Errata to "Higher Index Theory"

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Please do send more (rwillett.hawaii.edu). Errata to the errata are also welcome.

1. Page 55. In Exercise 1.9.9, u should be defined to be "$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$", not "$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$".

2. Page 67. In the last two lines of Definition 2.2.1, "if they there is" should be "if there is".

3. Page 67. Page 67, the second displayed line in the formula has a sign error, and should read

$$" \ u - 1 = 2e_0e_1 - e_0 - e_1 = (1 - 2e_0)(1 - e_1) \ ".$$

4. Page 68. In the sentence below line (2.3), "v_1" should be "$v_{\pi/2}$".

5. Page 70. In the fifth line of Construction 2.2.8, it should say "any quasi-idempotent a in $M_\infty(A)$", not "any quasi-idempotent e in $M_\infty(A)$".

6. Page 71. In Remark 2.2.10, the reference to "line (2.2.1)" should be to "Proposition 2.2.9".

7. Page 87. In the sentence above the fourth displayed line, "lifting Q" should be "lifting u".

8. Page 94. In the fourth displayed line from bottom, in the formula

$$\text{“ } \beta_A(e) : z \mapsto zp + (1_{M_n(A)} - e) \text{ ”},$$

the "p" should be an "e".

9. Page 97. In the last line, in the formula "$(\phi_j)_*([p_i] - [q_i]) = [p] - [q]$", the "$i$" subscripts should be "$j$".

10. Page 99. In the statement of Proposition 2.7.5, "$\alpha(a)vv^* = \alpha(a)$" should be "$\alpha(a)v^*v = \alpha(a)$".

11. Page 101. In the last displayed line, the definition of v as "$v := \sum_{n=2}^{\infty} v_n$" is nonsense[1]. What we meant to do was define $\alpha := \sum_{n=2}^{\infty} \mathrm{ad}_{v_n}$ (convergence in the pointwise-strong operator topology). Then replace "ad_v" everywhere it appears in the proof by "α".

12. Page 104. In the fourth displayed line

$$\text{“ } [v_i p_i v_i^*] - [v_i q v_i^*] = [p_i] - [q_i] \text{ ”},$$

the "q" should be "q_i".

13. Page 105. In Definition 2.7.14, wherever "π_A" occurs, it should be "π^A", and similarly for "π_B".

14. Page 100. In the last two lines of the proof of Proposition 2.7.5, "$\phi : K_*(M_2(A)) \to K_*(A)$" should be "$\phi : K_*(M_2(C)) \to K_*(C)$".

15. Page 119. In Construction 2.9.11, the element "v" is not defined. It is the unitary multiplier $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ implementing the grading on $M_2(A \otimes \mathcal{K})$.

[1] This one is particularly embarrassing.

16. Page 172. In Example 4.2.4 in the fifth line, the support of V_ϵ should be $\{((s,t),x) \in Y \times X \mid x = s \text{ and } t \leqslant \epsilon\}$.

17. Page 175. Three lines from the bottom, "$f(B(x;s))$" should be "$f(B(x;1))$".

18. Page 199. In the displayed line in condition (ii), "$\rho(x,y)$" should be "$d(x,y)$".

19. Page 202. In Definition 5.1.10, there is a parenthesis ")" missing in "$d_Y(f(x_1), f(x_2))$" in the definition of $\omega_f(r)$.

20. Pages 203. Lemma 5.1.12 does not work if X is the empty set, which should be excluded. Similarly, the empty set should be excluded in the considerations of functoriality throughout this section.

21. Page 218. Two lines from bottom, "$\{z \in \mathbb{Z} \mid \mathrm{Re}(z) > 1/2\}$" should be "$\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1/2\}$".

22. Page 222. In Definition 6.2.1 part (i), "$t_K \geqslant 0$" should be "$t_K \geqslant 1$".

23. Page 224. Lemma 6.2.7 does not work if X is the empty set, which should be excluded. Similarly, the empty set should be excluded from all considerations of functoriality in this section and Section 6.3. This does not apply to Proposition 6.3.3, where the empty set is an example, and to Section 6.4, where it is important that the empty set is an allowed object.

24. Page 236. In Definition 6.4.3 part (i), "$t_K \geqslant 0$" should be "$t_K \geqslant 1$".

25. Page 238. In Definition 6.4.8, "$t_K \geqslant 0$" should be "$t_K \geqslant 1$" both places it appears.

26. Page 239. In the proof of Lemma 6.4.11, in the displayed formula

$$\text{“ } \chi_K \phi((T_t)) = \left(0 \oplus \bigoplus_{n=1}^{\infty} \chi_K T_t\right) \text{ ”},$$

the "ϕ" should be "β".

27. Pages 251-2. In Remark 6.5.2, "$t_K \geqslant 0$" should be "$t_K \geqslant 1$" in all three places it appears.

28. Page 261. In the second line of the proof of Proposition 6.6.2, "$t_K \geqslant 0$" should be "$t_K \geqslant 1$" .

29. Page 261. Just below line (6.9), it is claimed that "the same proof as Lemma 6.4.11 shows that $K_*(C^*_{L,0}(H_X)^G) = 0$". This is misleading as the necessary proof is a little more complicated. It can be fixed by redefining the map β using the formula

$$(T_t) \mapsto \left(0 \oplus \bigoplus_{n=1}^{\infty} T_{t+n} \right);$$

this ensures that the image of (T_t) under β is locally compact for all 'time' t. As the function $t \mapsto T_t$ is uniformly continuous, we still see that $\alpha + \beta$ and β induce the same map on K-theory, via conjugation by an isometry, and a homotopy (compare for example [2, Proof of Proposition 3.5 on page 662]); the proof then goes through as in 6.4.11.

30. Pages 261-2. There is a mistake in the proof of Proposition 6.6.2. To explain the problem, we need some notation.

Let $\mathcal{A}$ be the $*$-subalgebra of $L^*(H_X)^G$ defined in the same way as $\mathbb{C}_L[H_X]^G$ (see Definition 6.6.1), but where the uniform continuity condition in t is dropped, and replaced with: "the function $t \mapsto T_t$ is continuous, and for each compact $K \subseteq X$, the functions $t \mapsto \chi_K T_t$ and $t \mapsto T_t \chi_K$ are uniformly continuous". Let A be the C^*-completion of $\mathcal{A}$ in the norm it inherits from $L^*(H_X)^G$, and define $A_0 := L^*_0(H_X)^G \cap A$. Then instead of the isomorphism in line (6.10) on page 261, the proof of Proposition 6.6.2 actually shows that the natural map

$$\frac{A}{A_0} \to \frac{L^*(H_X)^G}{L^*_0(H_X)^G}$$

is an isomorphism.

To complete the proof of Proposition 6.6.2, it remains to show that the canonical inclusion $C^*_L(H_X)^G \to A$ induces an isomorphism on K-theory. This can be established using the same argument as that given for Theorem 3.4 in [3].

Unfortunately, this makes the argument quite technical, and we do not know a simpler way to fix this issue: an alternative would be to show that the functors defined using $C^*_L(H_X)^G$ and A are equivariant homology theories (this is technical) in an appropriate sense and that they agree for spaces induced from actions of finite subgroups of G; this is not obviously any easier, however.

31. Page 262. In the top line, "VTV^*" should be "V^*TV".

32. Page 271. In Exercise 6.8.11, "$H_X \oplus X$" should be "$H_X \oplus H$".

33. Page 271. In Exercise 6.8.15, $C^*_{L,og}(H_X)$ is defined to be the $*$-algebra of all bounded uniformly continuous functions from $[1, \infty)$ to the Roe algebra $C^*(H_X)$ such that $\mathrm{prop}(T_t) \to 0$ as $t \to \infty$. It should be defined to be the *completion* of this $*$-algebra. In case it helps to be more explicit: the metric d in this exercise is the given proper metric on X.

34. Page 283. In Definition 7.2.1, the map f is a bijection *onto its image*, not strictly a bijection as stated.

35. Page 284. In Definition 7.2.2, "such that each σ_i contains (x_{i-1}, x_i)" should say "such that each σ_i contains $\{x_{i-1}, x_i\}$".

36. Page 287. The last sentence of Definition 7.2.8 should say "The *Rips complex* $P_r(Z)$ of Z is defined to be the space $S_r(Z)$ equipped with the metric d_{P_r}."

37. Page 289. In the displayed chain of inequalities about three quarters of the way down, the second line should say

$$\text{“ } \geqslant \min\{1, a\} \left(\sum_{i=0}^{n} d_{\sigma(F)}(x_i, y_i) + \sum_{i=0}^{n-1} d_{\sigma(F)}(y_i, x_{i+1}) \right) \text{ ”}$$

(the sums and the set braces are currently missing).

38. Page 291. The last displayed line says

$$\text{“ } h : Y \times [0,1] \to (1-t)f_0(y) + tf_1(y) \text{ ”}$$

and should say

$$\text{“ } h : Y \times [0,1] \to P_s(Z), \quad (y,t) \mapsto (1-t)f_0(y) + tf_1(y) \text{ ”}.$$

39. Page 298. In definition 7.4.1, the last sentence should finish “and contractible universal cover EG”.

40. Page 300. In the proof of Theorem 7.4.4, the reference about four fifths of the way down to Theorem 7.4.4 should be to Theorem 7.4.3.

41. Page 302. Just above Definition 7.4.6, we should have “$X = P_r(Z)$”, not “$X = P_r(X)$”.

42. Page 302. The last word of Definition 7.4.6 should be “equivalence”, not “equivariant”.

43. Page 302. In the proof of Theorem 7.4.7, the displayed line says

$$\begin{aligned}\text{“ } Z = &\{\mu \in \ell^1(G) \mid \mu \geqslant 0 \text{ and } \|\mu\| \leqslant 1\} \setminus \\ &\{\mu \in \ell^1(G) \mid \mu \geqslant 0 \text{ and } 1/2 < \|\mu\|\} \text{ ”}\end{aligned}$$

and should say

$$\begin{aligned}\text{“ } Z = &\{\mu \in \ell^1(G) \mid \mu \geqslant 0 \text{ and } \|\mu\| \leqslant 1\} \setminus \\ &\{\mu \in \ell^1(G) \mid \mu \geqslant 0 \text{ and } \|\mu\| \leqslant 1/2\} \text{ ”}.\end{aligned}$$

As a result the explanation for local compactness is wrong: it should say that Z is locally compact as it is the complement of a closed set in a compact set.

44. Page 303. In the statement of Theorem 7.4.8, the left hand side says “$\lim_{Y \subseteq \underline{E}G} K^G_*(\underline{E}G)$” and should say “$\lim_{Y \subseteq \underline{E}G} K^G_*(Y)$”.

45. Page 315. In the bottom line "1908s" should be "1980s".

46. Page 440, and 454-455. The statement of Proposition 12.1.10 parts (ix) and (x) on page 440 can (and should) be strengthened as follows; the strengthened version is needed for the proof of Lemma 12.3.9 on page 454-455. Point (ix) should be strengthened to: "For all $\epsilon_1 > 0$ there exists $R_1 > 0$ such that for all $R \geqslant R_1$, all $s \in [1, \infty)$ and all $x \in E$ we have that

$$\|(\Psi(B_{s,x})^2 - 1)(1 - \chi_{x,R})\| < \epsilon_1.$$"

Point (x) should be strengthened to: "For all $\epsilon_2 > 0$ and all $r > 0$ there exists $R_2 > 0$ such that for all $R \geqslant R_2$, all $s \in [2d, \infty)$, and all $x, y \in E$ with $|x - y| \leqslant r$ we have that

$$\|(\Psi(B_{s,x}) - \Psi(B_{s,y}))(1 - \chi_{x,R})\| < \epsilon_2.$$"

The (delicate) proof of these statements can be found in Appendix A of [1], due to Bao, Chen, and Zhang.

47. Pages 522-523. Everywhere "Y" appears in the proof of Lemma A.3.14, it should be "X".

48. Page 523. In the third line of the displayed chain of inequalities near the top of the page, "$d_Y(y, g^{-1}h)$" should be "$d_Y(y, g^{-1}hy)$" (and "Y" should be "X" as in erratum 47).

References

[1] H. Bao, X. Chen, and J. Zhang. Strongly quasi-local algebras and their K-theories. *J. Noncommut. Geom.*, 17:241–285, 2023. 7

[2] Y. Qiao and J. Roe. On the localization algebra of Guoliang Yu. *Forum Math.*, 22(4):657–665, 2010. 4

[3] R. Willett and G. Yu. Controlled KK-theory I: a Milnor exact sequence. arXiv:2011.10906, 2020. 5