LECTURE NOTES: TRANSFINITE INDUCTION FOR MEASURE THEORY (CORRECTED AUG 30, 2013)

MATH 671 FALL 2013 (PROF: DAVID ROSS, DEPARTMENT OF MATHEMATICS)

Start with a quick discussion of *cardinal* and *ordinal* numbers. Cardinals are a measure of size, ordinals of ordering. Ideally you should find an real introduction to this somewhere, either in a Math 454 text or in a mid-20th-century analysis text.

1. Cardinality

Definition 1.1. *A is* equinumerous *with B provided there is a bijection from A onto B.*

Other, equivalent notation/terminology for "A is equinumerous with B":

(1) A and B *have the same cardinality*
(2) $A \approx B$
(3) $\mathrm{card}(A) = \mathrm{card}(B)$

We seem to be referring to a concept called "cardinality" without really defining it. Technically, we should view all the above as simply complicated ways of expressing a relationship between the two sets A and B. Early treatments of set theory tried to define the cardinality of a set to be its equivalence class under the relation $\approx$, but since $\approx$ is defined on the class of all sets (which is not itself a set), this leads to technical problems related to the classical set-theoretic paradoxes. Nevertheless, it is not a terrible way to think of cardinalities.

Some important cardinalities:

- $\aleph_0$ =cardinality of $\mathbb{N}$. Sets with this cardinality (ie, which can be put into a 1-1 correspondence with $\mathbb{N}$) are called *countably infinite*.
- $2^{\aleph_0} = \mathfrak{c}$ =cardinality of $\mathbb{R}$ (*the continuum*). Any math grad student should be able to prove that this is also the cardinality of $\mathcal{P}(\mathbb{N})$.
- $\aleph_1$ =the first uncountable cardinal. Implicit in this terminology is the assertion that there is a set A with the property that A is is uncountable (=infinite but not countably infinite) and that if B is any other uncountable set then there is a 1-1 function from A into B.

  The assertion that $\aleph_1 = 2^{\aleph_0}$ is Cantor's *continuum hypothesis*, and it is independent of the usual axioms of set theory. Few mathematicians believe it.

There is a whole beautiful theory of cardinal arithmetic, which we won't go into here; see any Math 454 text. One rather useful fact is that the union of finitely many infinite sets has the same cardinality as the largest of the sets; in fact the union of κ many sets whose cardinality is at most κ itself has cardinality at most κ. This is a hugely nontrivial result in general, but you should be able to prove the countable version (the union of countably many countable sets is countable).

Finally, an important and useful fact that has a surprisingly tricky proof:

Theorem 1.1. *(Cantor-Schroeder-Bernstein) If A has the same cardinality as a subset of B and vice versa, then* $\mathrm{card}(A) = \mathrm{card}(B)$*.*

2. Ordinals

Just as cardinality is meant to be a measure of a set's size, ordinality is meant to be a measure of the order structure of a set. When discussing a queue of people, for example, we might say that a person is 5th in line. This has cardinality implications - the set of people from the front to/including him has 5 people in it - but also an ordinality implication: 5 indicates his place in the line's order.

Intuitively, an ordinal is a well-ordered set which represents all well ordered sets with a given 'order type' (that is, it represents all well-ordered sets which are order-isomorphic to it). Ordinals are meant to generalize the natural numbers, and the order relation on ordinals will just be set membership $\in$.

Here is one way to define them:

Definition 2.1. *An* ordinal number *is a well-ordered set* $(\alpha, <)$ *satisfying:*

$$\forall a \in \alpha \; a = \{x \in \alpha \; : \; x < a\}$$

Note that this means that for any $x, a \in \alpha$, $x \in a \iff x < a$, that is, the well-order on α is really just set membership, and every ordinal is the set of its predecessors.

This definition is comprehensive enough that from it one can deduce many properties of ordinals; this is a major part of Math 454.

The ones we will need are:

Lemma 2.1. *If α is an ordinal, then:*

1. *Every $a \in \alpha$ is an ordinal*
2. *$\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal*
3. *If α and β are ordinals, then $\alpha \cap \beta$ is an ordinal.*
4. *(Trichotomy) If α and β are ordinals then either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$*
5. *If $(X, <)$ is a well-ordered set, then there is a unique order-isomorphism E from X onto α for some ordinal α*
6. *If two ordinals are order-isomorphic, then they are equal.*
7. *Every ordinal α is either a* successor ordinal*, that is, $\alpha = \beta + 1$ for some β, or it is a* limit ordinal*, $\alpha = \bigcup\{\beta : \beta < \alpha\}$.*

John von Neumann defined the natural number $0 = \emptyset$ to be the first ordinal, and recursively $n + 1 := \{0, 1, \ldots, n\}$ (which agrees with the definition of "+1" given in the above lemma). In other words, the class of all ordinals has $\mathbb{N}$ as its initial segment.

We can define things by recursion and prove things by induction on any ordinal exactly the same way we do on the natural numbers. For example, the order-isomorphism E from a well-ordered set $(X, <)$ to an ordinal α has a cute recursive definition, namely:

$$E(x) := \{E(y) \; : \; y < x\}$$

This is pretty terse; you should try it on an imaginary well-ordered set to see what it means. Note, for example, that if x_0 is the smallest element of X, then $E(x_0) = \emptyset$.

To get a sense of why transfinite induction works, suppose $\phi(x)$ is some statement we're trying to prove holds for all ordinals x (or maybe just all x below some β), and you prove

(1) $\phi(0)$ is true, and
(2) Whenever $\phi(\gamma)$ is true for all $\gamma < \alpha$ then $\phi(\alpha)$ is true.

(Note that the first condition is actually implied by the second, so redundant.) The claim is that $\phi(\alpha)$ must be true for all α. Otherwise there is a least exception α, then the antecedent of the second condition is true, hence $\phi(\alpha)$ is true, a contradiction.

Alternately, we prove

(1) $\phi(0)$ is true, and
(2) Whenever $\phi(\alpha)$ is true then $\phi(\alpha+1)$ is true, and
(3) Whenever $\phi(\alpha)$ is true for all $\alpha < \lambda$ then $\phi(\lambda)$ is true, for limit ordinals λ.

These two approaches correspond to what we call *weak* and *strong* induction in a course like Math 321.

Assuming there exist uncountable ordinals (there do!), there must be a least one, which we call ω_1. An interesting property of ω_1 is that if $A \subseteq \omega_1$ is a countable set of ordinals, that is, $A = \{\alpha_n\}_{n\in\mathbb{N}}$ with each $\alpha_n < \omega_1$, then for some β we have $\alpha_n < \beta < \omega_1$ for all n. (This is a consequence of the fact that the union of a countable set of countable sets is countable, another general fact any math grad student should be able to prove.)

3. Generating a $\sigma-$algebra

We are now ready to generate a $\sigma - $ algebra.

Lemma 3.1. *Suppose* $\{\emptyset, \Omega\} \subseteq \mathcal{C} \subseteq \mathcal{P}(\Omega)$. *Define:*

$$\mathcal{A}_0 := \mathcal{C}; \tag{1}$$

$$\mathcal{A}_{\alpha+1} := \mathcal{A}_\alpha \cup \{A^{\complement} : A \in \mathcal{A}_\alpha\} \cup \{\bigcup_n A_n : \{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_\alpha\} \tag{2}$$

$$\mathcal{A}_\lambda := \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha, \qquad \lambda \textit{ a limit ordinal} \tag{3}$$

Then $\mathcal{A}_{\omega_1} = \sigma(\mathcal{C})$

Proof. $\subseteq$) Let $\mathcal{A}$ be any $\sigma -$ algebra containing $\mathcal{C}$. Of course, $\mathcal{A}_0 \subseteq \mathcal{A}$. Suppose $\mathcal{A}_\alpha \subseteq \mathcal{A}$. If $A \in \mathcal{A}_\alpha \subseteq \mathcal{A}$ then $A^{\complement} \in \mathcal{A}$. If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_\alpha \subseteq \mathcal{A}$ then (since $\mathcal{A}$ is a $\sigma -$ algebra) $\bigcup_n A_n \in \mathcal{A}$ It follows that $\mathcal{A}_{\alpha+1} \subseteq \mathcal{A}$. Finally, suppose $\mathcal{A}_\alpha \subseteq \mathcal{A}$ for all $\alpha < \lambda$. Then $\mathcal{A}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha \subseteq \mathcal{A}$. By induction, $\mathcal{A}_\alpha \subseteq \mathcal{A}$ for all ordinals α, including $\alpha = \omega_1$. Since $\mathcal{A}$ was arbitratry, $\mathcal{A}_{\omega_1} \subseteq \sigma(\mathcal{A})$

$\supseteq$) We need to show that $\mathcal{A}_{\omega_1}$ is a $\sigma -$ algebra. $\{\emptyset, \Omega\} \subseteq \mathcal{C} \subseteq \mathcal{A}_{\omega_1}$. If $A \in \mathcal{A}_{\omega_1}$ then for some $\alpha < \omega_1, A \in \mathcal{A}_\alpha$, so $A^{\complement} \in \mathcal{A}_{\alpha+1} \subseteq \mathcal{A}_{\omega_1}$. If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_{\omega_1}$ then each $A_n \in \mathcal{A}_{\alpha_n}$ for some n, and (see discussion above) there is some $\beta < \omega_1$ with $\alpha_n < \beta$ for all n, and so $\bigcup_n A_n \in \mathcal{A}_\beta \subseteq \mathcal{A}_{\omega_1}$, proving the claim. □

4. Borel Sets

The Borel subsets of $\mathbb{R}$ is the $\sigma -$ algebra $\mathcal{B}_{\mathbb{R}}$ generated by the open subsets of $\mathbb{R}$. Since any open subset of $\mathbb{R}$ is the union of open intervals with rational endpoints (and therefore a *countable* union of such intervals), $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{C})$ where $\mathcal{C}$ is the countable collection of such open intervals.

Define $\mathcal{A}_{\omega_1}$ as above. We prove by induction that for all $\alpha \leq \omega_1, \mathrm{card}(\mathcal{A}_\alpha) \leq 2^{\aleph_0}$. $\mathrm{card}(\mathcal{A}_0) = \aleph_0$. Assume $\mathrm{card}(\mathcal{A}_\alpha) \leq 2^{\aleph_0}$. Every element of $\mathcal{A}_{\alpha+1}$ is either an element of $\mathcal{A}_\alpha$, the complement of an element of $\mathcal{A}_\alpha$, or a set determined by a function from $\mathbb{N}$ into $\mathcal{A}_\alpha$. The cardinality of such functions is also $2^{\aleph_0}$.[1] Thus $\mathcal{A}_{\alpha+1}$ is the union of three sets each of cardinality at most $2^{\aleph_0}$, so it has cardinality at most $2^{\aleph_0}$. Finally, $\mathcal{A}_{\omega_1}$ is the union of at most $2^{\aleph_0}$ many sets each of cardinality at most $2^{\aleph_0}$, so it too has cardinality at most $2^{\aleph_0}$. In fact, every singleton set of the form $\{r\}$ for r a real number is Borel (why?), so $\mathcal{B}_{\mathbb{R}} = \mathcal{A}_{\omega_1}$ has cardinality of the continuum.

We'll see another way to prove this later in the semester.

You won't find this fact stated, let alone proved, in many intro analysis or intro probability texts, and that is a pity.

By the way, Cantor showed that if X is any set, then $\mathrm{card}(X)$ is strictly less than $\mathrm{card}(\mathcal{P}(X)$. Since there are only as many Borel sets as there are real numbers, these cardinality considerations show that there are subsets of $\mathbb{R}$ which are not Borel.

[1] You can show this by finding a 1-1 function from (the set of functions from $\mathbb{N}$ to $\mathbb{R}$) into $\mathbb{R}$, then applying the Cantor-Schroeder-Bernstein Theorem.