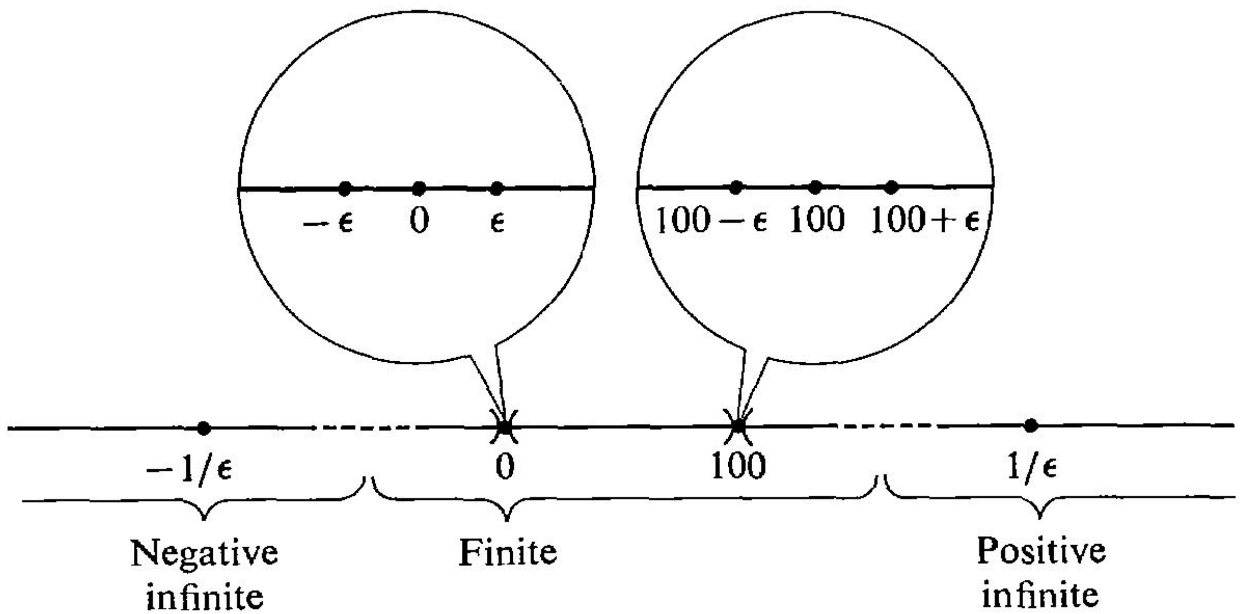
$$\boxed{\text{For every positive real number } x \text{ there is an } N \text{ in } \mathbb{N} \text{ such that } Nx > 1.}$$

By the transfer property,

$$\boxed{\text{For every positive } x \text{ in } {}^*\mathbb{R} \text{ there is an } N \text{ in } {}^*\mathbb{N} \text{ such that } Nx > 1.}$$

Note that this is true for our ϵ as well; while $N\epsilon < 1$ for every N in $\mathbb{N}$, there will be elements of ${}^*\mathbb{N}$ which are less than $1/\epsilon$!

Since $^*\mathbb{R}$ (*sometimes called the set of "hyperreal numbers"*) is, like the usual set of real numbers, *closed* under the basic arithmetic *operations*, it *also* contains *negative* infinitesimals (like $-\epsilon$), infinite numbers (like $1/\epsilon$), and many *other* objects:



In particular, *as we have seen there are elements of* $^*\mathbb{N}$ which are *bigger than every element of* $\mathbb{N}$; in *other words, there are infinite integers.*

Some notation:

If two numbers x and y differ by an infinitesimal, write $x \approx y$

The set of infinitesimals is therefore $\{x \text{ in } {}^*\mathbb{R} \ : \ x \approx 0\}$.

It turns out that *every* finite hyperreal s differs infinitesimally from some unique standard real r; call r the *standard part* of s, $r = \mathrm{st}(s)$.

In other words, st() takes any finite hyperreal to the closest standard real number.

Example Return to the problem of finding the slope of the curve $y = x^n$ (i.e., $\frac{d}{dx}(x^n)$)

Recall the computation:

$$\frac{(x+o)^n - x^n}{o} = \frac{(x^n + nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \cdots) - x^n}{o}$$

$$= \cdots \text{some algebra} \cdots = nx^{n-1} + \frac{n(n-1)}{2}ox^{n-2} + \cdots$$

For any given value of x and o, this will be a hyperreal number. If x is a standard real number, and o an infinitesimal, then all the terms but the first one are infinitesimal, so the standard part of this expression is just nx^{n-1}.

In a nonstandard developement of Calculus, one can therefore just define the derivative of a function f at a (standard) real number x by:

$$\frac{d}{dx}f(x) = \mathrm{st}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right), \qquad \text{where } \epsilon \approx 0, \epsilon \neq 0$$

provided this standard part is defined and does not depend on the choice of $\epsilon \approx 0$.

To make this practical, one needs arithmetic rules for manipulating infinitesimals (and infinite real numbers); these are actually quite simple to state and use.

EG (from *Elementary Calculus* by H.J. Keisler)

Let a and b be finite hyperreal numbers. Then

(i) $st(-a) = -st(a)$.

(ii) $st(a+b) = st(a) + st(b)$.

(iii) $st(a-b) = st(a) - st(b)$.

(iv) $st(ab) = st(a) \cdot st(b)$.

(v) *If* $st(b) \neq 0$, *then* $st(a/b) = st(a)/st(b)$.

(vi) $st(a^n) = (st(a))^n$.

(vii) *If* $a \geq 0$, *then* $st(\sqrt[n]{a}) = \sqrt[n]{st(a)}$.

(viii) *If* $a \leq b$, *then* $st(a) \leq st(b)$.

Other notions from Calculus become equally simple.

For example, the following standard definition of continuity (from any modern Calculus text) is meant to capture the intuitive idea that a function's graph does not have a break at x:

Definition - Weierstrassian

f is continuous at x provided for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever y is a real number with $0 < |y - x| < \delta$, $f(y) - f(x)| < \epsilon$.

It is often hard for freshmen to parse this expression! Compare with:

Definition - nonstandard

f is continuous at x provided whenever $y \approx x$, $f(y) \approx f(x)$.

Applications outside of Calculus

The nonstandard model has turned out to be a surprisingly useful construct, not just for the foundations of mathematics, but for research in many areas of pure and applied mathematics.

The general practice of using these large models for work in mathematics has come to be called "nonstandard analysis" (after the Robinson's article and a later book), though it should be *emphasized* (for Bishop Berkeley's sake) that while the methodology is not classical, the results it is used to prove are classical (and, in particular, true).

The most interesting applications are based on the ubiquity of "hyperfinite sets"

(**Definition:** A set E in $^*V$ is *hyperfinite* if there is a *one-to-one correspondence between E and $\{0, 1, 2, \ldots, H\}$ for some H in $^*\mathbb{N}$. Equivalently, if the formal statement "X is finite" holds for E in $^*V$.)

Lemma: If A is an infinite set in V then there is a hyperfinite set $\hat{A}$ in ${}^*V$ such that *every* element of A is in $\hat{A}$

Proof: Consider the statements: (i) X is finite; (ii) a is in X (one such statement for every element a of A)

Given any finite number of these statements, a corresponding finite number $\{a_1, \ldots, a_n\}$ of elements of A are mentioned, so $X = \{a_1, \ldots, a_n\}$ satisfies those statements. By the saturation principle there therefore a set X in ${}^*V$ satisfying all the statements simultaneously; let $\hat{A}$ be this X.

Corollary: There is a hyperfinite set containing $\mathbb{R}$.

> "Nonstandard analysis is the art of making infinite sets finite by extending them." —M. Richter

The advantage of hyperfinite sets is that *we* can use finitary counting methods with them, *even* if they *are* infinite.

The duality between *discrete*/hyperfinite *and* continuous/infinite has been *exploited* in areas such as:

- Stochastic analysis (Brownian motion is the standard part of a random walk on a hyperfinite time line; stochastic differential *equations* are solved as difference *equations* against these random walks)
- Mathematical *economics* (hyperfinite sets of infinitesimal traders)
- Mathematical physics (*eg*, "Loeb measure" solutions of the Boltzmann *equation*)
- Number theory ("approximate" $\mathbb{N}$ by the set $\{0, 1, \ldots, H\}$, H infinite integer)
- Probability theory (probability measures are built as counting measures on hyperfinite sets)
- 9^{th} *grade* philosophy of mathematics:
  - Zeno's Paradoxes (W.I. McLaughlin, *Scientific American*, Nov. 1994)
  - $0.99999\cdots = 1$
  - Dartboard problem

Final thoughts

- The initial *objections* to infinitesimals *were ontological; are* infinitesimals real?
- The 'modern' solution *doesn't resolve* this question, only shows that they *can* be introduced formally - a very Leibnizian solution.
- What a *given* infinitesimal "is", as an *object, depends on* how the nonstandard model is *constructed*.
- This has been *raised* as an *objection* to nonstandard analysis, but *post-*Dedekind *the same question* can be *raised* with *respect* to the real numbers!
- (In fact, the nonstandard model *can be* used *to define* $\mathbb{R}$ from the rational numbers.)

- From the point of view of a mathematical logician, there has been a fundamental shift in the nature of mathematics over the centuries: from being about numbers and geometric objects, to being about *statements* about numbers and geometric objects, and finally (in the 20th century) to being about statements about mathematics.
- It is difficult to maintain a Platonist view of mathematical objects like the real line when faced with this shift.
- On the other hand, nearly all mathematicians adopt the useful fiction in practice that such mathematical objects (and far more esoteric ones) are 'real'; there is no reason not to do this with infinitesimals as well.
- Infinitesimals represent an opportunistic approach to mathematical argument - use anything that works. However, whereas the infinitesimal arguments of Archimedes et al were just used for *discovery*, the same arguments in the modern context constitute a complete and correct *proof*.