Lecture notes: Measurable Functions/Random Variables

Math 671 Fall 2013 (Prof. Ross, UH Dept. of Math)

Definition 0.1. *Let* $(\Omega, \mathcal{A})$ *and* $(\Lambda, \mathcal{B})$ *be* $\sigma -$ *algebras.* $f : \Omega \to \Lambda$ *is* measurable *(or* $\mathcal{A}$*-*$\mathcal{B}$*measurable for definiteness) if* $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$

- Usually Λ will be a topological space (usually $\mathbb{R}^n$) and $\mathcal{B}$ the Borel $\sigma -$ algebra.
- In fact, Λ will usually be $\mathbb{R}$, and if we say "f is a measurable function" without mentioning other context then assume f is $\mathbb{R}$-valued.
- If $\Omega = \mathbb{R}$ then we will usually assume $\mathcal{A}$ is the Borel $\sigma -$algebra, and a measurable f on Ω is called a *Borel* function.
- Some useful facts:
  (1) For a function $f : (\Omega, \mathcal{A}) \to \mathbb{R}$, measurability is equivalent to any of the following conditions:
    (a) $f^{-1}(-\infty, r) \in \mathcal{A}$ for all $r \in \mathbb{R}$ (or even $r \in \mathbb{Q}$)
    (b) $f^{-1}(-\infty, r] \in \mathcal{A}$ for all $r \in \mathbb{R}$ (or even $r \in \mathbb{Q}$)
    (c) $f^{-1}(r, \infty) \in \mathcal{A}$ for all $r \in \mathbb{R}$ (or even $r \in \mathbb{Q}$)
    (d) $f^{-1}[r, \infty) \in \mathcal{A}$ for all $r \in \mathbb{R}$ (or even $r \in \mathbb{Q}$)
  (2) A continuous function from one topological space to another is (Borel-Borel)-measurable.
  (3) (compare to Lemma 3.4 in text) If $f : (\Omega_2, \mathcal{A}_2) \to (\Omega_3, \mathcal{A}_3)$, $g : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ are both measurable, then $f \circ g$ is measurable.

(4) If $f, g : (\Omega, \mathcal{A})$ are measurable then so are $f+g, cf, f-g$, and fg, where c is a constant.

(5) If $A \in \mathcal{A}$ then I_A is measurable from $(\Omega, \mathcal{A}) \to \mathbb{R}$. In fact, any *simple function* of the form $\sum_{i=1}^{n} a_i I_{A_i}$ is measurable.

- The proofs for (some version of) all of these is in the text, and you should make sure you can prove them.
- It will often be convenient to assume that our functions are $\overline{\mathbb{R}}$-valued; this doesn't affect anything above.
- Remarks on text's $m\sum, b\sum, (m\sum)^+$
- A continuous function from one topological space to another is (Borel-Borel)-measurable.
- Supose $f_n : (\Omega, \mathcal{A}) \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable functions. Then

$$\inf_n f_n, \qquad \sup_n f_n, \qquad \liminf_n f_n, \qquad \limsup_n f_n$$

and $I_{\{\lim_n f_n \text{ exists}\}}$ are all measurable.

- A (Real-valued) *random variable* is a measurable function X from a probability space to $\mathbb{R}$.
- (Notation) If X is a random variable, we often write

$$[X = r] \text{ for } \{\omega \in \Omega : X(\omega) = r\}$$

$$[X < r] \text{ for } \{\omega \in \Omega : X(\omega) < r\}$$

$$[X \in A] \text{ for } \{\omega \in \Omega : X(\omega) \in A\} = X^{-1}(A)$$