1 Introduction

In 1900, David Hilbert outlined 23 mathematical problems to the International Congress of Mathematicians in Paris. Here is the first:

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1. Cantors Problem von der Mächtigkeit des Continuums

Zwei Systeme, d.h. zwei Mengen von gewöhnlichen reellen Zahlen (oder Punkten) heißen nach Cantor aequivalent oder von gleicher Mächtigkeit, wenn sie zu einander in eine derartige Beziehung gebracht werden können, daß einer jeden Zahl der einen Menge eine und nur eine bestimmte Zahl der anderen Menge entspricht. Die Untersuchungen von Cantor ber solche Punktmengen machen einen Satz sehr wahrscheinlich, dessen Beweis jedoch trotz eifrigster Bemühungen bisher noch Niemanden gelungen ist; dieser Satz lautet:

Jedes System von unendlich vielen reellen Zahlen d. h. jede unendliche Zahlen- (oder Punkt)menge ist entweder der Menge der ganzen natrlichen Zahlen 1, 2, 3, ... oder der Menge sämmtlicher reellen Zahlen und mithin dem Continuum, d.h. etwa den Punkten einer Strecke aequivalent; im Sinne der Aeqivalenz giebt es hiernach nur zwei Zahlenmengen, die abzählbare Menge und das Continuum.

Aus diesem Satz würde zugleich folgen, daß das Continuum die nächste Mächtigkeit ber die Mächtigkeit der abzählbaren Mengen hinaus bildet; der Beweis dieses Satzes wrde mithin eine neue Brcke schlagen zwischen der abzählbaren Menge und dem Continuum.

Babelfish.altavista.com translation:

Two systems, i.e. two quantities of usual real numbers (or points) are called after CAN gate equivalent or from same power, if they can be brought too each other into a such relationship that to each number quantity and only a certain number of the other quantity corresponds. The investigations of CAN gate over such point sets very probably make a sentence, whose proof succeeded however despite most eager efforts so far still nobody; this sentence reads:

Each system of infinitely many real numbers i.e. each infinite payment (or Punkt)menge is either the quantity of the whole natural numbers of 1, 2, 3... or the quantity of more saemmtlicher real numbers and therefore the Continuum, i.e. about the points of a distance equivalent; in the sense of the Aeqivalenz it giebt from this only two number sets, the countable quantity and the Continuum.

From this sentence it would follow at the same time that the Continuum forms next power beyond the power of the countable quantities; the proof of this sentence would therefore strike a new bridge between the countable quantity and the Continuum.

2 Formulation of the problem

- A and B have the same cardinality, $A \approx B$, if A and B can be put into a one-to-one correspondence.
- $A \lessapprox B$ is there is a one-to-one function $f : A \to B$
- **Proposition:** If $A \approx B$ and $B \approx A$ then $A \approx B$
- **Theorem (Cantor-Schröder-Bernstein):** If $A \lessapprox B$ and $B \lessapprox A$ then $A \approx B$
- A is "countably infinite" (or $\text{card}(A) = \aleph_0$) if $A \approx \mathbb{N}$
- A has cardinality $\mathfrak{c}$ (or $\text{card}(A) = 2^{\aleph_0}$) if $A \approx \mathbb{R}$
- **Theorem (Cantor):** $\mathbb{R}$ is uncountable, that is, $\mathbb{R} \not\approx \mathbb{N}$
- **CH:** If $A \subseteq \mathbb{R}$ is infinite, then $A \approx \mathbb{N}$ or $A \approx \mathbb{R}$

3 The cardinality $\aleph_0$

EXAMPLES:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,...
($\mathbb{N}$ itself)

0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5,...
(The integers $\mathbb{Z}$)

a, b, aa, ab, ba, bb, aaa, aab,...
('Words' on the alphabet {a,b})

Words on any finite alphabet

Words on 0, 1, 2, 3,...,9, -, /
(The rational numbers $\mathbb{Q}$)

The algebraic real numbers

The computable real numbers

$A \times B$ **if** $A,\ B$ **are countably infinite**

4 The cardinality $\mathfrak{c}$

EXAMPLES:

$\mathbb{R}$ **itself**

(a, b) if $a < b \in \mathbb{R}$

[a, b) if $a < b \in \mathbb{R}$

${}^{\mathbb{N}}\{0, 1\}$ **(=countable sequences of 0s and 1s)**

$[0, 1) \times [0, 1)$

$A \times B$ **if** $A,\ B$ **have cardinality** $\mathfrak{c}$

${}^{\mathbb{N}}\mathbb{R}$ **(=countable sequences of real numbers)**

$\mathbb{R} \setminus A$ **for any countable A**
($\therefore$ irrationals, transcendental reals, $\ldots$)

5 Timeline of CH

1845 Cantor born.

1874 Cantor, *Journal für die reine und angewandte Mathematik* ; proof that $\mathbb{R} \not\approx \mathbb{N}$

1878 Cantor, Ein Beitrag zur Mannigfaltigkeitslehre, *Journal für die reine und angewandte Mathematik* ; articulates the Continuum Problem. (Lots of other stuff, for example first correct proof that $[0,1]^2 \approx [0,1]$)

1900 Hilbert problems.

1916 Hausdorff proves CH true for Borel sets

1918 Cantor dies.

1934 W. Sierpinski, Hypothése du Continu

1936 K. Gödel proves CH consistent with ZFC (note: 1938, suggests CH true)

1947 K. Gödel, What is Cantor's continuum problem?, *Amer. Math. Monthly* (note: 1944, suggests CH false)

1963 P. Cohen proves $\neg$CH consistent with ZFC

6 Some consequences of CH

Freiling $\exists f \in {}^{\mathbb{R}}\mathcal{P}_\omega(\mathbb{R}) \forall x, y \in \mathbb{R}, x \in f(y) \vee y \in f(x)$

Sierpinski $\mathbb{R}^2$ is the union of graphs of countably many functions

Sierpinski There is a countable sequence $f_k : \mathbb{R} \rightarrow \mathbb{R}$ such that for any uncountable $E \subseteq \mathbb{R}$, $f_k[E] = \mathbb{R}$ for all but finitely many k.

A useful formulation of CH:

- **Theorem**(ZF) There is a linearly ordered set $(I, <)$ such that (a) I is uncountable, and (b) $\forall \alpha \in I$, $\operatorname{seg}(\alpha) := \{\beta \in I \ : \ \beta \leq \alpha\}$ is countable
- For each $\alpha \in I$ let $g_\alpha : \mathbb{N} \rightarrow \operatorname{seg}(\alpha)$ be an onto function.
- CH is equivalent to: $\mathbb{R} \approx I$ (in which case we can write $\mathbb{R} = \{x_\alpha \ : \ \alpha \in \ I\}$)

7 Further Reading

- J. Dauben, *Georg Cantor: His mathematics and philosophy of the infinite*, 1979
- C. Freiling, Axioms of symmetry, *J. Symbolic Logic*, 1986
- L. Gillman, Two classical surprises concerning the axiom of choice and the continuum hypothesis, *American Math. Monthly*, June-July 2002
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- P. Maddy, Believing the axioms I, *J. Symbolic Logic*, June 1988
- W. Sierpinski, *Hypothése du continu*, 1934
- N. Vilenkin, *Stories about sets*, 1968