The Math of Rational Choice - Math 100 Spring 2015
Part 2

Fair Division

Situations where "fair division" procedures are useful:

- Inheritance; dividing assets after death
- Divorce: dividing up the money, books, records (umm...cassette tapes? CDs? iTunes?), dogs
- Campus budgeting
- Dividing cake or pizza
- Picking teams

Three slightly different problems:

I. Divide something into n equal pieces, where "equal" means according to some objective criteria.

II. Fair share: Divide something into n shares such that each of n agents believes they received at least $\frac{1}{n}$ of the booty.

III. Envy-free division: Divide something into n shares such that each of n agents believes nobody received more than they did.

Problem: Granularity

Puzzle: King Lear willed all his worldly posessions to his three daughters. He willed 50% (or $= \frac{1}{2}$) to his daughter Goneril, 25% ($= \frac{1}{4}$) to his daughter Regan, and $\frac{1}{6}$ to his daughter Cordelia. When he died, it turned out all he owned was 11 magic rings. Nobody knew what to do, since 11 is not divisible by 2, 4, or 6.

Fortunately for them, Sauron was passing by on his way back to Mordor; he too had a magic ring, and he engineered the following solution:

1. First, he put his ring in the pile, so there were 12 in all.
2. He gave 50% of the 12, or 6 rings, to Goneril
3. He gave 25% of the 12, or 3 rings, to Regan
4. He gave $\frac{1}{6}$ of the 12, or 2 rings, to Cordelia
5. Since $6 + 3 + 2 = 11$, there was a ring left over, Sauron's own ring, which he retrieved before saying his goodbyes and heading on to Mordor.

Question: What happened?

A) That Sauron is such a nice young man, he secretly added to the pot.

B) Sauron's ring ruled all the other rings, and forced them to divide evenly.

C) Besides being a bad father, Lear was a rotten mathematician.

D) Someone got ripped off.

E) *King Lear* sounds cool, I'm signing up for a Shakespeare class next semester for sure!

This granularity (often called *discreteness*) is a problem with all fair allocations problems.

For the rest of the discussion, we will assume that the goods to be allocated are either *continuous* (like money), or the granularity is small enough to not matter much.

Some famous problems of equal division:

1. The Ham Sandwich Theorem (Steinhaus, Banach 193?)
   (Closely related to the Hairy Billiard Ball Theorem)

   > Given that there are three sets in 3-dimensional space of finite size; is there a plane simultaneously cutting each of the three sets into two parts of equal measure?
   >
   > Can you always slice a ham sandwich (with one straight cut) so that the two slices of bread and the ham inside are all cut exactly in half by volume?

2. Problem of the Nile (Fisher, 1938)

Each year the Nile would flood, thereby irrigating or perhaps devastating parts of the agricultural land of a predynastic Egyptian village. The value of different portions of the land would depend upon the height of the flood. In question was the possibility of giving to each of the k residents a piece of land whose value would be 1/k of the total land value no matter what the height of the flood.

Assumptions:

Assets: We have some assets (booty) to divide. These might be tangible physical objects (cake, pizza, jewelry) or conceptual/representational objects (money, status).

The assets might be continuous (money, land, cake) or discrete (cars, candy, people).

Alost all the methods we discuss will work best for continuous (or nearly continuous) assets

Parties: We have agents/players vying to share the booty. These might be people, institutions, states, etc.

Values: Each agent has a internalized value system for judging the value of booty. This might be individual or shared.

EXAMPLE: Cut/choose to choose cake (a DIVIDER-CHOOSER method)

2 Agents: Person A cuts the cake into two pieces, person B gets to choose the piece

Clicker question: Which is it better to be?

A) Person who cuts

B) Person who chooses

Strategy for DIVIDER is to choose 2 pieces as close in size or value as possible

(Why might value not equal size? EG: half-mushroom half-sausage pizza)

So: whichever piece the DIVIDER gets seems fair (1/2 the value) to him or her

CHOOSER can pick either piece. Since at least one piece is at least 50% of the value in CHOOSER's eyes, can make sure to get a fair piece

Result is a fair piece in each person's eyes

result is ENVY-FREE since neither player thinks the other got a better piece

No necessarily a completely equal division, since CHOOSER could get a piece that they judge to be much better.

Better to be the CHOOSER. (Always randomize when using this method this!)

What about 3 or more people?

Obvious generalization doesn't work! (Example in class and readings)

Main methods we will consider:

Lone Divider Method Hugo Steinhaus 1930s (*Scottish Book*)

Lone Divider Method Generalization to $n > 3$ people by Harold Kuhn

Last Diminisher Banach and Knaster (1948)

Moving Knife Dubins and Spanier (1961)

(Might consider others, such as Selfridge and Conway (1960),...)

Lone Divider – 3 people

1. Divider D divides cake into three pieces (of equal value to D).
2. Choosers A and B *bid* on the cake. That is, each lists which pieces they consider to be worth at least 1/3 to them. (This is done in secret.)

   Note each must list at least 1 piece, and listing only 1 when 2 are acceptable might hurt them.
3. If A and B can each get a (different) piece that is acceptable to them, then give each such a piece and give D the remaining piece. Note that this can happen if either A or B (or both) label at least 2 pieces as acceptable, as well as if A and B have only one acceptable piece but they are different.
4. If A and B each only have 1 piece labeled as acceptable, and it is the same piece: give one of the non-contested pieces to D, smush the two remaining pieces of cake back together, and let A and B choose using cut/choose.

Lone Divider – 3 people, alternate sequential form without the simultaneous secret bids

1. Divider D divides cake into three pieces (of equal value to D).
2. If Chooser A says 2 or more pieces are acceptable, pieces are chosen by players in the order B, A, D; game over.
3. Otherwise: if Chooser B says 2 or more pieces are acceptable, pieces are chosen by players in the order A, B, D; game over.
4. Otherwise: D gets a piece deemed "unacceptable" by both A and B, smush the two remaining pieces of cake back together, and let A and B choose using cut/choose.

Example 1:

value	p_1	p_2	p_3
D	$0.33\bar{3}$	$0.33\bar{3}$	$0.33\bar{3}$
A	0.40	0.10	0.50
B	0.45	.20	.35

Example 2:

value	p_1	p_2	p_3
D	$0.33\bar{3}$	$0.33\bar{3}$	$0.33\bar{3}$
A	0.25	0.45	0.30
B	0.50	.26	.24

Example 3:

value	p_1	p_2	p_3
D	$0.33\bar{3}$	$0.33\bar{3}$	$0.33\bar{3}$
A	0.25	0.45	0.30
B	0.35	.60	.05

Example 4:

value	p_1	p_2	p_3
D	$0.33\bar{3}$	$0.33\bar{3}$	$0.33\bar{3}$
A	0.25	0.45	0.30
B	0.30	.60	.10

Lone Divider – $n > 3$ people

1. Divider D divides cake into n pieces (of equal value to D).
2. the $n - 1$ Choosers bid on pieces
3. If there is a way to allocate $n - 1$ pieces to the Choosers without conflict, do so; give the last piece to the Divider.
4. Otherwise: By some very deep mathematics ("*Marriage lemma*") it turns out that there is ne subset of the players of size $k < n$ contesting fewer than k pieces, and the rest of the players can be 'matched' to all but one of the remaining pieces without conflict. That means there is at least one uncontested piece (might be hard to determine!). Give it to the Divider, smush the rest of the pieces back together, let the $n - 1$ choosers *use this same procedure* to try to divide the remainder.

Selfridge-Conway

This is a modification of Lone Divider that makes the choice Envy-Free

1. Divider D divides cake into three pieces (of equal value to D).
2. If Chooser A says 2 or more pieces are both acceptable *and equal*, pieces are chosen by players in the order B, A, D; game over.
3. Otherwise: Chooser A trims the largest piece to equal the next largest, sets trimmed part aside. Player B then chooses a piece, Player A chooses a piece (must take the trimmed piece if B did not), then D gets the remaining piece.
4. If there was trimming, it gets divided as follows: whichever chooser got the untrimmed piece cuts into 3 pieces, the *other* chooser gets first choice, then D, then the cutting chooser.

Example: I will post a worked example.

Last Diminisher

Another generalization of cut/choose to 3 or more people:

1. Assume the N people are ordered (randomly?), A, B, C, D,...
2. A cuts a piece s/he considers fair. Passes it on to B.
3. If B thinks it is $\leq$ fair then s/he passes it on, otherwise trims it down to "fair" and passes it on. (Warning: text says $<$)
4. If C thinks it is $\leq$ fair then s/he passes it on, otherwise trims it down to "fair" and passes it on.
5. This continues to the last player.
6. All trimmings go back to the original assets. (For example, if dividing a pizza, they get reattached...?)
7. The piece goes to the last player who thought it was fair (either because they trimmed it or because it arrived fair to him/her).
8. That player leaves and the procedure is now repeated with the remaining $N-1$ players (and assets that each thinks is worth $\geq \frac{N-1}{N}$)

Examples See online readings

Note: Convince yourself that this is not an Envy-Free procedure

Lone Chooser AKA *Fink Protocol*

1. Assume the N people are ordered (randomly?), A, B, C, D,...
2. A and B divide the assets in 2 by cut/choose. Now they each think they have at least 50%
3. A and B each divide their piece into 3 pieces they consider fair. Player C chooses one piece from each group. Each of A, B, and C now think they have $2/6 = 1/3$ of the assets.
4. A, B, and C each divide their assets into 4 pieces they consider fair. Player D chooses 1 pieces from each group. Each of A, B, C, and D now have $3/12 = 1/4$ of the assets.
5. And so on, until you've run out of people.

Note: Convince yourself that this is not an Envy-Free procedure. (A might think that B did not do the triple cut fairly at stage 2.)

Note: Lots of cuts, even by the third person. Very simple, but not practical for many things.

The Method of Markers (Lucas, 1975)

Useful for dividing items that can be "lined up"

(Team selection, jewelry, Halloween candy, ...)

Requires

1. Individual items should be close in value, and
2. There should be many more items than there are players

Procedure (for N players):

1. Assets are lined up, say left to right.
2. Each player (secretly) sets N-1 markers, dividing the assets into N blocks of (approximately) equal value
3. The player whose first marker comes first gets the first block. (If there is a tie, pick one of the tied players at random.) Remove all his remaining markers.
4. The player whose second block ends first gets that second block. Items to the left of that block go into a 'leftover' pile, and his remaining markers are removed.
5. The player whose third block ends first gets that second block. Items to the left of that block go into a 'leftover' pile, and his remaining markers are removed.
6. etc
7. At the end, the leftovers are divided up by some method, even randomly will do

Example: Class

Sealed Bids: (Steinhaus and Knaster 1948?)

Great for dividing discontinuous assets, such as inheritance

Assumes

- players can assign cash value to assets to be divided,
- players are willing to take cash instead of assets
- players have sufficient cash on hand (to fulfill their bids)

Procedure: 1. Each player 'bids' secretly on all items (thereby valuing all items)

2. Their bids/values are formed into a table. Start by assuming that the player with the highest bid wins the appropriate item

3. Each player's bids are added up (=his/her value of the estate), then divided by N to form their estimate of their *Fair Market Share*; record this in the table

4. Each player might now owe some money to the estate, determined by their won items minus their fair market share (or might be owed money by the estate). Settle this difference.

5. The estate now has a positive (or zero) balance; divide that evenly between the players.

Works because every player is simultaneously a buyer and a seller, so will set honest prices

Example: Class