Sol'n HW4

9. Give a precise definition of what it means for the variable x to occur free as the ith symbol in the wff α. (If x is the ith symbol of α but does not occur free there, then it is said to occur *bound* there.)

(Sorry for this annoying problem!)

Suppose we're already defined $len(\varphi)$ = length of φ, φ a term or WFF.

Now we define $\Theta(i,\varphi) = \begin{cases} T & \text{if } x \text{ is free as } i^{th} \text{ symbol in } \varphi \\ F & \text{o/w} \end{cases}$

Where φ is a WFF or term and $i \in \mathbb{N}$:

$\Theta(i, c) = F$, c a constant symbol

$\Theta(i, v) = \begin{cases} T, & v = x \\ F, & v \neq x \end{cases}$, v a variable symbol

$\Theta(i, f(\tau_1, \ldots, \tau_n)) = T$ provided $i = 2 + (j-1) + \sum_{m=1}^{j-1} len(\tau_m) + k$, where $1 \le j \le n$ and $\Theta(k, \tau_j) = T$

for "f(" # of commas to j^{th} term

$\Theta(i, (\tau_1 = \tau_2)) = T$ provided $\Theta(i-1, \tau_1)$ or $\Theta(i-1-len(\tau_1)-1, \tau_2) = T$

$\Theta(i, P(\tau_1, \ldots, \tau_n)) = T$ provided $i = 2 + (j-1) + \sum_{m=1}^{j-1} len(\tau_m) + k$, where $1 \le j \le n$ and $\Theta(k, \tau_j) = T$

for "P(" # of commas to j^{th} term

$\Theta(i, (\neg\varphi)) = T$ provided $\Theta(i-2, \varphi) = T$

$\Theta(i, (\varphi \to \psi)) = T$ provided $\Theta(i-1, \varphi) = T$ or $\Theta(i-1-len(\varphi)-1, \psi = 1$

$\Theta(i, \forall v \varphi) = F$ if $v = x$, $= T$ if $v \neq x$ and $\Theta(i-2, \varphi)$

OK, how to define $len(\varphi)$ = length of φ, φ a term or WFF:

$len(c) = len(v) = 1$, c a constant symbol, v a variable symbol

$len(f(\tau_1, \ldots, \tau_n)) = 2 + n + \sum_{i=1}^{n} len(\tau_i)$

$len((\tau_1 = \tau_2)) = 3 + len(\tau_1) + len(\tau_2)$

$len(P(\tau_1, \ldots, \tau_n)) = 2 + n + \sum_{i=1}^{n} len(\tau_i)$

$len((\neg\varphi)) = 3 + len(\varphi)$

$len((\varphi \to \psi)) = 3 + len(\varphi) + len(\psi)$

$len(\forall x \varphi) = 2 + len(\varphi)$

... and that's it! Whew!

2. Let $\mathcal{L} = \{P\}$ be the language with precisely one unary predicate symbol P. Suppose $A = \{1, 2, 3, \cdots, N\}$ for some natural number N. How many different $\mathcal{L}$-structures are there with domain A?

Every distinct $P_{\mathfrak{A}} \subseteq A$ gives a different $\mathcal{L}$-structure. Thus the # of different $\mathcal{L}$-structures with universe A is $\mathrm{Card}(\mathcal{P}(A)) = 2^n$

4. Let $\mathcal{L} = \{E\}$ be the language with precisely one binary predicate symbol. Consider the $\mathcal{L}$-structures $\mathfrak{A} = \{\mathbb{N}, <_{\mathbb{N}}\}$, $\mathfrak{B} = \{\mathbb{Q}, <_{\mathbb{Q}}\}$, and $\mathfrak{C} = \{\mathbb{Z}, <_{\mathbb{Z}}\}$. For each pair of these models, find a sentence true in one but not the other.

$\exists x \forall y\, E(x,y)$ is true in $\mathfrak{A}$, not in $\mathfrak{B}, \mathfrak{C}$.

$\forall x \forall y \left(E(x,y) \to \left(\exists z\, E(x,z) \wedge E(z,y)\right)\right)$ is true in $\mathfrak{B}$, not in $\mathfrak{C}$

6. A model $\mathfrak{A}$ for the language $\mathfrak{L}$ (from problem 5) is defined by

$$\mathfrak{A} = \langle \mathbb{N}, 0_{\mathbb{N}}, f_{\mathfrak{A}}, <_{\mathbb{N}} \rangle,$$

where $f_{\mathfrak{A}}(x) = x + 1$ for all $x \in \mathbb{N}$. For each of the formulas in 5(a)-(e) above, if the formula is a sentence state whether or not it holds in $\mathfrak{A}$.

The sentences are (a), (d), (e):

(a) $\forall x \exists y P(y, x)$ In $\mathfrak{A}$ this asserts that every x is bigger than some y, but 0 is not bigger than any y, so false in $\mathfrak{A}$

(d) $\exists x(x = f(x))$ In $\mathfrak{A}$ this says some x is fixed by f, but x never $= x+1$ in $\mathbb{N}$, so false in $\mathfrak{A}$

(e) $\forall x P(x, f(x))$ Says for all x, $x < x+1$, which is true in $\mathfrak{A}$

7. A model $\mathfrak{B}$ for the language $\mathfrak{L}$ (from problem 5) is defined by

$$\mathfrak{B} = \langle \mathbb{Q}, 0_{\mathbb{Q}}, f_{\mathfrak{B}}, <_{\mathbb{Q}} \rangle,$$

where $f_{\mathfrak{B}}(x) = x/2$ for all $x \in \mathbb{Q}$. For each of the formulas in 5(a)-(e) above, if the formula is a sentence state whether or not it holds in $\mathfrak{B}$.

(a) $\forall x \exists y P(y, x)$ Every x in $\mathbb{Q}$ $\underline{\underline{\text{is}}}$ bigger than some y, so true in $\mathfrak{A}$

(d) $\exists x(x = f(x))$ Since $0 = 0/2$, this is true in $\mathfrak{A}$

(e) $\forall x P(x, f(x))$ If $x > 0$, $x \not< x/2$, so false in $\mathfrak{A}$.