Math 671 - Assignment 11 - Due Dec. 6

Turn in problems 1, 2, 5, and 6.

1. Prove that the characteristic function for any distribution is uniformly continuous.

2. Let $X_1, X_2, \ldots$ be iid random variables with the $\Gamma(a, b)$ distribution. Show (without appealing to a law of large numbers) that $\overline{X}_n \Rightarrow \gamma$ for some constant γ. Extra points for doing this two radically different ways.

3. Let $Y_n = Z_1^2 + \cdots + Z_n^2$ where the random variables Z_i are independent standard normals, $Z_i \sim \mathcal{N}(0, 1)$. Show that Y_n is distributed as a "χ^2 distribution on n degrees of freedom", that is, it is a $\Gamma(n/2, 2)$ random variable. Conclude that $\frac{Y_n - n}{\sqrt{2n}} \Rightarrow \mathcal{N}(0, 1)$.

4. (Might be hard!) Let X have mean μ and variance σ^2.

   (a) Prove: $P(X - \mu \geq \alpha) \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}$ (This is Cantelli's Inequality)

   (b) Conclude: $P(|X - \mu| \geq \alpha) \leq \frac{2\sigma^2}{\sigma^2 + \alpha^2}$. When is this better than Chebyshev's inequality?

   (c) Show that Cantelli's inequality is sharp. (Hint: Let $X \sim \textit{Bernoulli}(p)$)

5. Suppose X is a (discrete) random variable with $P(X \in \mathbb{Z}) = 1$. Show that for all $n \in \mathbb{Z}$,
$$P(X = n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-itn} \phi_X(t) dt$$

6. Suppose X is a (discrete) random variable with characteristic function $\phi(t)$. Show that the following are equivalent:

   (a) $\phi(2\pi) = 1$

   (b) $P(X \in \mathbb{Z}) = 1$

   (c) ϕ is periodic with period 2π

   (Note we did part of this in class.)