Decompositions and K-theory

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Abstract

We introduce a notion of decomposability for a C^*-algebra, and use it as a tool to study K-theory groups. The decomposability notion is motivated by the theory of nuclear dimension as introduced by Winter and Zacharias, and by the theory of dynamical complexity introduced by Guentner, Yu, and the author. A major inspiration for our K-theoretic results comes from recent work of Oyono-Oyono and Yu in the setting of controlled K-theory of filtered C*-algebras; we do not, however, use that language in this paper.

We give two main applications. The first is a vanishing result for K-theory that is relevant to the Baum-Connes conjecture. The second is a permanence result for the Künneth formula in C^*-algebra K-theory: roughly, this says that if A can be decomposed into a pair of subalgebras (C, D) such that C, D, and $C \cap D$ all satisfy the Künneth formula, then A itself satisfies the Künneth formula.

Contents

1 Introduction

A general Mayer-Vietoris sequence

Let C and D be C^*-subalgebras of a C^*-algebra A, and consider the following diagram

$$K_1(C \cap D) \xrightarrow{\iota} K_1(C) \oplus K_1(D) \xrightarrow{\sigma} K_1(A) \overset{\partial}{\dashrightarrow} K_0(C \cap D) \xrightarrow{\iota} K_0(C) \oplus K_0(D)$$

of K-theory groups, where the solid arrows labeled ι and σ are defined respectively by

$$\iota : \alpha \mapsto (\alpha, -\alpha) \quad \text{and} \quad \sigma : (\beta, \gamma) \mapsto \beta + \gamma,$$

and the dashed arrow labeled ∂ may or may not exist. For example, if C and D are ideals in A such that $A = C + D$, then one can canonically fill in the dashed arrow so that the sequence above becomes part of the usual six-term exact Mayer-Vietoris sequence in C^*-algebra K-theory, i.e. the C^*-algebraic analogue of the Mayer-Vietoris sequence associated to a cover by two open sets in classical algebraic topology.

The main technical tools developed in this paper are partial exactness results for this sequence that hold under more general 'local decomposability' assumptions. These work even for many simple C^*-algebras, where decompositions into ideals are not possible. Looking at the diagram above in more detail,

$$K_1(C \cap D) \xrightarrow{\iota} \underbrace{K_1(C) \oplus K_1(D)}_{(III)} \xrightarrow{\sigma} \underbrace{K_1(A)}_{(II)} \overset{\partial}{\dashrightarrow} \underbrace{K_0(C \cap D)}_{(I)} \xrightarrow{\iota} K_0(C) \oplus K_0(D) \tag{1}$$

we establish partial exactness results at each of the three places marked (I), (II), and (III), under progressively more stringent assumptions. Exactness at point (I) is the easiest to prove, and is automatic: if $\iota(\alpha) = 0$ for some $\alpha \in K_0(C \cap D)$, one can always canonically construct a class in $K_1(A)$ that is the 'reason' for its being zero in some sense.

For exactness in the positions marked (II) and (III) in line (1), we need more assumptions. Here are the technical definitions.

Definition 1.1. Let A be a C^*-algebra, and let $\mathcal{C}$ be a set of pairs (C, D) of C^*-subalgebras of A. Then A *decomposes* over $\mathcal{C}$ if for any $\delta > 0$ and any finite subset $\mathcal{F}$ of A there exists a positive contraction h in the multiplier algebra of A and a pair $(C, D) \in \mathcal{C}$ such that:

(i) $\|[h, a]\| < \delta$ for all $a \in \mathcal{F}$;

(ii) $d(ha, C) < \delta$ and $d((1-h)a, D) < \delta$ for all $a \in \mathcal{F}$.

In words: condition (i) says that h is almost central; and condition (ii) says that h almost multiplies A into C and $1-h$ almost multiplies A into D. The pair $\{h, 1-h\}$ should be thought of as a 'local partition of unity' on A, splitting it into two 'parts' C and D that are simpler than the original. We discuss examples below, but keep the discussion on an abstract level for now.

This decomposability notion seems to have interesting connections to the structure theory of C^*-algebras, but we will not discuss that in this paper other than briefly in Appendix A. To get applications to K-theory we seem to also need assumptions on the intersections $C \cap D$. Here is one such assumption.

Definition 1.2. Let A be a C^*-algebra, and let $\mathcal{C}$ be a set of pairs (C, D) of C^*-subalgebras of A. Then A is *excisively decomposable* over $\mathcal{C}$ if for any $\delta > 0$ and any finite subset $\mathcal{F}$ of A there exists a positive contraction h in the multiplier algebra of A and a pair $(C, D) \in \mathcal{C}$ satisfying the properties in Definition 1.1 above, and that in addition satisfies the following:

(iii) $d((1-h)ha, C \cap D) < \delta$ and $d((1-h)h^2a, C \cap D) < \delta$ for all $a \in \mathcal{F}$.

These conditions allow us to prove a version of exactness at position (II) in line (1): roughly this says that if A is excisively decomposable over $\mathcal{C}$, then for any class $[u]$ in $K_1(A)$ one can find a pair $(C, D) \in \mathcal{C}$ and build a class $\partial(u) \in K_0(C \cap D)$ such that if $\partial(u) = 0$, then $[u]$ is in the image of σ.

The first of our main results is as follows. It is substantially easier to prove than our results on the Künneth formula.

Theorem 1.3. *Say that A excisively decomposes over a class $\mathcal{C}$ of pairs such that for all (C, D) in $\mathcal{C}$, C, D, and $C \cap D$ all have trivial K-theory. Then A has trivial K-theory.*

This result is already quite powerful: for example, it allows one to reprove the main theorem on the Baum-Connes conjecture of Guentner, Yu, and the author from [16] without the need for the controlled K-theory methods used there.

In order to get our results on the Künneth formula, we need an exactness property at position (III) in line (1); unfortunately, this needs the stronger assumption on A defined below.

Definition 1.4. Let A be a C^*-algebra and $\mathcal{C}$ a set of pairs (C, D) of C^*-subalgebras of A. The set $\mathcal{C}$ is *strongly excisive* if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any C^*-algebra B, if $c \in C \otimes B$ and $d \in D \otimes B$ satisfy $\|c - d\| < \delta$, then there exists $x \in (C \cap D) \otimes B$ with $\|x - c\| < \epsilon$ and $\|x - d\| < \epsilon$.

The C^*-algebra is *strongly excisively decomposable over* $\mathcal{C}$ if it decomposes over $\mathcal{C}$, and if $\mathcal{C}$ is strongly excisive.

Annoyingly, strong excisiveness seems a strong assumption. It is satisfied for example if for any pair $(C, D) \in \mathcal{C}$, C or D is an ideal (or more generally, a hereditary subalgebra); however, this is too much to ask if one wants applications that go beyond well-understood cases. Nonetheless, we are able to show that it is satisfied in some interesting cases: see the discussion of examples below.

We now discuss some background on the Künneth formula before describing our main result in that direction.

The Künneth formula

The main application of the results in this paper is to the external product map

$$\times : K_*(A \otimes B) \to K_*(A) \otimes K_*(B)$$

in C^*-algebra K-theory. This product map can be seen as a special case of the Kasparov product, or can be defined in an elementary way as described for example in [19, Section 4.7]. A C^*-algebra A is said to satisfy the *Künneth* *formula* if for any C^*-algebra B with free abelian K-groups, the product map above is an isomorphism.

Study of the Künneth formula seems to have been initiated by Atiyah [1] in the commutative case, and in general by Schochet [30]. In particular, these authors showed (in the relevant contexts) that A satisfies the Künneth formula in the above sense if and only if for any B there is a canonical short exact sequence

$$0 \to \mathrm{Tor}(K_*(A), K_*(B)) \to K_*(A) \otimes K_*(B) \xrightarrow{\times} K_*(A \otimes B) \to 0.$$

This short exact sequence is a useful computational tool, so it is desirable to know for which C^*-algebras the Künneth formula holds.

Atiyah essentially showed that all commutative C^*-algebras satisfy the Künneth formula. It follows that any C^*-algebra[1] that is KK-equivalent to a commutative C^*-algebra satisfies the Künneth formula. The class of such C^*-algebras is exactly the class satisfying the UCT[2], whence the UCT implies the Künneth formula. Remarkably, Lin recently announced a proof that the UCT holds for all nuclear C^*-algebras; given this, the Künneth formula holds for all nuclear C^*-algebras as well. The connection to the UCT provides another motivation for studying the Künneth formula: the Künneth formula can be viewed as a weak form of the UCT, and one can see satisfying the Künneth formula as evidence for satisfying the UCT.

The class of C^*-algebras satisfying the Künneth formula is strictly larger than the class satisfying the UCT, however. This follows from combining work of Chabert, Echterhoff, and Oyono-Oyono [7], of Lafforgue [21], and of Skandalis [31]. Indeed, it follows from the 'going down functor' machinery of [7] that if G is any group that satisfies the Baum-Connes conjecture with coefficients, then $C^*_r(G)$ satisfies the Künneth formula. Thanks to [21], this applies in particular when G is a hyperbolic group. On the other hand, results of [31] imply[3] that if G is an infinite, hyperbolic, property (T) group, then $C^*_r(G)$ does not satisfy the UCT.

Other results extending the range of validity of the Künneth formula include work of Bönicke and Dell'Aiera [4], which extends the results of [7]

[1] For this and the next paragraph, all C^*-algebras are separable.

[2] This is implicit in the original work of Rosenberg and Schochet [29], and was made explicit by Skandalis in [31, Proposition 5.3].

[3] The result as stated here is not exactly in [31], but it follows from Skandalis's ideas, plus more recent advances in geometric group theory: see [18, Theorem 6.2.1] for a discussion of the version stated.

from groups to groupoids; and work of Oyono-Oyono and Yu [25] which uses the methods of controlled K-theory developed by those authors [24]. The latter was the main inspiration for this paper, and we say more on this below.

Despite all these positive results, there are known to be C^*-algebras that do not satisfy the Künneth formula. The only way we know to produce such examples is based on the existence of non K-exact C^*-algebras: see the discussion in [7, Remark 4.3 (1)]. We do not know of an exact C^*-algebra that does not satisfy the Künneth formula.

Having got through the above discussion, here is our main theorem.

Theorem 1.5. *Let A be a C^*-algebra. Assume that A strongly excisively decomposes over a class $\mathcal{C}$ of pairs of C^*-subalgebras such that for each $(C, D) \in \mathcal{C}$, C, D, and $C \cap D$ satisfy the Künneth formula. Then A satisfies the Künneth formula.*

Examples

In Appendix A, we show that if A is a (separable) C^*-algebra of nuclear dimension one, then A decomposes over a class that consists of subhomogeneous C^*-algebras with one-dimensional spectrum. However, it is not at all clear if one can also get excisive decompositions, so we do not get applications to the Künneth formula out of this. The results of Appendix A are included as a sort of 'plausibility test' to show that our notion of decomposability is natural from the point of view of the structure theory of C^*-algebras, not as these results can be used directly to get K-theoretic applications.

In Appendix B, we show that decompositions of appropriate groupoids as introduced in [16, Appendix A] give rise to excisive decompositions of the associated reduced groupoid C^*-algebras. We use this to show that a large class of reduced groupoid C^*-algebras satisfy the Künneth formula. This gives a new proof that a large class of amenable groupoids – those with a strong form of finite dynamical complexity as in [16, Appendix A] – satisfy the Künneth formula. Similar results have been proved recently[4] by Oyono-Oyono using the methods of controlled K-theory.

[4]But not so recently – they came before our results! Still, we hope having two different approaches is interesting.

Inspiration and motivation

This paper was inspired by the work of Oyono-Oyono and Yu in [25] on the Künneth formula in controlled K-theory. It owes a great deal to their work, both conceptually and in some technical details: in particular, the key idea to use a sort of approximate Mayer-Vietoris sequence comes directly from [25], and the difficult proof of Proposition 5.6 is based closely on their work. A major difference of our work from [25] in that we do not use controlled K-theory, only usual K-theory groups. We do not use filtrations on our C^*-algebras, and we do not need (nor do we get results on) a 'controlled' version of the Künneth formula. It is not clear to us what the difference is between the range of validity of our results and those of [25]; we suspect that there is a large overlap.

We were motivated largely by the sort of decompositions that arise in the theory of nuclear dimension [35]. We hope the results are reasonably natural from the point of 'pure C^*-algebra theory' (as opposed to the sort of C^*-algebra theory that is based around examples associated to metric spaces or dynamical systems).

Outline of the paper

Section 2 introduces a general notion of 'boundary classes', and shows that such classes have good properties with respect to the sequence of maps in line (1): roughly, we prove a weak form of exactness at position (II) in line (1). The discussion in Section 2 leaves open how one might construct boundary classes: this is done in Section 3 using decompositions. It is now straightforward to prove Theorem 1.3, our first main goal of the paper.

Until the appendices, the remainder of the paper deals with our approach to the Künneth formula.

In Section 4, we prove exactness at position (I) in line (1); this is simpler than exactness at position (II), but is postponed until later as it is not needed for the proof of Theorem 1.3. We also collect together some other technical results on the boundary map that are needed later. Exactness at position (III) in line (1) is handled in Section 5: this is the most difficult of our exactness properties, both to prove and to use.

Section 6 recalls some basic facts we need about product maps and proves that these interact well with our boundary classes. Section 7 recalls well-known material about the inverse Bott map that we need for the technical proofs. Finally, in Sections 8 and 9, we are finally ready to attack Theorem 1.5, proving the surjectivity and injectivity halves respectively.

The paper is completed by two appendices dealing with examples. The first of these, Appendix A shows that C^*-algebras of nuclear dimension one are natural examples of (not necessarily excisively) decomposable C^*-algebras. Appendix B gives examples of (strongly) excisive decompositions from groupoid theory, and applications to the Künneth formula.

Notation and conventions

Throughout, if A is a C^*-algebra (or more generally, Banach algebra), then $\widetilde{A}$ denotes A itself if A is unital, and the unitization of A if it is not unital. If X is a subspace of a C^*-algebra A, then $\widetilde{X}$ is the subspace of $\widetilde{A}$ spanned by X and the unit. There is a minor ambiguity here about what happens when C is a C^*-subalgebra of A, and C has its own unit which is not the unit of A: we adopt the convention that in this case, $\widetilde{C}$ means the C^*-subalgebra of A generated by C and the unit of $\widetilde{A}$. This convention will always, and only, apply to C^*-subalgebras called C, D and $C \cap D$ (plus suspensions and matrix algebras of these), so we hope it causes no confusion.

We use 1_n and 0_n to denote the unit and zero element of $M_n(\widetilde{A})$ when it seems helpful to avoid ambiguity, but drop the subscripts whenever things seem more readable without. We use the usual 'top-left corner' identification of $M_n(A)$ with $M_m(A)$ for $n \leqslant m$, usually without comment. We also use the usual 'block sum' convention that if $a \in M_n(A)$, and $b \in M_m(A)$, then

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(A).$$

The symbol $\otimes$ as applied to C^*-algebras always denotes the spatial tensor product. If X is a closed subspace of a C^*-algebra A and B is a C^*-algebra, then $X \otimes B$ denotes the closure of the algebraic tensor product $X \odot B$ inside $A \otimes B$. For a C^*-algebra A, $SA := C_0(\mathbb{R}) \otimes A$ is its suspension, $S^2A := S(SA)$ its double suspension, and for a closed subspace X of A, $SX := C_0(\mathbb{R}) \otimes X$. Also, $\mathcal{K}$ denotes the compact operators, so $A \otimes \mathcal{K}$ is the stabilisation of $\mathcal{K}$.

Contrary to the practice in some introductory texts on C^*-algebra K-theory such as [28] and [33], we will work with K-theory classes defined by idempotents and invertibles, rather than just by projections and unitaries. This is because one typically has more concrete formulas available in that context, allowing more control. Readers unfamiliar with this approach can find the necessary background in [2, Chapters II, III and IV], for example.

We have attempted to keep the paper self-contained and elementary, using nothing more complicated than K-theory for C^*-algebras up to the Bott periodicity theorem.[5] The price of keeping things elementary in this sense is that we seemed to occasionally be forced into technicalities[6], particularly in the proliferation of nested quantifiers. For this reason, we also tried to find a 'softer', or more conceptual, proof that proceeds via the construction of an appropriate machine, but completely failed! We would be very interested to see a softer version :) .

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2 Boundary classes

In this section, we work in the context of general Banach algebras. This is not needed for our applications, but we hope it clarifies what goes into the results; it also makes no difference to the proofs.

Definition 2.1. Let A be a Banach algebra, and let C and D be Banach subalgebras. We define maps on K-theory by

$$\iota : K_*(C \cap D) \to K_*(C) \oplus K_*(D), \quad \kappa \mapsto (\kappa, -\kappa).$$

and

$$\sigma : K_*(C) \oplus K_*(D) \to K_*(A), \quad (\kappa, \lambda) \mapsto \kappa + \lambda.$$

[5] Modulo the comments above about invertibles and idempotents: this just means we work in a more elementary Banach algebra context much of the time

[6] and split infinitives

With notation as above, assume for a moment that C and D are (closed, two-sided) ideals in A such that $A = C + D$. Then there is a Mayer-Vietoris boundary map $\partial : K_1(A) \to K_0(C \cap D)$ that fits into a long exact sequence

$$\cdots \xrightarrow{\iota} K_1(C) \oplus K_1(D) \xrightarrow{\sigma} K_1(A) \xrightarrow{\partial} K_0(C \cap D) \xrightarrow{\iota} K_0(C) \oplus K_0(D) \xrightarrow{\sigma} \cdots .$$

Our aim in this section is to get analogous results for more general Banach subalgebras C and D: for at least some classes $[u] \in K_1(A)$, we want to (non-canonically) construct a 'boundary class' $\partial(u) \in K_0(C \cap D)$ that has similar exactness properties with respect to ι and σ.

First, we need a technical lemma about approximating 'almost idempotents' by idempotents. Some variant of this (with sharper estimates!) is presumably well-known.

Lemma 2.2. *For any $\epsilon, c > 0$ there exists $\delta \in (0, 1/16)$ with the following property. Let A be a Banach algebra and $e \in A$ satisfy $\|e^2 - e\| < \delta$ and $\|e\| \leqslant c$. Let χ be the characteristic function of $\{z \in \mathbb{C} \mid Re(z) > 1/2\}$. Then $\chi(e)$ (defined via the holomorphic functional calculus) is a well-defined idempotent, and satisfies $\|\chi(e) - e\| < \epsilon$.*

Proof. First note that if $\delta \in (0, 1/16)$ and if $z \in \mathbb{C}$ satisfies $|z^2 - z| < \delta$, then $|z||z-1| < \delta$, and so either $|z| < \sqrt{\delta}$, or $|z-1| < \sqrt{\delta}$. Hence by the polynomial spectral mapping theorem, if $\|e^2 - e\| < \delta$, then the spectrum of e is contained in the union of the balls of radius $\sqrt{\delta}$ and centered at 0 and 1 respectively. As $\sqrt{\delta} < 1/2$, it follows that χ is holomorphic on the spectrum of e. Hence $\chi(e)$ makes sense under the assumptions, and is an idempotent by the functional calculus.

Let now $r = 2\sqrt{\delta} < 1/2$, and let γ_0 and γ_1 be positively oriented circles centered on 0 and 1 respectively, and of radius r. Then by the above remarks, if $\|e^2 - e\| < \delta$ we have that $\gamma_0 \cup \gamma_1$ is a positively oriented contour on which χ is holomorphic, and that has winding number one around each point of the spectrum of e. Hence by definition of the holomorphic functional calculus

$$\chi(e) - e = \frac{1}{2\pi i} \int_{\gamma_0 \cup \gamma_1} (\chi(z) - z)(z - e)^{-1} dz.$$

Estimating the norm of this using that $|\chi(z) - z| = r$ for $z \in \gamma_0 \cup \gamma_1$ gives

$$\|\chi(e) - e\| \leqslant \frac{1}{2\pi} \int_{\gamma_0 \cup \gamma_1} r \|(z - e)^{-1}\| |dz|. \tag{2}$$

Let us estimate the term $\|(z-e)^{-1}\|$ for $z \in \gamma_0 \cup \gamma_1$. Set $w = 1 - z$. Then we have that $w - e$ is also invertible, and

$$\begin{aligned} \|(z-e)^{-1}\| &= \|(w-e)(w-e)^{-1}(z-e)^{-1}\| \\ &\leqslant (c + |w|)\|\|((z^2 - z) - (e^2 - e))^{-1}\| \\ &\leqslant (c+2)\|\|((z^2 - z) - (e^2 - e))^{-1}\|. \end{aligned} \tag{3}$$

Now, we have that for $z \in \gamma_0 \cup \gamma_1$,

$$|z^2 - z| = |z||z-1| \geqslant \frac{1}{2} r = \sqrt{\delta} > \delta > \|e^2 - e\|.$$

Hence using the Neumann series inverse formula

$$((z^2 - z) - (e^2 - e))^{-1} = \frac{1}{z^2 - z}\Big(1 - \frac{e^2 - e}{z^2 - z}\Big)^{-1} = \frac{1}{z^2 - z} \sum_{n=0}^{\infty} \Big(\frac{e^2 - e}{z^2 - z}\Big)^n$$

we get the estimate

$$\|((z^2 - z) - (e^2 - e))^{-1}\| \leqslant \frac{1}{|z^2 - z| - \|e^2 - e\|} \leqslant \frac{1}{\frac{1}{2}r - \delta} = \frac{1}{\sqrt{\delta} - \delta}.$$

Combining this with line (3), we see that for $z \in \gamma_0 \cup \gamma_1$,

$$\|(z-e)^{-1}\| \leqslant \frac{c+2}{\sqrt{\delta} - \delta}.$$

To complete the proof, substituting the above estiumate into line (2) gives that

$$\|\chi(e) - e\| \leqslant \frac{1}{2\pi} \int_{\gamma_0 \cup \gamma_1} \frac{r(c+2)}{\sqrt{\delta} - \delta} |dz| = \frac{1}{2\pi}\big(\text{Length}(\gamma_0) + \text{Length}(\gamma_1)\big)\frac{r(c+2)}{\sqrt{\delta} - \delta}.$$

Substituting in $\text{Length}(\gamma_0) = \text{Length}(\gamma_1) = 2\pi r$ and $r = 2\sqrt{\delta}$ we get

$$\|\chi(e) - e\| \leqslant \frac{4\sqrt{\delta}(c+2)}{1 - \sqrt{\delta}},$$

which is enough to complete the proof. □

Definition 2.3. Let A be a Banach algebra, let B be a Banach subalgebra of A, let $a \in A$, and let $\epsilon > 0$. The element a is *ϵ-in* B, denoted $a \in_\epsilon B$, if there exists $b \in B$ with $\|a - b\| \leqslant \epsilon$.

The following lemma uses standard ideas. We give details as arguments with general idempotents (as opposed to projections) are maybe not so common in the C^*-algebraic literature.

Lemma 2.4. *Let A be a Banach algebra and B a Banach subalgebra. Then for all $c > 0$ and all $\epsilon \in (0, \frac{1}{4c+6})$ there exists $\delta > 0$ with the following property. Say $e \in M_n(A)$ is an idempotent which is δ-in $M_n(B)$ and such that $\|e\| \leqslant c$. Then there is an idempotent $f \in M_n(B)$ with $\|e - f\| < \epsilon$. Moreover, the class $[f] \in K_0(B)$ does not depend on the choice of ϵ, δ, or f.*

Proof. Let $\delta > 0$, to be chosen depending on c and ϵ in a moment, and assume that e is δ-in $M_n(B)$ so there is $b \in M_n(B)$ with $\|b - e\| < \delta$. Then

$$\|b^2 - b\| \leqslant \|e\|\|b - e\| + \|b\|\|b - e\| + \|b - e\| \leqslant (2c + \delta + 1)\delta.$$

Let χ be the characteristic function of the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1/2\}$. Then for suitably small δ (depending only on c and ϵ), we may apply Lemma 2.2 to get that $\|b - \chi(b)\| < \epsilon/2$. Setting $f = \chi(b)$ and assuming also that $\delta < \epsilon/2$ we get that

$$\|e - f\| \leqslant \|e - b\| + \|b - f\| < \epsilon$$

as desired.

To see that $[f] \in K_0(B)$ does not depend on the choice of f, let $f' \in M_n(B)$ be another idempotent with $\|e - f'\| < \epsilon$. Then $\|f - f'\| < 2\epsilon < 1/(2c + 3)$. As $\|f\| \leqslant c + 1$, we see that

$$\|f - f'\| < \frac{1}{2c + 3} \leqslant \frac{1}{\|2f - 1\|},$$

whence [2, Proposition 4.3.2] implies that f and f' are similar, and so in particular define the same K-theory class. □

Definition 2.5. Let $c > 0$, let $\epsilon \in (0, \frac{1}{4c+6})$, and let $\delta > 0$ be as in Lemma 2.4. Let A be a Banach algebra, and B be a Banach subalgebra of A, and say $e \in M_n(A)$ is an idempotent that is δ-in $M_n(B)$. Then we write $\{e\}_B \in K_0(B)$ for the class of any idempotent $f \in M_n(B)$ with $\|e - f\| < \epsilon$.

The key technical definition is as follows.

Definition 2.6. Let $c > 0$, let $\epsilon \in (0, \frac{1}{4c+6})$, and let $\delta > 0$ be as in Lemma 2.4. Let A be a Banach algebra, let C and D be Banach subalgebras of A, let $u \in M_n(\widetilde{A})$ be an invertible element for some n. An element $v \in M_{2n}(\widetilde{A})$ is a (δ, c, C, D)-*lift* of u if it satisfies the following conditions:

(i) $\|v\| \leqslant c$ and $\|v^{-1}\| \leqslant c$;

(ii) $v \in_\delta M_{2n}(\widetilde{D})$;

(iii) $v \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \in_\delta M_{2n}(\widetilde{C})$;

(iv) $v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \in_\delta M_{2n}(\widetilde{C \cap D})$;

(v) with notation as in Definition 2.5, the K-theory class

$$\left\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right\}_{\widetilde{C\cap D}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K_0(\widetilde{C \cap D})$$

is actually in the subgroup $K_0(C \cap D)$.

We may now use such lifts to construct 'boundary classes'.

Proposition 2.7. *Let $c > 0$, let $\epsilon \in (0, \frac{1}{4c+6})$. Then there is $\delta > 0$ satisfying the conclusion of Lemma 2.4, and with the following properties. Let A be a Banach algebra, and let $u \in M_n(\widetilde{A})$ be an invertible with $\|u\| \leqslant c$ and $\|u^{-1}\| \leqslant c$. Assume there exist Banach subalgebras C and D of A and a (δ, c, C, D)-lift v of u. Then the K-theory class*

$$\partial_v u := \left\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right\}_{\widetilde{C\cap D}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K_0(C \cap D)$$

has the following properties.

(i) If ι is as in Definition 2.1, then $\iota(\partial_v u) = 0$ in $K_0(C) \oplus K_0(D)$.

(ii) If $\partial_v u = 0$, then there is $l \in \mathbb{N}$ and an invertible $x \in_\epsilon M_{n+l}(\widetilde{D})$ such that $(u \oplus 1_l)x^{-1} \in_\epsilon M_{2n}(\widetilde{C})$. In particular, if σ is as in Definition 2.1, then $\sigma([(u \oplus 1_l)x^{-1}], [x]) = [u]$ in $K_1(A)$.

Proof. Let us first consider $\iota(\partial_v u)$. Note first that as v is δ-in $M_{2n}(\widetilde{D})$, there is $w \in M_{2n}(\widetilde{D})$ such that $\|w - v\| < \delta$. In particular, w is invertible for δ suitably small. It follows by definition of the left hand side that

$$\left\{v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}\right\}_{\widetilde{D}} = \left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1}\right]$$

in $K_0(\widetilde{D})$ for all suitably small δ. Hence as elements of $K_0(\widetilde{D})$,

$$\left\{v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}\right\}_{\widetilde{D}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1}\right] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

However, as w is in $M_{2n}(\widetilde{D})$, $\left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $K_0(\widetilde{D})$, so the above is the zero class in $K_0(\widetilde{D})$, hence also in $K_0(D)$.

On the other hand, our assumption that $v \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$ is δ-in $M_{2n}(\widetilde{C})$ implies similarly that for all δ suitably small, we have

$$\begin{aligned} \partial_v u &= \left\{v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}\right\}_{\widetilde{C}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} v^{-1} v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} v \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \right\}_{\widetilde{C}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which is zero as a class in $K_0(C)$. We have shown that the image of $\partial_v u$ in both $K_0(C)$ and $K_0(D)$ is zero, whence $\iota(\partial_v u) = 0$ as claimed.

Throughout the rest of the proof, whenever we write 'δ_n', it is implicit that this is a positive number, depending only on c and δ, and that tends to zero when δ tends to zero as long as c stays in a bounded set.

Now let us assume that $\partial_v u = 0$. This implies that there exists $l \in \mathbb{N}$ and an invertible element w of $M_{2n+l}(\widetilde{C \cap D})$ such that

$$\left\| w\left(v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \oplus 1_l\right) w^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1_l \right\| < \delta_1$$

for some $\delta_1 > 0$. Write $v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, and let

$$v_1 := \begin{pmatrix} v_{11} & 0 & v_{12} & 0 \\ 0 & 1_l & 0 & 0 \\ v_{21} & 0 & v_{22} & 0 \\ 0 & 0 & 0 & 1_l \end{pmatrix} \in_\delta M_{n+l+n+l}(\widetilde{D})$$

(writing the matrix size as $n+l+n+l$ is meant to help understand the size of the various blocks) and if

$$w = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} \in M_{n+n+l}(\widetilde{C \cap D})$$

let

$$w_1 := \begin{pmatrix} w_{11} & 0 & w_{12} & w_{13} \\ 0 & 1_l & 0 & 0 \\ w_{21} & & w_{22} & w_{23} \\ w_{31} & 0 & w_{32} & w_{33} \end{pmatrix} \in M_{n+l+n+l}(\widetilde{C \cap D}).$$

Then in $M_{(n+l)+(n+l)}(\widetilde{C})$ we have

$$\left\| w_1 v_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_1^{-1} w_1 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| < \delta_2$$

for some δ_2. This implies that for δ suitably small there exist invertible $x, y \in M_{n+l}(\widetilde{D})$ and δ_3 such that

$$\left\| w_1 v_1 - \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\| < \delta_3.$$

Now, by assumption

$$v \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \in_\delta M_{2n}(\widetilde{C}).$$

Write $u_1 := u \oplus 1_l \in M_{n+l}(\widetilde{A})$. Then

$$v_1 \begin{pmatrix} u_1^{-1} & 0 \\ 0 & u_1 \end{pmatrix} \in_\delta M_{(n+l)+(n+l)}(\widetilde{C}).$$

and thus as w_1 is in $M_{2(n+l)}(\widetilde{C})$, we have that

$$w_1 v_1 \begin{pmatrix} u_1^{-1} & 0 \\ 0 & u_1 \end{pmatrix} \in_{\delta_4} M_{2(n+l)}(\widetilde{C})$$

for some δ_4. Hence in particular, xu_1^{-1} is invertible for δ suitably small, is δ_4-in $M_{n+l}(\widetilde{C})$, and has norm bounded above by some absolute constant depending only on c. We now have that for δ suitably small (depending only on ϵ and c), $u_1 x^{-1}$ is ϵ-in $M_{n+l}(\widetilde{C})$ and that x is ϵ-in $M_{n+l}(\widetilde{D})$, completing the proof. $\square$

Definition 2.8. With notation as in Proposition 2.7, we call $\partial_v(u) \in K_0(C \cap D)$ the *boundary class* associated to the data (u, v, C, D).

3 Decompositions and the vanishing theorem

Our main goal in this section is to show that locally excisive decompositions as in Definitions 1.1 and 1.2 can be used to build lifts as in Definition 2.6, and thus allow us to build boundary classes.

It would be possible to get analogous results for general Banach algebras, but it would make the statements and proofs more technical. As our applications are all to the K-theory of C^*-algebras, at this stage we therefore specialise to that case.

First, it will be convenient to give a technical variation of Definitions 1.1 and 1.2.

Definition 3.1. Let A be a C^*-algebra, let $X \subseteq A$ be a subspace, and let $\delta > 0$. Then an *excisive δ-decomposition* of X is a triple

$$(h, C, D)$$

consisting of a positive contraction h in the multiplier algebra of A, and C^*-subalgebras C and D of A such that

(i) $\|[h, x]\| \leqslant \delta\|x\|$ for all $x \in X$;

(ii) hx and $(1-h)x$ are $\delta\|x\|$-in C and D respectively for all $x \in X$;

(iii) $h(1-h)x$ and $h^2(1-h)x$ are $\delta\|x\|$-in $C \cap D$ for all $x \in X$.

Finally, we say that A *excisively decomposes* over a class $\mathcal{C}$ of pairs of C^*-subalgebras if for any $\delta > 0$ and finite dimensional subspace X of A there exists an excisive δ-decomposition (h, C, D) of X with (C, D) in $\mathcal{C}$.

Remark 3.2. The conditions on multiplying into the intersection in (iii) from Definition 3.1 might look odd for two reasons. First, they are asymmetric in h and $1-h$: this is a red herring, however, as it would be essentially the same to require that $h(1-h)x$ and $h(1-h)^2x$ are both $\delta\|x\|$-in $C \cap D$. Second, there are two conditions for $C \cap D$, and only one each for C and D. This seems attributable to the fact that one needs (at least) two polynomials to generate $C_0(0,1)$ as a C^*-algebra, but only one each for $C_0(0,1]$ and $C_0[0,1)$.

Remark 3.3. It is tempting to believe thast that if A is decomposable over a class $\mathcal{C}$ as in Definition 1.1, then some additional perturbation argument shows that is also excisively decomposable over $\mathcal{C}$. We do not believe this is true due to the following example, inspired by a suggestion of Ian Putnam (we warn the reader that we did not check the details of what follows). It seems by adapting Proposition A.1 that one can show that if A has nuclear dimension one and real rank zero, then it decomposes over the class $\mathcal{C}$ of pairs of its finite dimensional C^*-subalgebras. In particular, this would apply to any Kirchberg algebra (see [5, Theorem G] and [27, Proposition 4.1.1]). However, if A excisively decomposes over a class of pairs of finite-dimensional C^*-algebras, then a mild elaboration of Proposition 3.7 below shows that $K_1(A)$ is torsion free. As there are Kirchberg algebras with non-trivial torsion K_1 group (see [27, Section 4.3]), this (if correct!) would show that being decomposable over a collection $\mathcal{C}$ and being excisively decomposable over $\mathcal{C}$ are not the same.

We need a basic lemma about how excisive decompositions behave under tensor products. If X is a subspace of a C^*-algebra A, recall that we write $X \otimes B$ for the norm closure of the subspace of $A \otimes B$ generated by elementary tensors $x \otimes b$ with $x \in X$ and $b \in B$.

Lemma 3.4. *Say A is a C^*-algebra, and X is a finite-dimensional subspace of A. Then there exists a constant $M_X > 0$ depending only on X such that if (h, C, D) is an locally excisive δ-decomposition of X, and if B is any C^*-algebra, then $(h \otimes 1, C \otimes B, D \otimes B)$ is a locally excisive $M_X\delta$-decomposition of $X \otimes B$.*

Proof. Let $x_1, ..., x_n$ be a basis for X consisting of unit vectors, and let $\phi_1, ..., \phi_n \in A^*$ be linear functionals dual to this basis, so $\phi_i(x_j) = \delta_{ij}$ (here δ_{ij} is the Kronecker δ function). Let $N = \sup_{i=1}^n \|\phi_i\|$. Note first that any $a \in X \otimes B$ can be written

$$a = \sum_{i=1}^n x_i \otimes b_i$$

for some unique $b_1, ..., b_n \in B$, and that we have for each i

$$\|b_i\| = \|(\phi_i \otimes \mathrm{id})(a)\| \leqslant \|\phi_i\| \|a\| \leqslant N\|a\|.$$

To see property (i), note that for any $a = \sum_{i=1}^n x_i \otimes b_i \in X \otimes B$ we have

$$\|[h \otimes 1, a]\| \leqslant \sum_{i=1}^n \|[h, x_i] \otimes b_i\| \leqslant \sum_{i=1}^n \delta \|x_i\| \|b_i\| \leqslant \delta n N \|a\|.$$

To see property (ii) and (iii), let us look at ha for some $a \in X \otimes B$; the cases of $(1-h)a$, $h(1-h)a$, and $h^2(1-h)a$ are similar. For each each $i \in \{1, ..., n\}$

choose $c_i \in C$ with $\|hx_i - c_i\| < \delta$. Then if $a = \sum_{i=1}^n x_i \otimes b_i \in X \otimes B$ is as above and $c = \sum_{i=1}^n c_i \otimes b_i \in C \otimes B$ we have

$$\|(h \otimes 1)a - c\| \leqslant \sum_{i=1}^{n} \|(hx_i - c_i) \otimes b_i\| \leqslant \sum_{i=1}^{n} \delta \|b_i\| \leqslant \delta n M \|a\|$$

and are done. $\square$

Corollary 3.5. *Say A is a C^*-algebra that excisively decomposes over $\mathcal{C}$, and let B be a C^*-algebra. Then $A \otimes B$ excisively decomposes over the set $\{(C \otimes B, D \otimes B) \mid (C, D) \in \mathcal{C}\}$.*

Proof. Let X be a finite-dimensional subspace of $A \otimes B$, and let $\delta > 0$. As the unit sphere of X is compact, there is a finite dimensional subspace Y of A such that for any x in the unit sphere of X there exists y in the unit sphere of $Y \otimes B$ such that $\|y - x\| < \delta/2$. Let M_Y be as in Lemma 3.4, and let (h, C, D) be an excisive $\delta/(2M_Y)$-decomposition for Y. Then Lemma 3.4 implies that $(h \otimes 1, C \otimes B, D \otimes B)$ is an excisive δ-decomposition for X. $\square$

Here is the second basic lemma we need.

Lemma 3.6. *Say A is a C^*-algebra, X_0 is a finite-dimensional subspace of A, and $N \geqslant 2$. Then there exists a finite-dimensional subspace X of A containing X_0, such that for any $\delta > 0$ there exists $\delta' > 0$ such that if (h, C, D) is a locally excisive δ'-decomposition of X, then (h, C, D) also satisfies the following properties:*

(i) $\|[h, x]\| \leqslant \delta \|x\|$ *for all* $x \in X_0$*;*

(ii) for all $n \in \{1, ..., N\}$, $h^n x$ (respectively, $h^n(1-h)x$, and $h^n(1-h)x$) is $\delta\|x\|$-in C (respectively D, and $C \cap D$) for all $x \in X_0$;

In words, we can bootstrap property (ii) from Definition 3.1 up to a stronger version of itself. That something like this is possible was pointed out to me by Aaron Tikuisis and Wilhelm Winter: I was originally working directly with the bootstrapped version, which led to messier definitions.

Proof. We just sketch the idea, which is elementary, and leave the details to the reader. Take a basis of X consisting of contractions, and write each of these as a sum of four positive contractions. Let X' be the space of spanned by all these positive contractions, say $\{a_1, ..., a_n\}$. Ley Y be spanned by all

m^{th} roots of all of $a_1, ..., a_n$ for $m \in \{1, ..., N+1\}$. Clearly if $\delta' \leqslant \delta$, then as Y contains X, we have the almost commutation property in the statement.

Let us now look at $h^n x$ for $x \in X$. It suffices to look at $h^n a$ for some $a \in \{a_1, ..., a_n\}$. Then using the almost commutation property, we have that $h^n a$ is close to $(ha^{1/n})^n$, so for δ' suitably small, we get what we want. Similarly, if $a \in \{a_1, ..., a_n\}$, if we write $g = h - 1$, then

$$h^n(1-h) = (1+g)^n(-g)a = -\sum_{k=0}^{n} \binom{n}{k} g^{k+1} a,$$

and again using the almost commutation property, this is close to

$$\sum_{k=0}^{n} \binom{n}{k} (ga^{1/(k+1)})^{k+1},$$

so we get the right property for δ' suitably small. The corresponding property for the intersection is similar, once we realise that for all $n \geqslant 1$, $h^n(1-h)$ can be written as a polynomial in $h(1-h)$ and $h^2(1-h)$ (proof by induction on n, for example): we leave the details of this to the reader. $\square$

For the remainder of this section, we will apply Lemma 3.4 to tensor products $M_n(A) = A \otimes M_n(\mathbb{C})$ without further comment. We will also abuse notation, writing things like 'hu' for an element $u \in M_n(A)$, when we really mean '$(h \otimes 1_n)u$'.

The next proposition is the key technical result of this section. It says that we can use excisive decompositions to build boundary classes as in Definition 2.8. For the statement, recall the notion of a (ϵ, c, C, D)-lift from Definition 2.6 above.

Proposition 3.7. *Let A be a C^*-algebra and let $\alpha \in K_1(A)$ be a K_1-class. Then there exist n and an invertible element $u \in M_n(\widetilde{A})$, $c > 0$, and a finite-dimensional subspace X of A such that for any $\epsilon > 0$ there exists $\delta > 0$ such that the following hold.*

(i) The class $[u]$ equals α.

(ii) If (h, C, D) is a locally excisive δ-decomposition of X, and if $a = h + (1-h)u$ and $b = h + u^{-1}(1-h)$ then

$$v := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is an (ϵ, c, C, D)-lift for u.

First we have an ancillary lemma.

Lemma 3.8. *Let A be a C^*-algebra and let u be an invertible element of $\widetilde{A}$ such that $u = 1 + y$ and $u^{-1} = 1 + z$ with y, z elements of A with norms bounded by some $c > 0$. Let $\delta > 0$ and let h be a positive contraction in $M(A)$ such that $\|[h, x]\| \leqslant \delta \|x\|$ for all $x \in \{y, z\}$. Define*

$$a := h + (1-h)u \quad \textit{and} \quad b := h + u^{-1}(1-h).$$

Then $ba - 1$ and $ab - 1$ are both within $2(c^2 + c)\delta$ of $(y + z)h(1 - h)$.

Proof. Using that y and z commute, we have that

$$\begin{aligned}
[a, b] &= (1-h)yz(1-h) - z(1-h)^2 y \\
&= [(1-h), z]y(1-h) + z(1-h)[y, (1-h)] \\
&= [z, h]y(1-h) + z(1-h)[h, y],
\end{aligned}$$

whence $\|ab - ba\| \leqslant 2c^2\delta$. Hence it suffices to show that $ab - 1$ is within $2c\delta$ of $h(1-h)(y+z)$. Using that $yz = -y - z$, we see that

$$ab - 1 = (1-h)yh + hz(1-h)$$

and using that $\|[y, h]\| \leqslant \delta\|y\|$ and $\|[z, h]\| \leqslant \delta\|z\|$, we are done. □

Proof of Proposition 3.7. Let $u \in M_n(\widetilde{A})$ be any invertible element such that $[u] = \alpha$. Using that $GL_n(\mathbb{C})$ is connected, up to a homotopy we may assume that u and u^{-1} are of the form $1 + y$ and $1 + z$ respectively with $y, z \in A$. Let X_0 be the subspace of A spanned by all matrix entries of all monomials of degree between one and three with entries from $\{y, z\}$. Let X be as in Lemma 3.6 for this X_0 and $N = 4$. Let then $\epsilon > 0$ be given, and let $\delta > 0$ be fixed, to be determined by the rest of the proof. Let (h, C, D) be an excisive δ-decomposition of X.

Throughout the proof, anything called 'δ_n' is a constant depending on X, δ and $\max\{\|y\|, \|z\|\}$, and with the property that δ_n tends to zero as n tends to zero, assuming the other inputs are held constant. Note that Lemma 3.4 implies that (abusing notation) there is δ_1 such that $(h, M_n(C), M_n(D))$ is an excisive δ_1 decomposition of $M_n(A)$ for all n. We check the properties from Definition 2.6. Property (i) is clear from the formula for v (which implies a similar formula for v^{-1}).

For property (ii), one computes

$$v = \begin{pmatrix} a(2-ba) & ab-1 \\ 1-ba & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} a(1-ba) & ab-1 \\ 1-ba & 0 \end{pmatrix}. \tag{4}$$

As $a = 1 + (1-h)y$ and $b = 1 + z(1-h)$, we have that a and b are both δ_2-in $M_n(\widetilde{D})$ for some δ_2. Hence also $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is δ_2-in $M_{2n}(\widetilde{D})$. On the other hand, Lemmas 3.8 and 3.6 implies that $1-ba$ and $1-ab$ are δ_3-in $M_{2n}(\widetilde{D})$ for some δ_3 by choice of X. It follows from this and that a is δ_2-in $M_n(\widetilde{D})$ that $\begin{pmatrix} a(1-ba) & ab-1 \\ 1-ba & 0 \end{pmatrix}$ is δ_4-in $M_{2n}(\widetilde{D})$ for some δ_4.

For part (iii), we compute

$$v \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} au^{-1} & 0 \\ 0 & bu \end{pmatrix} + \begin{pmatrix} a(1-ba)u^{-1} & (ab-1)u \\ (1-ba)u^{-1} & 0 \end{pmatrix}. \tag{5}$$

We have that $au^{-1} = 1 + hz$ and that $\|bu - (1+yh)\| < \delta_5$ for some δ_5. Hence the first term in line (5) is δ_6-in $M_{2n}(\widetilde{C})$ for some δ_6. For the second term, using Lemma 3.8 we have that up to some δ_7, $(1-ba)u^{-1}$ and $(ab-1)u$ equal

$$(y+z)h(1-h)(1+z) \quad \text{and} \quad (y+z)h(1-h)(1+y).$$

On the other hand $\|a(1-ba)u^{-1} - (1+hz)(y+z)h(1-h)\| < \delta_7$ for some δ_8. The claim follows from all of this and the choice of X.

For parts (iv) and (v), note that

$$\begin{aligned} v^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} b & 1-ba \\ ab-1 & a(2-ba) \end{pmatrix} \\ &= \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 1-ba \\ ab-1 & a(1-ba) \end{pmatrix}. \end{aligned}$$

Using this and the formula in line (4) we have that $v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ equals

$$\begin{pmatrix} ab-1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a(1-ba)b & 0 \\ (1-ba)b & 0 \end{pmatrix} + \begin{pmatrix} 0 & a(1-ba) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a(1-ba)^2 \\ 0 & (1-ba)^2 \end{pmatrix}. \tag{6}$$

Now, using Lemma 3.8 and the fact that h almost commutes with y and z, every term appearing in within some δ_9 of something of the form $(1-h)hp_C(h)q_C(y,z)$, where p_C is a polynomial of degree at most 3 in h (possibly with a constant term), q_C is a noncommutative polynomial of degree at most 3 with no constant term, and moreover the coefficients in p_C and q_D are universally bounded. Hence by choice of X, all the terms are δ_{10}-in $M_n(C \cap D)$, for some δ_{10}. This completes the proof. $\square$

We are now ready for the proof of Theorem 1.3 from the introduction.

Theorem 3.9. *Say that A excisively decomposes over a class $\mathcal{C}$ of pairs such that for all (C,D) in $\mathcal{C}$, C, D, and $C \cap D$ all have trivial K-theory. Then A has trivial K-theory.*

Proof. It suffices to show that $K_1(A) = K_1(SA) = 0$. For $K_1(A)$, let $\alpha \in K_1(A)$ be an arbitrary class. Then using Proposition 3.7 we may build a boundary class $\partial_v(u) \in K_0(C \cap D)$. As $K_0(C \cap D) = 0$, this class $\partial_v(u)$ is zero. Hence by Proposition 2.7 it is in the image of $\sigma : K_1(C) \oplus K_1(D) \to K_1(A)$. However, $K_1(C) = K_1(D)$ by assumption, so we are done with this case.

The case of $K_1(SA)$ is almost the same. Indeed, Corollary 3.5 implies that SA excisively decomposes over the set $\{(SC, SD) \mid (C,D) \in \mathcal{C}\}$, and we have that SC, SD, and $SC \cap SD = S(C \cap D)$ all have trivial K-theory. $\square$

We remark that Theorem 1.3 can be used to simplify the proof of the main theorem of [16], in particular obviating the need for filtrations and controlled K-theory in the proof, and replacing the material of [16, Section 7] entirely. Further applications are certainly possible, but we will not pursue this here.

4 More on boundary classes

In this section we collect together further technical results on boundary classes that are needed for the proof of Theorem 1.5 on the Künneth formula. We state results for Banach algebras when it makes no difference to the proof, and C^*-algebras when the proof is simpler in that case.

The first result corresponds to exactness at position (I) in line (1) from the introduction. For the statement, recall the notion of lifts from Definition 2.6.

Proposition 4.1. *Let A be a Banach algebra and let C and D be Banach subalgebras of A. Assume that $p, q \in M_n(\widetilde{C \cap D})$ are idempotents such that $[p]-[q] \in K_0(C \cap D)$. Assume moreover that with ι as in Definition 2.1 that $\iota([p]-[q]) = 0$.*

Then there exist $k \in \mathbb{N}$, an invertible element u of $M_{n+k}(\widetilde{A})$, an invertible element v of $M_{2(n+k)}(\widetilde{A})$ and $c > 0$ such that for any $\delta > 0$, v and v^{-1} are (δ, c, C, D)-lifts of u and u^{-1} respectively, and such that $\partial_v u = [p]-[q]$ and $\partial_{v^{-1}}(u^{-1}) = [q]-[p]$.

Proof. As $\iota([p]-[q]) = 0$, there exist natural numbers $l \leqslant k$ and invertible elements $u_C \in M_{n+k}(\widetilde{C})$, $u_D \in M_{n+k}(\widetilde{D})$ such that

$$u_C(p \oplus 1_l)u_C^{-1} = q \oplus 1_l = u_D(p \oplus 1_l)u_D^{-1}.$$

Define

$$u := (1 - p \oplus 1_l)u_C^{-1} + (p \oplus 1_l)u_D^{-1} \in M_{n+k}(\widetilde{A}),$$

so u is invertible with inverse $u^{-1} = u_C(1-p\oplus 1_l) + u_D(p \oplus 1_l)$, as the reader can directly check. Define now

$$v := \begin{pmatrix} (p \oplus 1_l)u_D^{-1} & p \oplus 1_l - 1 \\ 1 - q \oplus 1_l & u_D(p \oplus 1_l) \end{pmatrix} \in M_{2(n+k)}(\widetilde{D}).$$

Note that v is invertible: indeed, direct computations show that

$$v^{-1} := \begin{pmatrix} u_D(p \oplus 1_l) & 1 - q \oplus 1_l \\ 1 - p_n \oplus 1_l & (p \oplus 1_l)u_D^{-1} \end{pmatrix}.$$

We also compute that

$$v \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} p \oplus 1_l & (1 - p \oplus 1_l)u_C^{-1} \\ u_C(1 - p \oplus 1_l) & q \oplus 1_l \end{pmatrix},$$

which is an element of $M_{2(n+k)}(\widetilde{C})$, so at this point we have properties (i), (ii), and (iii) from Definition 2.6.

To complete the proof, we compute using the formulas above for v and v^{-1} that

$$v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} = \begin{pmatrix} p \oplus 1_l & 0 \\ 0 & 1 - q \oplus 1_l \end{pmatrix},$$

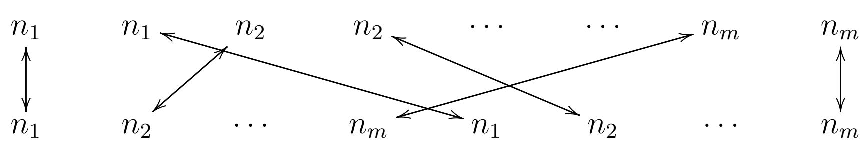
which is in $M_{2(n+k)}(\widetilde{C \cap D})$. Moreover, as a class in $K_0(\widetilde{C \cap D})$,

$$\left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}\right] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = [p] - [q],$$

so in particular this class is in $K_0(C \cap D)$, completing the proof that v satisfies the conditions from Definition 2.6, and that $\partial_v(u) = [p] - [q]$.

The computations with v^{-1} and u^{-1} replacing v and u are similar: we leave them to the reader. $\square$

Lemma 4.2. *Let A be a Banach algebra, let $c > 0$, and let $\epsilon \in (0, \frac{1}{4c+6})$. Let $\delta > 0$ satisfy the conclusion of Proposition 2.7. Assume that for $i \in \{1, ..., m\}$, $u_i \in M_{n_i}(\widetilde{A})$ is an invertible elements such that $\|u\| \leqslant c$ and $\|u^{-1}\| \leqslant c$, and let C and D be Banach subalgebras of A such that for each i there is a (δ, c, C, D)-lift v_i of u_i. Let $s \in M_{2(n_1+\cdots+n_m)}$ be the self-inverse permutation matrix defined by the following diagram in the sizes of the matrix blocks*



and define

$$v_1 \boxplus \cdots \boxplus v_m := s(v_1 \oplus \cdots \oplus v_m)s$$

Then $v := v_1 \boxplus \cdots \boxplus v_m$ is a (δ, c, C, D)-lift of $u := u_1 \oplus \cdots \oplus u_m$, and

$$\partial_v u = \sum_{i=1}^{n} \partial_{v_i}(u_i)$$

in $K_0(C \cap D)$.

Proof. We leave the direct and elementary checks involved in this to the reader. $\square$

We conclude this section with a technical result on inverses that we will need later.

Lemma 4.3. *Assume that the assumptions of Proposition 3.7 are satisfied. Then on shrinking δ, we may assume that v^{-1} is also an (ϵ, c, C, D)-lift of u^{-1}, and moreover that*

$$\partial_v(u) = -\partial_{v^{-1}}(u^{-1})$$

as elements of $K_0(C \cap D)$.

Proof. Checking that

$$v^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

satisfies the properties from Definition 2.6 with respect to u^{-1} is essentially the same as checking the corresponding properties for v and u in the proof of Proposition 3.7. We leave the details to the reader.

It remains to establish the formula $\partial_v(u) = -\partial_{v^{-1}}(u^{-1})$. For $t \in [0,1]$, define

$$v_t := \begin{pmatrix} 1 & ta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -tb & 1 \end{pmatrix} \begin{pmatrix} 1 & ta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Analogous computations we used to establish to property (iii) in the proof of Proposition 3.7 show that $v_t^{-1} v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} v_t$ is in $M_{2n}(\widetilde{C \cap D})$ up to an error we can make as small as we like depending on δ, and that the difference

$$v_t^{-1} v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} v_t - v_t^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t$$

is in $M_{2n}(C \cap D)$, again up to an error that we can make as small as we like by making δ small (and keeping c and X fixed). Hence for all $t \in [0,1]$ we get that the classes

$$\left\{ v_t^{-1} v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} v_t \right\}_{\widetilde{C \cap D}} - \left\{ v_t^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t \right\}_{\widetilde{C \cap D}}$$

of $K_0(C \cap D)$ are well-defined. They are moreover all the same, as the elements defining them are homotopic. However, the above equals $\delta_v(u)$ when $t = 0$, and equals $-\delta_{v^{-1}}(u^{-1})$ when $t = 1$, so we are done. $\square$

5 Decompositions and the summation map

In this section, we prove a technical result, based very closely on [25, Lemma 2.9], and corresponding to exactness at position (III) in line (1) from the introduction.

The statements are a little involved. The basic idea of the result is that it gives conditions under which one has some sort of exactness

$$K_1(C \cap D) \xrightarrow{\iota} K_1(C) \oplus K_1(D) \xrightarrow{\sigma} K_1(A)$$

of the sequence of maps from Definition 2.1. Roughly, it says that if X is a finite dimensional subspace of A and if (h, C, D) is a suitable decomposition of X as in Definition 1.1, then if $\sigma(\kappa, \lambda) = 0$ *and there is a 'reason' for this element being zero in the subspace* X, then (κ, λ) is in the image of ι. This result is weak: it seems the quantifiers are in the wrong order for it to be useful; nonetheless, it plays a crucial role in the injectivity half of theorem 1.5.

We need a stronger version of excisiveness. The following definition is closely related to the so-called CIA property as used in the definition of 'nuclear Mayer-Vietoris pairs' in [25, Definition 4.8].

Definition 5.1. Let $f : (0, \infty) \to (0, \infty)$ be a function such that $f(t) \to 0$ as $t \to 0$, which we call a *decay function.* A pair (C, D) of C^*-subalgebras of a C^*-algebra A is f*-excisive* if for any C^*-algebra B and any $c \in C \otimes B$, $d \in D \otimes B$ there exists $x \in (C \cap D) \otimes B$ such that

$$\max\{\|x - c\|, \|x - d\|\} \leqslant f(\|c - d\|).$$

A C^*-algebra A *strongly excisively decomposes* over a class $\mathcal{C}$ of pairs of C^*-subalgebras if there is a decay function f such that A decomposes over $\mathcal{C}$ and each $(C, D) \in \mathcal{C}$ is f-excisive.

Remark 5.2. If A strongly excisively decomposes over $\mathcal{C}$, then it is straightforward to check that it excisively decomposes over $\mathcal{C}$ in the sense of Definition 3.1.

The following lemma is very similar to Lemma 3.4 and Corollary 3.5.

Lemma 5.3. *Say A is a C^*-algebra, and X is a finite-dimensional subspace of A. Then there exists a constant $M_X > 0$ depending only on X such that if (h, C, D) is an f-excisive δ-decomposition of X, and if B is any C^*-algebra, then $(h \otimes 1, C \otimes B, D \otimes B)$ is an f-excisive $M_X\delta$-decomposition of $X \otimes B$.*

Moreover, say A is a C^*-algebra, f a decay function, and B is a C^*-algebra. Then for any finite dimensional subspace X of $A \otimes B$ and $\delta > 0$ there is a finite-dimensional subspace Y of A and $\delta' > 0$ such that if (h, C, D) is an f-excisive δ'-decomposition of Y, then $(h \otimes 1, C \otimes B, D \otimes B)$ is an f-excisive δ-decomposition of X.

Proof. The first part is essentially the same as Lemma 3.4. For the second part, let $\delta > 0$. As the unit sphere of X is compact, there is a finite-dimensional subspace Y of A such that every point of the unit sphere of X

is within $\delta/2$ of a point of the unit sphere of $Y \otimes B$. Using the first part, it id not difficult to see that this Y works for $\delta' = \delta/(2M_Y)$. $\square$

We need two preliminary lemmas before we get to the main result.

For the first lemma, recall that invertible elements of C^*-algebras of the form $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ are equal to zero in K-theory for purely algebraic reasons (compare [22, Lemma 2.5 and Lemma 3.1]). The following lemma thus says that any invertible element u of a C^*-algebra that is zero in K_1 by virtue of a homotopy is also zero in K_1 for purely algebraic reasons, up to an arbitrarily good approximation[7].

Lemma 5.4. *Let $c, \epsilon > 0$. Then there exists $\delta > 0$ with the following property. Let X be a subspace of a C^*-algebra A and let $\{u_t\}_{t \in [0,1]}$ be a homotopy of invertibles in $M_n(\widetilde{A})$ such that $u_1 = 1_n$, such that each u_t and u_t^{-1} is δ-in $\{1 + x \in M_n(\widetilde{A}) \mid x \in M_n(X)\}$, and each is of norm at most c.*

Then there exists $m \in \mathbb{N}$ and invertible elements $a \in_\delta \{1 + x \in M_{mn}(\widetilde{A}) \mid x \in M_{nm}(X)\}$ and $b \in_\delta \{1 + x \in M_{(m+1)n}(\widetilde{A}) \mid x \in M_{(m+1)n}(X)\}$ such that a, b, a^{-1} and b^{-1} all have norm at most c, and such that the difference

$$\begin{pmatrix} u_0 & 0 \\ 0 & 1_{(2m+1)n} \end{pmatrix} - \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

in $M_{2(m+1)n}(\widetilde{A})$ has norm at most ϵ.

Proof. Let $\delta > 0$ (to be chosen later), and choose a partition $0 = t_0 < \ldots < t_m = 1$ of the interval $[0,1]$ with the property that for any i, $\|u_{t_{i+1}} - u_{t_i}\| < \delta$. Define

$$a := \begin{pmatrix} u_{t_1}^{-1} & 0 & \ldots & 0 \\ 0 & u_{t_2}^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_{t_m}^{-1} \end{pmatrix} \in M_{mn}(\widetilde{A}).$$

[7] But not exactly! – otherwise the algebraic and topological K_1 groups of a C^*-algebra would always be the same.

$$b := \begin{pmatrix} u_{t_0} & 0 & \dots & 0 \\ 0 & u_{t_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{t_m} \end{pmatrix} \in M_{(m+1)n}(\widetilde{A}).$$

Then we have that

$$\begin{pmatrix} u_0 & 0 \\ 0 & 1_{(2m+1)n} \end{pmatrix} - \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

equals

$$\begin{pmatrix} 0_{(m+1)n} & 0 & 0 & \cdots & 0 \\ 0 & 1 - u_{t_1}u_{t_0}^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 - u_{t_2}u_{t_1}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 - u_{t_m}^{-1} \end{pmatrix}.$$

Recalling that $u_{t_m} = 1$, the latter element has norm bounded above by

$$\max_i \|1 - u_{t_{i+1}}u_{t_i}^{-1}\| = \max_i \|u_{t_i} - u_{t_{i+1}}\|\|u_{t_i}^{-1}\| < \delta c,$$

which we can make as small as we like by decreasing the size of δ. $\square$

The next lemma uses decompositions and the Whitehead trick to split up an element of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ using decompositions as in Definition 3.1.

Lemma 5.5. *Say A is a C^*-algebra and X a finite-dimensional subspace of A. Then there is a finite-dimensional subspace Y of A such that for any $\epsilon > 0$ there exists $\delta > 0$ so that the following holds. Assume that $a \in M_n(\widetilde{A})$ is an invertible element such that a and a^{-1} have norm at most c, and are δ-in the set $\{1 + x \in M_n(\widetilde{A}) \mid x \in M_n(X)\}$. Assume that (h, C, D) is a δ-decomposition of Y. Then there are homotopies $\{v_t^C\}_{t\in[0,1]}$ and $\{v_t^D\}_{t\in[0,1]}$ of invertible elements such that:*

(i) for each t, $v_t^C \in_\epsilon \{1 + c \mid c \in M_{2n}(C)\}$ and $v_t^D \in_\epsilon \{1 + d \mid d \in M_{2n}(D)\}$;

(ii) $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = v_0^C v_0^D$;

(iii) $v_1^C = v_1^D = 1_{2n}$*;*

(iv) for each t *the norms of* v_t^C *and* v_t^D *are both at most* $(3+c)^5$*.*

Proof. Let Y_0 be the subspace of A spanned by all monomials of degree between one and four with entries from X. Let Y be as in Lemma 3.6 for this Y_0 and $N = 4$. Let then $\epsilon > 0$ be given, and let $\delta > 0$ be fixed, to be determined by the rest of the proof. Let (h, C, D) be an excisive δ-decomposition of X.

Write $a = 1 + x$ and $a^{-1} = 1 + y$ with $x, y \in_\delta M_n(X)$. Consider the usual ('Whitehead trick') product decomposition

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad (7)$$

Set $x^C := 1 + hx$ and $x^D := (1-h)x$, so that $x^C + x^D = a$. Similarly, set $y^C := 1 + hy$ and $y^D = (1-h)y$, so that $y^C + y^D = a^{-1}$. For any element z of a C^*-algebra, set

$$X(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y(z) := \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix}.$$

Then using that $X(z_1 + z_2) = X(z_1)X(z_2)$ and similarly for Y, the product in line (7) equals

$$X(x^D)X(x^C)Y(-y^C)Y(-y^D)X(x^C)X(x^D) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Rewriting further, this equals the product of

$$v^C := X(x^D)X(x^C)Y(-y^C)X(x^C) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X(-x^D),$$

and

$$v^D := X(x^D) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(-x^C)Y(-y^D)X(x^C)X(x^D) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We claim this v^C and v^D have the properties required of v_0^C and v_0^D in the statement. The norm estimates are clear, as is the equation $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} =$

$v_0^C v_0^D$. For the remainder of the proof, any constant called δ_n depends only on c, X, and δ, and tends to zero as δ tends to zero.

We first claim that v^C is ϵ-in the set $\{1 + c \mid c \in M_{2n}(C)\}$ for δ suitably small. Note that using Lemma 5.3, h commutes with x and y up to some error δ_1. Using this, plus the fact that $xy = yx = -y - x$, one computes that

$$X(x^C)Y(-y^C)X(x^C)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

is within some δ_2 of an element of the form

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}h & 0\\ 0 & h\end{pmatrix} Z_1,$$

where all entries of Z_1 are products of a noncommutative polynomial in x and y of degree at most two and with no constant term, with a polynomial in h of degree at most two. Hence up to error some δ_3, we have that v^C agrees with

$$X(x^D)\Big(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}h & 0\\ 0 & h\end{pmatrix} Z\Big)X(-x^D),$$

and that up to some δ_4, this is the same as

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}h & 0\\ 0 & h\end{pmatrix} Z_2,$$

where every entry of Z_2 is a product of a noncommutative polynomial in x and y of degree at most four and with no constant term, with a polynomial in h of degree at most four. The claim follows from this, and the choice of X.

The computations showing that v^D is ϵ-in the set $\{1 + d \mid d \in M_{2n}(D)\}$ for δ suitably small are similar. Indeed, we first we note that

$$Y(-y_D) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1-h & 0\\ 0 & 1-h\end{pmatrix}\begin{pmatrix}0 & 0\\ -y & 0\end{pmatrix},$$

whence $X(-x^C)Y(-y^D)X(x^C)$ is within δ_5 of an element of the form

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1-h & 0\\ 0 & 1-h\end{pmatrix} Z_3$$

where every entry of Z_3 is a product of a noncommutative polynomial in x and y of degree at most two and with no constant term, with a polynomial in h of degree at most two. Hence $X(-x^C)Y(-y^D)X(x^C)X(x^D)$ is within δ_6 of an element of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-h & 0 \\ 0 & 1-h \end{pmatrix} Z_4,$$

where every entry of Z_3 is a product of a noncommutative polynomial in x and y of degree at most three and with no constant term, with a polynomial in h of degree at most three. The same is true therefore of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(-x^C)Y(-y^D)X(x^C)X(x^D) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We thus get that v^D is within δ_7 of an element of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-h & 0 \\ 0 & 1-h \end{pmatrix} Z_5,$$

where every entry of Z_5 is a product of a noncommutative polynomial in x and y of degree at most four and with no constant term, with a polynomial in h of degree at most four.

To construct homotopies with the required properties, define $x_t^C := 1 + (1-t)hx$, $x_t^D := (1-t)(1-h)x$, $y_t^C := 1+(1-t)hy$, and $y_t^D; = (1-t)(1-h)y$. Define moreover

$$v_t^C := X(x_t^D)X(x_t^C)Y(-y_t^C)X(x_t^C) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X(-x_t^D)$$

and

$$v_t^D := X(x_t^D) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(-x_t^C)Y(-y_t^D)X(x_t^C)X(x_t^D) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using precisely analogous computations to those we have already done, one sees that these elements have the claimed properties: we leave the remaining details to the reader. □

Here is the key technical result of this section.

Proposition 5.6. *Let A be a C^*-algebra, let $f : (0,\infty) \to (0,\infty)$ be a function such that $f(t) \to 0$ as $t \to 0$, let $\epsilon > 0$, let $c > 0$, and let X be a finite-dimensional subspace of A. Then there exists a finite-dimensional subspace Y of A and $\delta > 0$ with the following property.*

Assume that for some $n \in \mathbb{N}$ there is a homotopy $\{u_t\}_{t\in[0,1]}$ of invertible elements in $M_n(\widetilde{A})$ with $u_1 = 1_n$, and such that each u_t and u_t^{-1} are δ-in the set $\{1 + x \in M_n(\widetilde{A}) \mid x \in M_n(X)\}$, and have norm at most some c. Then if (h, C, D) is a δ-decomposition of X with (C, D) f-excisive then the following holds.

Say $l < n$ and $u_C \in M_{n-l}(\widetilde{C})$ and $u_D \in M_{n-l}(\widetilde{D})$ are invertible, such that they and their inverses have norm at most c, and such that $\|u_0 - u_C u_D \oplus 1_l\| < \delta$. Then there exists $k \in \mathbb{N}$ and an invertible element $x \in M_k(\widetilde{C \cap D})$ such that if $[x] \in K_1(C \cap D)$ is the corresponding class, then with notation as in Definition 2.1,

$$\iota[x] = ([u_C], [u_D]) \in K_1(C) \oplus K_1(D).$$

Proof. Applying Lemma 5.4 to the homotopy $\{u_t\}$ we get $m \in \mathbb{N}$ and invertible elements $a \in_\delta \{1 + x \in M_n(\widetilde{A}) \mid x \in M_{mn}(X)\}$ an $b \in_\delta \{1 + x \in M_{(m+1)n}(\widetilde{A}) \mid x \in M_n(X)\}$ such that

$$\begin{pmatrix} u_0 & 0 \\ 0 & 1_{(2m+1)n} \end{pmatrix} - \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

has norm at most δ. Let Y_a and Y_b have the properties in Lemma 7 with respect to a and b, and let $Y := Y_a + Y_b$, a finite dimensional subspace of A. Let then c and ϵ be given, and let δ be fixed, to be determined by the rest of the proof. Let (h, C, D) be a δ-decomposition of Y with (C, D) f-excisive.

As usual, throughout the proof any constant called δ_n depends on f, c, Y, and δ, and tends to zero as δ tends to zero. Applying (a very slight variation of) Lemma 7 to $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$, we get elements $v_t^{C,a}$ and $v_t^{D,a}$, and $v_t^{C,b}$ and $v_t^{D,b}$ for $t \in [0,1]$ satisfying the conditions there for some δ_1. Moreover, if we write $v^{C,a} := v_1^{C,a}$ and similarly for the other terms, then

$$\begin{aligned} \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} &= v^{D,a} v^{C,a} v^{C,b} v^{D,b} \\ &= \underbrace{v^{D,a} v^{C,a} v^{C,b} (v^{D,a})^{-1}}_{=:v^C} \underbrace{v^{D,a} v^{D,b}}_{=:v^D}. \end{aligned}$$

Note that v^C and v^D are δ_2-in $M_{2n}(\widetilde{C})$ and $M_{2n}(\widetilde{D})$ respectively, that they define the trivial class in $K_1(C)$ and $K_1(D)$ respectively, and that they and their inverses have norm at most $(3+c)^{20}$.

Let u_C and u_D have the properties in the statement. Replacing u_C and u_D by their block sums with 1_l, we may (for notational simplicity) assume that $l = 0$. Now, we have that $u_C u_D$ and $v_C v_D$ are within some δ_3 of each other. Hence $1 - v_C^{-1} u_C$ and $1 - v_D u_D^{-1}$ are within some δ_4 of each other. Applying our f-excisiveness assumption, there exists an element y in some matrix algebra over $C \cap D$ that is within some δ_5 of both. Set $x = 1 + y$. Then x is an invertible element of some matrix algebra over $\widetilde{C \cap D}$ (as long as δ is suitably small) that is close to both $v_C^{-1} u_C$ and to $v_D u_D^{-1}$. Hence for suitably small δ, we have that as classes in $K_1(C)$

$$[x] = [v_C^{-1} u_C] = [u_C],$$

where the second equality follows as v_C represents the trivial class in $K_1(C)$. Similarly, in $K_1(D)$,

$$[x] = [v_D u_D^{-1}] = [u_D^{-1}].$$

It follows from the last two displayed lines that

$$\iota[x] = ([u_C], [u_D])$$

as required. □

6 The product map

In this section we recall some facts about the product map

$$\times : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

and discuss how it interacts with the boundary classes of Definition 2.8.

We first recall concrete formulas for some of the special cases of this product. See for example [19, Section 4.7] for background on this, and [19, Proposition 4.8.3] for the particular '$K_1 \otimes K_0$' formula that we use.

Fix some standard identification $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \cong M_{nm}(\mathbb{C})$ (compatibly as n and m vary), and use this to identify $M_n(A) \otimes M_m(B)$ with $M_{nm}(A \otimes B)$ for any C^*-algebras A and B. Any two such identifications differ by an inner automorphism, so the choice does not matter on the level of K-theory. We will use these identifications without comment from now on.

We recall a basic lemma that is useful for setting up products in the non-unital case: see [19, Lemma 4.7.2] for a proof.

Lemma 6.1. *For a non-unital C^*-algebra A, let $\epsilon_A : \widetilde{A} \to \mathbb{C}$ denote the canonical quotient map. For non-unital C^*-algebras A and B, define ϕ to be the $*$-homomorphism*

$$(\epsilon_A \otimes id) \oplus (id \otimes \epsilon_B) : \widetilde{A} \otimes \widetilde{B} \to A \oplus B.$$

(where we have identified $A \otimes \mathbb{C}$ with A and similarly for B to make sense of this). Then the map

$$K_*(A \otimes B) \to K_*(\widetilde{A} \otimes \widetilde{B})$$

induced by the canonical inclusion $A \otimes B \to \widetilde{A} \otimes \widetilde{B}$ is an isomorphism onto Kernel(ϕ_*).

A precisely analogous statement holds if A is unital and B is non-unital on replacing ϕ by the canonical quotient map $id \otimes \epsilon_B : A \otimes \widetilde{B} \to A$, and similarly if A is not unital, and B is unital. $\square$

Definition 6.2. Let A and B be unital C^*-algebras, and let $p \in M_n(A)$ and $q \in M_n(B)$ be idempotents. Then the *product* of the corresponding K-theory classes $[p] \in K_0(A)$ and $[q] \in K_0(B)$ is defined to be

$$[p] \times [q] := [p \otimes q] \in K_0(A \otimes B).$$

Similarly, if A and B are unital and $u \in M_n(A)$ is invertible and $p \in M_m(B)$ an idempotent, define

$$u \boxtimes p := u \otimes p + 1 \otimes (1 - p) \in M_{nm}(A \otimes B).$$

Note that $u \boxtimes p$ is invertible, with inverse $u^{-1} \boxtimes p$. The *product* of $[u] \in K_1(A)$ and $[p] \in K_0(B)$ is defined to be

$$[u] \times [p] := [u \boxtimes p] \in K_1(A \otimes B).$$

One checks that these formulas defined on generators extend to well-defined homomorphisms

$$\times : K_0(A) \otimes K_0(B) \to K_0(A \otimes B) \quad \text{and} \quad \times : K_1(A) \otimes K_0(B) \to K_1(A \otimes B).$$

Assume now that A and B are non-unital. Then one checks that for either $(i,j) = (0,0)$, or $(i,j) = (1,1)$, the canonical composition

$$K_i(A) \otimes K_j(B) \to K_i(\widetilde{A}) \otimes K_j(\widetilde{B}) \xrightarrow{\times} K_{i+j}(\widetilde{A} \otimes \widetilde{B})$$

takes image in the subgroup $\mathrm{Kernel}(\phi_*)$ of the right hand side, where ϕ is as in Lemma 6.1 (and similarly if just one of A or B is non-unital, and the other has unit). Using the identification $\mathrm{Kernel}(\phi_*) \cong K_{i+j}(A \otimes B)$ of Lemma 6.1, we thus get a general product map

$$\times : K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes B)$$

if $(i,j) \in \{(1,0),(0,0)\}$.

For the next definition, for any C^*-algebra, let

$$\beta^{-1} : K_*(S^2A) \to K_*(A)$$

be the inverse of the Bott periodicity isomorphism.

Definition 6.3. Let A and B be C^*-algebras. Define

$$K(A) \otimes_1 K(B) := \big(K_1(A) \otimes K_0(B)\big) \oplus \big(K_1(SA) \otimes K_0(SB)\big).$$

Define a ‘product’ map

$$\pi : K(A) \otimes_1 K(B) \to K_1(A \otimes B)$$

to be the composition

$$\begin{aligned} \big(K_1(A) \otimes K_0(B)\big) \oplus \big(K_1(SA) \otimes K_0(SB)\big) &\xrightarrow{\times \oplus \times} K_1(A \otimes B) \oplus K_1(S^2(A \otimes B)) \\ &\xrightarrow{\mathrm{id} \oplus \beta^{-1}} K_1(A \otimes B) \oplus K_1(A \otimes B) \\ &\xrightarrow{\mathrm{add}} K_1(A \otimes B) \end{aligned}$$

We define

$$K(A) \otimes_0 K(B) := \big(K_0(A) \otimes K_0(B)\big) \oplus \big(K_0(SA) \otimes K_0(SB)\big)$$

and

$$\pi : K(A) \otimes_0 K(B) \to K_0(A \otimes B)$$

completely analogously.

Note that naturality of the product map with respect to suspensions and Bott periodicity gives that the map π above identifies with the usual product map

$$\big(K_1(A) \otimes K_0(B)\big) \oplus \big(K_0(A) \otimes K_1(B)\big) \to K_1(A \otimes B)$$

under the usual canonical identifications relating suspensions to dimension shifts in K-theory, and similarly in the K_0 case.

We need a tensor product lemma. Recall that if C, D are C^*-subalgebras of a C^*-algebra A, and if B is another C^*-algebra, then there is a natural inclusion

$$(C \cap D) \otimes B \subseteq (C \otimes B) \cap (D \otimes B).$$

This inclusion need not be an equality above in general: see for example [20]. However, excisive pairs clearly behave well in this setting as in the following lemma.

Lemma 6.4. *Let (C, D) be an excisive pair of C^*-subalgebras as in Definition 5.1. Then the natural inclusion*

$$(C \cap D) \otimes B \subseteq (C \otimes B) \cap (D \otimes B)$$

is the identity.

Proof. Excisiveness implies that the image of the inclusion is dense. The image is a C^*-subalgebra, however, so closed. □

The next lemma is the key technical result of this section. Morally, it can be thought of as saying that if notation is as in Proposition 2.7 and if p an idempotent in some matrix algebra over B, then the diagram

$$\begin{tikzcd} K_1(A) \otimes K_0(B) \arrow[r, "\partial_v"] \arrow[d, "\times"] & K_0(C \cap D) \arrow[d, "\times"] \\ K_1(A \otimes B) \arrow[r, "\partial_{v \boxtimes p}"] & K_0((C \cap D) \otimes B) \end{tikzcd}$$

makes some sort of sense, and commutes, when one inputs the class $[u] \otimes [p] \in K_1(A) \otimes K_0(B)$.

Lemma 6.5. *Let A be a unital C^*-algebra, let $c > 0$, and let $\epsilon \in (0, \frac{1}{4c+6})$. Then there exists $\delta > 0$ satisfying the assumptions of Proposition 2.7, and with the following additional property. Assume that $u \in M_n(A)$ is invertible* *and that $v \in M_{2n}(A)$ is a (δ, c, C, D)-lift for u as in the conclusion of Proposition 2.7. Let B be a C^*-algebra, and let $p \in M_m(B)$ be an idempotent with $\|p\| \leqslant c$.*

Then (with notation as in Definition 6.2) $v \boxtimes p$ is a (ϵ, c, C, D)-lift for $u \boxtimes p$, and we have

$$\partial_v(u) \times [p] = \partial_{v \boxtimes p}(u \boxtimes p)$$

as classes in $K_0((C \cap D) \otimes B)$.

Proof. We leave it to the reader to check that $v \boxtimes p$ is a (ϵ, c, C, D)-lift of $u \boxtimes p$ for suitably small $\delta > 0$ (depending only on ϵ and c). Computing, we see that

$$\begin{aligned}
&\partial_{v \boxtimes p}(u \boxtimes p) \\
&= \Big\{ \big(v \otimes p + 1 \otimes (1-p)\big) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \big(v^{-1} \otimes p + 1 \otimes (1-p)\big) \Big\}_{\widetilde{(C \cap D)} \otimes B} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \Big\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \otimes p + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (1-p) \Big\}_{\widetilde{(C \cap D)} \otimes B} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Using that the two terms inside the curved brackets are orthogonal, we have

$$\begin{aligned}
&\Big\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \otimes p + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (1-p) \Big\}_{\widetilde{(C \cap D)} \otimes B} \\
&\qquad = \Big\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \otimes p \Big\}_{\widetilde{(C \cap D)} \otimes B} + \Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (1-p) \Big].
\end{aligned}$$

As

$$\Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (1-p) \Big] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = -\Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes p \Big],$$

we get that

$$\begin{aligned}
\partial_{v \boxtimes p}(u \boxtimes p) &= \Big\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \otimes p \Big\}_{\widetilde{(C \cap D)} \otimes B} - \Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes p \Big] \\
&= \left(\Big\{ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \Big\}_{\widetilde{C \cap D}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \times [p],
\end{aligned}$$

which is exactly $\partial_v(u) \times [p]$ as claimed. ☐

We also need compatibility results for the maps ι and σ of Definition 2.1 and the maps π of Definition 6.3. These are recorded by the following lemma.

Lemma 6.6. *Let C and D be an excisive pair of C^*-subalgebras of a C^*-algebra A, and let B be a C^*-algebra. Then for $i \in \{0,1\}$, the diagrams*

$$\begin{tikzcd} K(C \cap D) \otimes_i K(B) \arrow[r, "\iota\otimes id"] \arrow[d, "\pi"] & K(C) \otimes_i K(B) \oplus K(D) \otimes_i K(B) \arrow[d, "\pi"] \\ K_i((C \cap D) \otimes B) \arrow[r, "\iota"] & K_1(C \otimes B) \oplus K_1(D \otimes B) \end{tikzcd}$$

and

$$\begin{tikzcd} K(C) \otimes_i K(B) \oplus K(D) \otimes_i K(B) \arrow[r, "\sigma\otimes id"] \arrow[d, "\pi"] & K(A) \otimes_i K(B) \arrow[d, "\pi"] \\ K_i(C \otimes B) \oplus K_i(D \otimes B) \arrow[r, "\sigma"] & K_i(A \otimes B) \end{tikzcd}$$

commute (where we have the canonical identification of Lemma 6.4 amongst others to make sense of this).

Proof. This follows directly from naturality of the product maps and Bott maps in K-theory. □

7 The inverse Bott map

For a C^*-algebra A, let

$$\beta^{-1} : K_*(S^2A) \to K_*(A)$$

be the inverse Bott isomorphism. It will be convenient to have a model for β^{-1} based on an asymptotic family. In this section, we recall some facts about asymptotic families and their action on K-theory (in the 'naive', rather than E-theoretic, picture). We then discuss how the inverse Bott map can be represented by an asymptotic family with good properties.

Recall (see for example [13, Definition 1.3]) that an *asymptotic family* between C^*-algebras A and B is a collection of maps $\{\alpha_t : A \to B\}_{t\in[1,\infty)}$ such that:

(i) for each $a \in A$, the map $t \mapsto \alpha_t(a)$ is continuous and bounded;

(ii) for all $a_1, a_2 \in A$ and $z_1, z_2 \in \mathbb{C}$, the quantities

$$\alpha_t(a_1a_2) - \alpha_t(a_1)\alpha_t(a_2), \quad \alpha_t(a_1^*) - \alpha_t(a_1)^*$$

and

$$\alpha_t(z_1a_1 + z_2a_2) - z_1\alpha_t(a_1) - z_2\alpha_t(a_2)$$

all tend to zero as t tends to infinity.

An asymptotic family $\{\alpha_t : A \to B\}$ defines a $*$-homomorphism

$$\alpha : A \to \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)}, \quad a \mapsto [t \mapsto \alpha_t(a)].$$

Conversely, choosing a continuous section $s : \frac{C_b([1,\infty),B)}{C_0([1,\infty),B)} \to C_b([1,\infty), B)$ (such exists by the Bartle-Graves selection theorem), any such homomorphism α determines an asymptotic family by the formula $\alpha_t(a) := (s(\alpha(a))(t)$. If s and s' are two different choices of section and $\{\alpha_t\}$ and $\{\alpha'_t\}$ the corresponding asymptotic families, then $\alpha_t(a) - \alpha'_t(a) \to 0$ as $t \to \infty$. Compare for example [13, pages 4-5]

We may use this correspondence to define the tensor product of an asymptotic family and a $*$-homomorphism. Say $\{\alpha_t : A \to B\}$ is an asymptotic family, and $\phi : C \to D$ a $*$-homomorphism with D nuclear. Then we get a natural $*$-homomorphism

$$\frac{C_b([1,\infty), B)}{C_0([1,\infty), B)} \otimes D \to \frac{C_b([1,\infty), B \otimes D)}{C_0([1,\infty), B \otimes D)}$$

using that $\cdot \otimes D$ agrees with the maximal tensor product $\cdot \otimes_{\max} D$ (compare [13, Proposition 4.3]), and therefore a $*$-homomorphism

$$\alpha \otimes \phi : A \otimes C \to \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)} \otimes D \to \frac{C_b([1,\infty), B \otimes D)}{C_0([1,\infty), B \otimes D)}.$$

Any choice of corresponding asymptotic family will be denoted $\{\alpha_t \otimes \phi : A \otimes C \to B \otimes D\}$, and any such choice satisfies

$$(\alpha_t \otimes \phi)(a \otimes c) - \alpha_t(c) \otimes \phi(c) \to 0 \quad \text{as} \quad t \to \infty$$

on elementary tensors.

An asymptotic family $\{\alpha_t : A \to B\}_{t \in [1,\infty)}$ canonically defines a map $\alpha_* : K_*(A) \to K_*(B)$. One way to define α_* uses the composition product in E-theory and the identification of $E_*(\mathbb{C}, A)$ with $K_*(A)$. However, there is also a more naive and direct way. This is certainly very well-known, but we are not sure exactly where to point in the literature for a description, so we describe it here for the reader's convenience.

Assume for simplicity that A and B are not unital (this is the only case we will need), and that $\{\alpha_t : A \to B\}$ is an asymptotic family. We extend $\{\alpha_t\}$ to unitisations and matrix algebras just as we would for a $*$-homomorphism. Note that as A and B are not unital, the extended asymptotic morphism on unitisations takes units to units.

If $e \in M_n(\widetilde{A})$ is an idempotent, then $\|\alpha_t(e)^2 - \alpha_t(e)\| \to 0$ as $t \to \infty$. Hence if χ is the characterisitic function of the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1/2\}$ then $\chi(\alpha_t(e))$ (defined using the holomorphic functional calculus) is a well-defined idempotent in $M_n(B)$ for all t suitably large. If $[e] - [f]$ is a formal difference of idempotents in $M_n(\widetilde{A})$ defining a class in $K_0(A)$, then one sees that for all t suitably large the formal difference

$$[\chi(\alpha_t(e))] - [\chi(\alpha_t(f))] \in K_0(\widetilde{B})$$

is in the kernel of the natural map $K_0(\widetilde{B}) \to K_0(\mathbb{C})$ induced by the canonical quotient $\widetilde{B} \to \mathbb{C}$. We define $\alpha_*([e] - [f]) := [\chi(\alpha_t(e))] - [\chi(\alpha_t(f))]$ for any suitably large t. The choice of t does not matter, as for any $t' \geqslant t$, the path $\{\chi(\alpha_s(e))\}_{s \in [t,t']}$ is a homotopy of idempotents, and similarly for f.

Similarly (and more straightforwardly), if $u \in M_n(\widetilde{A})$ is invertible, then as the extension of α_t to unitisations is unital, for all suitably large t, $\alpha_t(u) \in M_n(\widetilde{B})$ is invertible, and we get a well-defined class $\alpha_*[u] := [\alpha_t(u)]$ for any suitably large t. In this way, we get a well-defined homomorphism

$$\alpha_* : K_*(A) \to K_*(B).$$

Lemma 7.1. *Let $c, \epsilon > 0$. Then there is $\delta > 0$ with the following property.*

Let $\{\alpha_t : A \to B\}$ be an asymptotic family between non-unital C^*-algebras, and let (C_A, D_A) be a pair of C^*-subalgebras of A and (C_B, D_B) a pair of C^*-subalgebras of B such that for all $c \in C_A$ and $d \in D_A$,

$$d(\alpha_t(c), C_B) \quad \text{and} \quad d(\alpha_t(d), D_B)$$

tend to zero as t tends to infinity. Assume that $u \in M_{2n}(\widetilde{A})$ is an invertible element with $\|u\| \leqslant c$ and $\|u^{-1}\| \leqslant c$, and let v be a $(\delta/2, c/2, C_A, D_A)$-lift of u. Then for all suitably large t, $\alpha_t(v) \in M_{2n}(\widetilde{B})$ is a (δ, c, C_B, D_B)-lift of $\alpha_t(u)$, and moreover

$$\partial_{\alpha_t(v)}(\alpha_t(u)) = \alpha_*(\partial_v(u))$$

in $K_0(C_B \cap D_B)$ for all suitably large t.

Proof. Note first that as the extension of $\{\alpha_t\}$ to unitisations is unital, and as α_t is asymptotically multiplicaitve, $\alpha_t(u)$ and $\alpha_t(v)$ are invertible for all suitably large t.

We first claim that asymptotic families are 'asymptotically contractive' in the following sense: for any $a \in A$ and any $\epsilon > 0$ we have $\|\alpha_t(a)\| < \|a\| + \epsilon$ for all suitably large t. Indeed, let

$$\alpha : A \to \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)}, \quad a \mapsto [t \mapsto \alpha_t(a)]$$

be the corresponding $*$-homomorphism described above. As α is a $*$-homomorphism, it is contractive. Hence by definition of the quotient norm, for any $\epsilon > 0$ there is $b \in C_0([1,\infty), B)$ such that

$$\sup_{t \in [1,\infty)} \|\alpha_t(a) - b(t)\| < \|\alpha(a)\| + \epsilon \leqslant \|a\| + \epsilon.$$

The claim follows from this.

Now, it follows from the claim and the fact that for all $d \in D_A$, $d(\alpha_t(d), D_B)$ tends to zero as t tends to infinity, that we have that $\alpha_t(v)$ is δ-in $M_{2n}(\widetilde{D_B})$ for all suitably large t. Similarly, and using also the asymptotic multiplicativity and unitality of $\{\alpha_t\}$, we get that

$$\alpha_t(v) \begin{pmatrix} \alpha_t(u)^{-1} & 0 \\ 0 & \alpha_t(u) \end{pmatrix} \in_\delta M_{2n}(\widetilde{C_B})$$

for all suitably large t. The remaining conditions from Definition 2.6 follow similarly.

To see that $\partial_{\alpha_t(v)}(\alpha_t(u)) = \alpha_*(\partial_v(u))$ for t large enough, note that for suitably large t, the former is represented by

$$\left\{\alpha_t(v) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \alpha_t(v)^{-1}\right\}_{\widetilde{C_B \cap D_B}} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

For the latter, one starts by choosing an idempotent $f \in M_{2n}(\widetilde{C_A \cap D_A})$ suitably close to $v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}$ as in Lemma 2.4 so that

$$\left\{v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}\right\}_{\widetilde{C_A \cap D_A}} = [f]$$

in $K_0(\widetilde{C_A \cap D_A})$. Then $\alpha_*(\partial_v(u))$ is represented by

$$\chi(\alpha_t(f)) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

for t suitably large, where χ is as usual the characteristic function of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1/2\}$. Now, as $\|\alpha_t(f)\|$ is uniformly bounded in t and as $\|\alpha_t(f)^2 - \alpha_t(f)\| \to 0$, we may apply Lemma 2.2 to conclude that $\|\alpha_t(f) - \chi(\alpha_t(f))\| \to 0$. On the other hand, by making δ suitably small and t large, and using the 'asymptotic contractiveness' claim at the start of the proof, we can make $\alpha_t(f)$ as close as we like to

$$\alpha_t(v) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \alpha_t(v)^{-1}.$$

Comparing lines (8) and (9), the proof is complete. □

We need the fact that Bott periodicity is induced by an appropriate asymptotic morphism. The following lemma is well-known.

Lemma 7.2. *For any C^*-algebra A there is an associated asymptotic family*

$$\alpha_t : S^2A \rightsquigarrow A \otimes \mathcal{K}$$

with the following properties:

- *(i) the map α_* induced on K-theory by $\{\alpha_t\}$ is the inverse Bott map β^{-1};*
- *(ii) if B is a C^*-subalgebra of A and $\{\alpha_t^A\}$ and $\{\alpha_t^B\}$ are the asymptotic families associated to A and B respectively, then for all $b \in S^2B$, $\alpha_t^A(b) - \alpha_t^B(b) \to 0$ as $t \to \infty$;*
- *(iii) for any finite-dimensional subspace X of A and any element of S^2X,*

  $$\sup\{d(\alpha_t(x), X \otimes \mathcal{K}) \mid x \in S^2X, \|x\| \leqslant 1\}$$

  tends to zero as t tends to infinity;
- *(iv) fixing an inductive limit description $\mathcal{K} = \overline{\bigcup_{n=1}^\infty M_n(\mathbb{C})}$, then for all t and all $a \in S^2A$, $\alpha_t(a)$ has image in the $*$-subalgebra $\bigcup_{n=1}^\infty M_n(A)$ of $A \otimes \mathcal{K}$.*

Proof. There are several different ways to do this. We sketch one from [11] based on the representation theory of the Heisenberg group. As in [11, Section 4], one may canonically construct a continuous field of C^*-algebras over $[0,1]$ with the fibre at 0 equal to $S^2\mathbb{C}$, and all other fibres equal to $\mathcal{K}$. As explained in [8, Appendix 2.B] or [9, pages 101-2], such a deformation (non-canonically) gives rise to an asymptotic family $\{\alpha_t : S^2\mathbb{C} \to A \otimes \mathcal{K}\}$, and this family induces the map on K-theory described in general in [11, Section 3], which is shown in [11, Theorem 4.5] to be the inverse of the Bott periodicity isomorphism.

This gives us our asymptotic family $\{\alpha_t\}$ for the case $A = \mathbb{C}$. In the general case, we may take α_t^A to be a choice of asymptotic family $\{\alpha_t \otimes \mathrm{id}_A : S^2\mathbb{C} \otimes A \to \mathcal{K} \otimes A\}$ described at the start of the section.

Note that the construction is not canonical at two places: going from a deformation to an asymptotic family, and taking the tensor product. However, any two asymptotic families $\{\alpha_t\}$, $\{\alpha'_t\}$ constructed from different choices will satisfy $\alpha_t(a) - \alpha'_t(a) \to 0$ as $t \to \infty$ for all $a \in S^2A$. It follows that the asymptotic families so constructed satisfy (i), (ii), and (iii).

To make it also satisfy (iv), let $\{k_t\}_{t\in[1,\infty)}$ be a continuous family of positive contractions in $\bigcup M_n(A) \subseteq \mathcal{K}$ such that for all $k \in \mathcal{K}$, $k_t k k_t - k \to 0$ as $\to \infty$. For each $a \in A$, choose a homeomorphism $s_a : [1,\infty) \to [1,\infty)$ such that

$$\alpha_t(a) - (1 \otimes k_{s_a(t)})\alpha_t(a)(1 \otimes k_{s_a(t)}) \to 0$$

as $t \to \infty$. Replacing α_t with the map

$$a \mapsto (1 \otimes k_{s_a(t)})\alpha_t(a)(1 \otimes k_{s_a(t)}),$$

we get the result. ∎

8 Surjectivity of the product map

In this section, we are finally ready to prove the first half of Theorem 1.5.

Theorem 8.1. *Let A be a C^*-algebra, and say A strongly excisively decomposes over a class $\mathcal{C}$ of pairs of C^*-subalgebras such that for each $(C,D) \in \mathcal{C}$, C, D, and $C \cap D$ satisfy the Künneth formula. Then for any C^*-algebra B with free abelian K-theory, the product map*

$$\times : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

is surjective.

Proof. It suffices to show that the product maps

$$\pi : K(A) \otimes_0 K(B) \to K_0(A \otimes B) \quad \text{and} \quad \pi : K(A) \otimes_1 K(B) \to K_1(A \otimes B)$$

of Definition 6.3 are surjective for any B with $K_*(B)$ free. Replacing B with its suspension, it moreover suffices to show that the second of the maps above is surjective. Let then α be an arbitrary class in $K_1(A \otimes B)$.

Let $X \subseteq A \otimes B$ and $u \in M_n(\widetilde{A})$ be as in Proposition 3.7 for this α. Using Lemma 5.3, for any $\delta > 0$ there is an f-excisive decomposition $(h \otimes 1, C \otimes B, D \otimes B)$ of X. Fix such a decomposition for a very small $\delta > 0$ (how small will be determined by the rest of the proof).

Using Proposition 3.7 we may build an element $v \in_{\delta_1} M_{2n}(\widetilde{A \otimes B})$ with the properties stated there, for some constant δ_1 that tends to zero as δ tends to zero. We may use v to construct an element $\partial_v u \in K_1((C \cap D) \otimes B)$ as in Proposition 2.7 (here we use the identification $(C \cap D) \otimes B = C \otimes B \cap D \otimes B$ of Lemma 6.4), and have that if

$$\iota : K_0((C \cap D) \otimes B) \to K_0(C \otimes B) \oplus K_0(D \otimes B)$$

is the map from Definition 2.1, then $\iota(\partial_v u) = 0$.

Using that the product map π for $C \cap D$ is surjective, we may lift $\partial_v u$ to an element κ of $K(C \cap D) \otimes_1 K(B)$. With notation as in Definition 6.3, Lemma 6.6 gives that the following diagram

$$\begin{tikzcd}
K(C \cap D) \otimes_0 K(B) \arrow[r, "\iota \otimes \mathrm{id}"] \arrow[d, "\pi"] & K(C) \otimes_0 K(B) \oplus K(D) \otimes_0 K(B) \arrow[d, "\pi"] \\
K_0((C \cap D) \otimes B) \arrow[r, "\iota"] & K_0(C \otimes B) \oplus K_0(D \otimes B)
\end{tikzcd}$$

commutates. Hence

$$\pi((\iota \otimes \mathrm{id})(\kappa)) = \iota(\pi(\kappa)) = \iota(\partial_v u) = 0.$$

Using that the product maps for C and D are injective, this gives us that $(\iota \otimes \mathrm{id})(\kappa) = 0$.

Now, we may write

$$\kappa = \sum_{i=1}^{k} \kappa_i \otimes \lambda_i + \sum_{i=k+1}^{m} \kappa_i \otimes \lambda_i$$

for some $k \leqslant m$, where $\kappa_i \in K_0(C \cap D)$ for $i \leqslant k$, $\kappa_i \in K_0(S(C \cap D))$ for $i > k$, and similarly $\lambda_i \in K_0(B)$ for $i \leqslant k$ and $\lambda_i \in K_0(SB)$ for $i > k$. As $K_*(B)$ is free, we may assume moreover that the set $\{\lambda_1, ..., \lambda_n\}$ generates a free direct summand of $K_0(B) \oplus K_0(SB)$. We then have that

$$(\iota \otimes \mathrm{id})(\kappa) = \sum_{i=1}^{m} \iota(\kappa_i) \otimes \lambda_i = 0,$$

which forces $\iota(\kappa_i) = 0$ for each i by assumption that $\{\lambda_1, ..., \lambda_m\}$ generates a free direct summand of $K_0(B) \oplus K_0(SB)$. Applying Lemma 4.1 to each κ_i separately gives us $l_1, ..., l_m \in \mathbb{N}$ and invertible elements $w_1, ..., w_m$ with

$$w_i \in \begin{cases} M_{l_i}(\widetilde{A}) & i \leqslant k \\ M_{l_i}(\widetilde{SA}) & i > k \end{cases}$$

and corresponding lifts $v_1, ..., v_m$ with

$$v_i \in \begin{cases} M_{2l_i}(\widetilde{A}) & i \leqslant k \\ M_{2l_i}(\widetilde{SA}) & i > k \end{cases}$$

such that $\partial_{v_i}(w_i) = \kappa_i$ and $\partial_{v_i^{-1}}(w_i^{-1}) = -\kappa_i$. It will be important that there is $c > 0$ such that for $i \leqslant k$, each v_i is an (ϵ, c, C, D)-lift of u_i for *any* $\epsilon > 0$, and similarly for $i > k$, with SC and SD in place of C and D.

Now, write $\kappa_i = [p_i] - [q_i]$ for projections p_i and q_i in matrix algebras over $\widetilde{B}$ for $i \leqslant k$, and over $\widetilde{SB}$ for $i > k$. Let $\{\alpha_t : S^2(A \otimes B) \rightsquigarrow A \otimes B \otimes \mathcal{K}\}$ be an asymptotic family inducing the inverse Bott map as in Lemma 7.2. With notation as in Definition 6.2, let us define

$$\begin{aligned}\mathbf{u} := u \oplus (w_1^{-1} \boxtimes p_1) \oplus (w_1 \boxtimes q_1) \oplus \cdots \oplus (w_k^{-1} \boxtimes p_k) \boxplus (w_k \boxtimes q_k) \\ \oplus \alpha_t(w_{k+1}^{-1} \boxtimes p_{k+1}) \oplus \alpha_t(w_{k+1} \boxtimes q_{k+1}) \oplus \cdots \oplus \alpha_t(w_m^{-1} \boxtimes p_m) \oplus \alpha_t(w_m \boxtimes q_m),\end{aligned}$$

and with notation also as in Lemma 4.2 define

$$\begin{aligned}\mathbf{v} := v \boxplus (v_1^{-1} \boxtimes p_1) \boxplus (v_1 \boxtimes q_1) \boxplus \cdots \boxplus (v_k^{-1} \boxtimes p_k) \boxplus (v_k \boxtimes q_k) \\ \alpha_t(v_{k+1}^{-1} \boxtimes p_{k+1}) \boxplus \alpha_t(v_{k+1} \boxtimes q_{k+1}) \boxplus \cdots \boxplus \alpha_t(v_m^{-1} \boxtimes p_m) \boxplus \alpha_t(v_m \boxtimes q_m)\end{aligned}$$

which we can think of as elements of $M_{n_t}(\widetilde{A} \otimes \widetilde{B})$ and $M_{n_t}(\widetilde{A} \otimes \widetilde{B})$ respectively for some $n_t \in \mathbb{N}$ depending on t (recall from Lemma 7.2 that each α_t :

$S^2(A \otimes B) \to A \otimes B \otimes \mathcal{K}$ takes image in $M_{m_t}(A \otimes B)$ for some $m_t \in \mathbb{N}$ depending on t). Then Lemma 4.2 gives that as long as our original δ was sufficiently small, we have

$$\partial_{\mathbf{v}}(\mathbf{u}) = \partial_v(u) + \sum_{i=1}^{k} \partial_{v_i^{-1} \boxtimes p_i}(w_i^{-1} \boxtimes p_i) + \sum_{i=1}^{k} \partial_{v_i \boxtimes q_i}(w_i \boxtimes q_i)$$
$$+ \sum_{i=k+1}^{m} \partial_{\alpha_t(v_i^{-1} \boxtimes p_i)} \alpha_t(w_i^{-1} \boxtimes p_i) + \sum_{i=k+1}^{m} \partial_{\alpha_t(v_i \boxtimes q_i)} \alpha_t(w_i \boxtimes q_i).$$

On the other hand, Lemmas 6.5, 7.1, and 7.2 give that for suitably large t this equals

$$\begin{aligned}
&\partial_v(u) + \sum_{i=1}^{k} \partial_{v_i^{-1} \boxtimes p_i}(w_i^{-1} \boxtimes p_i) + \sum_{i=1}^{k} \partial_{v_i \boxtimes q_i}(w_i \boxtimes q_i) \\
&\qquad + \sum_{i=k+1}^{m} \alpha_*(\partial_{v_i^{-1} \boxtimes p_i}(w_i^{-1} \boxtimes p_i)) + \sum_{i=k+1}^{m} \alpha_*(\partial_{v_i \boxtimes q_i}(w_i \boxtimes q_i)) \\
&= \partial_v(u) + \sum_{i=1}^{k} \partial_{v_i^{-1}}(w_i^{-1}) \times [p_i] + \sum_{i=1}^{k} \partial_{v_i}(w_i) \times [q_i] \\
&\qquad + \sum_{i=k+1}^{m} \alpha_*(\partial_{v_i^{-1}}(w_i^{-1}) \times [p_i]) + \sum_{i=k+1}^{m} \alpha_*(\partial_{v_i}(w_i) \times [q_i]) \\
&= \partial_v(u) + \sum_{i=1}^{k} (-\kappa_i) \times [p_i] + \sum_{i=1}^{k} \kappa_i \times [q_i] \\
&\qquad + \sum_{i=k+1}^{m} \beta^{-1}((-\kappa_i \times [p_i]) + \sum_{i=k+1}^{m} \beta^{-1}(\kappa_i \times [q_i]) \\
&= \partial_v(u) - \sum_{i=1}^{k} \kappa_i \times \lambda_i - \sum_{i=k+1}^{m} \beta^{-1}(\kappa_i \times \lambda_i) \\
&= \partial_v(u) - \pi(\kappa),
\end{aligned}$$

and this is zero.

We have just shown that $\partial_{\mathbf{v}}(\mathbf{u}) = 0$. Noting that $[\mathbf{u}]$ defines a class in $K_1(A \otimes \widetilde{B})$ by Lemma 6.1, it follows at this point from Proposition 2.7 that (as long as the original $\delta > 0$ was suitably small) there exists $\mu \in$

$K_1(C \otimes \widetilde{B}) \oplus K_1(D \otimes \widetilde{B})$ such that $\sigma(\mu) = [\mathbf{u}]$. Moreover, if we define

$$\nu := \sum_{i=1}^{m} [w_i] \otimes [p_i] + \sum_{i=1}^{m} [w_i^{-1}] \otimes [q_i] = \sum_{i=1}^{m} [w_i] \otimes \lambda_i \in K(A) \otimes_1 K(B)$$

then we have by definition of $\mathbf{u}$ that

$$\sigma(\mu) = [\mathbf{u}] = [u] - \pi(\nu).$$

Using surjectivity of the product maps for C and D, and with notation as in Definition 6.3, we may lift μ to some $\xi \in K(C) \otimes_1 K(B) \oplus K(D) \otimes_1 K(B)$. Lemma 6.6 gives commutativity of the diagram

$$\begin{tikzcd} K(C) \otimes_1 K(B) \oplus K(D) \otimes_1 K(B) \arrow[r, "\sigma\otimes\mathrm{id}"] \arrow[d, "\pi"] & K(A) \otimes_1 K(B) \arrow[d, "\pi"] \\ K_1(C \otimes B) \oplus K_1(D \otimes B) \arrow[r, "\sigma"] & K_1(A \otimes B) \end{tikzcd},$$

which implies that

$$[u] = \pi(\nu) + \sigma(\mu) = \pi(\nu) + \sigma(\pi(\xi)) = \pi(\nu) + \pi((\sigma\otimes\mathrm{id})(\xi)) = \pi(\zeta + (\sigma\otimes\mathrm{id})(\xi)),$$

so we have that $[u]$ is in the image of the map π, and are done. □

9 Injectivity of the product map

Finally, in this section we complete the main part of the paper by proving the injectivity half of Theorem 1.5.

Theorem 9.1. *Let A be a C^*-algebra, and say A strongly excisively decomposes over a class $\mathcal{C}$ of pairs of C^*-subalgebras such that for each $(C, D) \in \mathcal{C}$, C, D, and $C \cap D$ satisfy the Künneth formula. Then for any C^*-algebra B with free abelian K-theory, the product map*

$$\times : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

is injective.

Proof. With notation as in Definition 6.3, it suffices to show that the maps

$$\pi : K(A) \otimes_0 K(B) \to K_0(A \otimes B) \quad \text{and} \quad \pi : K(A) \otimes_1 K(B) \to K_1(A \otimes B)$$

defined there are injective for any B with $K_*(B)$ free abelian. On replacing B with its suspension, it suffices just to show injectivity in the K_1 case.

Consider then an element $\kappa \in K(A) \otimes_1 K(B)$ such that $\pi(\kappa) = 0$. To complete the proof, it suffices to show that $\kappa = 0$. Fix a very small $\delta > 0$, to be determined by the rest of the proof.

We may assume κ is of the form

$$\kappa = \sum_{i=1}^{k} \kappa_i \otimes ([p_i] - [q_i]) + \sum_{i=k+1}^{m} \kappa_i \otimes ([p_i] - [q_i]),$$

where for some $n \in \mathbb{N}$, each κ_i is an element of $K_1(A)$ for $i \leqslant k$ or of $K_1(SA)$ for $i > k$, and each pair p_i, q_i consists of projections in $M_n(\widetilde{B})$ for $i < k$ or in $M_n(\widetilde{SB})$ for $i > k$, so that the difference is in $M_n(B)$ or $M_n(SB)$ as appropriate, and so that the collection $([p_i] - [q_i])_{i=1}^{n}$ constitutes part of a basis for the free abelian group $K_0(B) \oplus K_0(SB)$. Using Proposition 3.7, we may assume that for $i \leqslant k$ there is a finite-dimensional subspace $X_{0,i}$ of A and invertible $u_i \in M_n(\widetilde{A})$ with the properties stated there for κ_i and δ; and similarly for each $i > k$, a finite-dimensional subspace $X_{1,i}$ of SA and invertible $u_i \in M_n(\widetilde{SA})$ with the properties stated in Proposition 3.7 with respect to κ_i and δ.

With notation '$\boxtimes$' as in Definition 6.2, and with

$$\{\alpha_t : S^2(\widetilde{A} \otimes \widetilde{B}) \to \widetilde{A} \otimes \widetilde{B} \otimes \mathcal{K}\}$$

an asymptotic family for $\widetilde{A} \otimes \widetilde{B}$ as in Lemma 7.2 that realizes the inverse Bott periodicity isomorphism, define

$$u_t := \bigoplus_{i=1}^{k} u_i \boxtimes p_i \oplus \bigoplus_{i=1}^{k} u_i^{-1} \boxtimes q_i + \bigoplus_{i=k+1}^{m} \alpha_t(u_i \boxtimes p_i) \oplus \bigoplus_{i=k+1}^{m} \alpha_t(u_i^{-1} \boxtimes q_i). \quad (10)$$

Then for all t suitably large, $[u_t]$ defines a class in $K_1(\widetilde{A} \otimes \widetilde{B})$, which we may consider as a class in $K_1(A \otimes B)$ thanks to Lemma 6.1.

By definition of π, there is $t_0 \in [1, \infty)$ such that $\pi(\kappa) = [u_t]$ for all $t \geqslant t_0$, and so that the map

$$[t_0, \infty) \to \bigcup_{n=1}^{\infty} M_n(A), \quad t \mapsto u_t$$

is a continuous path of invertibles. As $[u_{t_0}] = \pi(\kappa) = 0$, we may assume moreover that there exist $l, p \in \mathbb{N}$ and a homotopy $\{w_s\}_{s\in[0,1]}$ of invertible elements in $M_p(\widetilde{A} \otimes \widetilde{B})$ such that $w_0 \oplus 1_l = u_{t_0}$, such that $w_1 = 1_p$, such that each w_s and w_s^{-1} are in $\{1 + x \in M_p(\widetilde{A} \otimes \widetilde{B}) \mid x \in M_p(A \otimes \widetilde{B})\}$. Let X_3 be a finite-dimensional subspace of $A \otimes \widetilde{B}$ such that all w_s and w_s^{-1} are δ-in $\{1 + x \in M_p(\widetilde{A} \otimes \widetilde{B}) \mid x \in M_p(X_3)\}$. Using part (iii) of Lemma 7.2, there is moreover a finite-dimensional subspace X_4 of $A \otimes \widetilde{B}$ such that for all $t \geqslant t_0$ there exists $n_t \in \mathbb{N}$ such that u_t and u_t^{-1} are δ-in $M_{n_t}(X_4)$.

Now, using Lemma 5.3, for any $\delta > 0$ there exists an f-excisive decomposition (h, C, D) such that $(1 \otimes h, SC, SD)$ is also an f-excisive δ-decomposition of X_1, and such that $(h \otimes 1, C \otimes \widetilde{B}, D \otimes \widetilde{B})$ is an f-excisive decomposition of both X_3 and X_4. If δ is small enough, Proposition 3.7 then lets us build for each i an invertible element v_i in either $M_{2n}(\widetilde{A})$ or $M_{2n}(\widetilde{SA})$ as appropriate, and with the properties stated there relative to each u_i and also so that v_i^{-1} has the relevant properties for u_i^{-1} as in Lemma 4.3. With notation as in Lemma 4.2, define also

$$v := \boxplus_{i=1}^{k} (v_i \boxtimes p_i) \boxplus \boxplus_{i=k}^{n} (v_i^{-1} \boxtimes q_i) \quad \text{and} \quad v_S := \boxplus_{i=k+1}^{m} (v_i \boxtimes p_i) \boxplus \boxplus_{i=k+1}^{m} (v_i^{-1} \boxtimes q_i)$$

elements of some matrix algebra over $\widetilde{A} \otimes \widetilde{B}$ and $\widetilde{SA} \otimes \widetilde{SB}$ respectively. Define also

$$u := \bigoplus_{i=1}^{k} u_i \boxtimes p_i \oplus \bigoplus_{i=1}^{k} u_i^{-1} \boxtimes q_i$$

and

$$u_S := \bigoplus_{i=k+1}^{m} u_i \boxtimes p_i \oplus \bigoplus_{i=k+1}^{m} u_i^{-1} \boxtimes q_i.$$

Then as long as $\delta > 0$, Lemmas 4.2 and 6.5 give boundary classes $\partial_v u \in K_0((C \cap D) \otimes \widetilde{B})$ and $\partial_{v_S}(u_S) \in K_0(S(C \cap D) \otimes \widetilde{SB})$.

Now, with β^{-1} the inverse Bott periodicity map, the element

$$(\mathrm{id} \oplus \beta^{-1})(\partial_v u, \partial_{v_S}(u_S)) \in K_0((C \cap D) \otimes \widetilde{B})$$

is necessarily zero. Indeed, using Lemmas 7.1 and 4.2, this element is represented by

$$\partial_v u + \partial_{\alpha_t v_S}(\alpha_t(u_S))) = \partial_{v \boxplus \alpha_t(v_S)}(u \oplus \alpha_t(u_S))$$

for suitably large t. With notation as in line (10), this equals $\partial_{v \boxplus \alpha_t(v_S)}(u_t)$. Now, we can drag a homotopy between u_t and 1 through the construction of Proposition 3.7 to produce a homotopy between this element and 1 (this uses our choice of (h, C, D), and the fact that there is a homotopy through invertibles between u_t and 1 that is close to $(1_{n_t} + M_{n_t}(X_3)) \cup (1_{n_t} + M_{n_t}(X_4))$ for some appropriate $n_t \in \mathbb{N}$).

Lemmas 6.5 and 4.2 give then that

$$\pi\left(\sum_{i=1}^{m} \partial_{v_i}(u_i) \otimes ([p_i] - [q_i])\right) = (\mathrm{id} \oplus \beta^{-1})(\partial_v u, \partial_{v_S}(u_S))$$

whence the class

$$\pi\left(\sum_{i=1}^{m} \partial_{v_i}(u_i) \otimes ([p_i] - [q_i])\right) \in K_0((C \cap D) \otimes \widetilde{B})$$

is zero also. Hence by injectivity of the product map for $C \cap D$, we have that

$$\sum_{i=1}^{m} \partial_{v_i}(u_i) \otimes ([p_i] - [q_i])$$

is zero in $K(C \cap D) \otimes_0 K(B)$. Using the assumption that the collection $([p_i]-[q_i])_{i=1}^{m}$ forms part of a basis for $K_0(B) \oplus K_0(SB)$, we get that $\partial_{v_i}(u_i) = 0$ in $K_0(C \cap D) \oplus K_0(S(C \cap D))$ for each i. Hence Proposition 2.7 gives us $j, l \in \mathbb{N}$ and invertible elements s_i in either $M_{j+l}(\widetilde{D})$ or $M_{j+l}(\widetilde{SD})$ as appropriate such that for each i we have that $(u_i \oplus 1_l)s_i^{-1}$ is in either $M_{j+l}(\widetilde{C})$ or $M_{j+l}(\widetilde{SC})$ as appropriate. Applying the same reasoning with the roles of u_i and u_i^{-1} interchanged, we similarly get invertible elements t_i in either $M_{j+l}(\widetilde{D})$ of $M_{j+l}(\widetilde{SD})$ as appropriate such that for each i, we have that $(u_i^{-1} \oplus 1_l)t_i^{-1}$ is in either $M_{j+l}(\widetilde{C})$ or $M_{j+l}(\widetilde{SC})$ as appropriate.

Now, consider the class $\lambda \in \big(K(C) \otimes_1 K(B)\big) \oplus \big(K(D) \otimes_1 K(B)\big)$ defined by $\lambda = (\lambda_C, \lambda_D)$ where

$$\lambda_C := \sum_{i=1}^{m} [(u_i \oplus 1_l)s_i^{-1}] \otimes [p_i] + \sum_{i=1}^{m} [(u_i^{-1} \oplus 1_l)t_i^{-1}] \otimes [q_i]$$

and

$$\lambda_D := \sum_{i=1}^{m} [s_i] \otimes [p_i] + \sum_{i=1}^{m} [t_i] \otimes [q_i],$$

and note that $\kappa = \sigma(\lambda)$. The image of λ under the product map

$$\times : K(C) \otimes_1 K(B) \oplus K(D) \otimes_1 K(B) \\ \to \big(K_1(C \otimes B) \oplus K_1(SC \otimes SB)\big) \oplus \big(K_1(D \otimes B) \oplus K_1(SD \otimes SB)\big)$$

is represented by the invertible element

$$x := \Bigg(\bigoplus_{i=1}^{m} \big((u_i \oplus 1_l)s_i^{-1} \boxtimes p_i\big) \oplus \bigoplus_{i=1}^{m} \big((u_i^{-1} \oplus 1_l)t_i^{-1} \boxtimes q_i\big) \ , \\ \bigoplus_{i=1}^{m} \big(s_i \boxtimes p_i\big) \oplus \bigoplus_{i=1}^{m} \big(t_i \boxtimes q_i\big) \Bigg).$$

We have that $\pi(\lambda)$ equals the image of the class above under the map

$$\mathrm{id} \oplus \beta^{-1} : \big(K_1(C \otimes B) \oplus K_1(SC \otimes SB)\big) \oplus \big(K_1(D \otimes B) \oplus K_1(SD \otimes SB)\big) \\ \to K_1(C \otimes B) \oplus K_1(D \otimes B),$$

which, with notation as in Lemma 7.2, is represented concretely by the invertible element $(\mathrm{id} \oplus \alpha_t)(x)$ for all suitably large t. On the other hand, using almost multiplicativity of the asymptotic family $\{\alpha_t\}$ and comparing this with the formula for u_t in line (10), we see that u_t can be made arbitrarily close to

$$(\mathrm{id} \oplus \alpha_t)\Bigg(\Big(\bigoplus_{i=1}^{m} \big((u_i \oplus 1_l)s_i^{-1} \boxtimes p_i\big) \oplus \bigoplus_{i=1}^{m} \big((u_i^{-1} \oplus 1_l)t_i^{-1} \boxtimes q_i\big) \Big) \\ \cdot \ \Big(\bigoplus_{i=1}^{m} \big(s_i \boxtimes p_i\big) \oplus \bigoplus_{i=1}^{m} \big(t_i \boxtimes q_i\big) \Big) \Bigg)$$

by increasing t (up to taking block sum with 1_q for some q depending on t).

Using the fact that for each fixed t there is $n_t \in \mathbb{N}$ such that, u_t is homotopic to the identity through invertibles that are δ-in

$$\{1 + x \in M_{n_t}(A \otimes \widetilde{B}) \mid x \in M_{n_t}(X_3) \cup M_{n_t}(X_4)\}$$

via the concatenation of the homotopies $\{u_s\}_{s \in [t_0,t]}$ and $\{w_s \oplus 1_{n_t - p}\}_{s \in [0,1]}$ and our assumption on (h, C, D), we are thus in a position to apply Proposition 5.6 to conclude that there exists a class $\mu \in K_1((C \cap D) \otimes \widetilde{B})$ such that

$\iota(\mu) = \pi(\lambda)$. Using surjectivity of the product map for $C \cap D$, we may lift μ to some element ν of $K(C \cap D) \otimes_1 K(\widetilde{B})$. Using Lemma 6.6, we have that

$$\pi(\lambda) = \iota(\mu) = \iota(\pi(\nu)) = \pi(\iota(\nu)).$$

Hence by injectivity of the product maps for C and D, this forces $\lambda = \iota(\nu)$. Finally, we have that $\kappa = \sigma(\lambda)$ and so

$$\kappa = \sigma(\lambda) = \sigma(\iota(\nu)).$$

However, $\sigma \circ \iota$ is clearly the zero map on K-theory, so we are done. □

A Nuclear dimension

In this appendix, we give examples of (a priori, non-excisive) decompositions coming from nuclear dimension one as in [35].

For the statement of the next result, if A is a C^*-algebra, let A_∞ denote the quotient $\prod_{\mathbb{N}} A / \oplus_{\mathbb{N}} A$ of the product of countably many copies of A by the direct sum. If (B_n) is a sequence of C^*-suablgebras of A, we let B_∞ denote the C^*-subalgebra $\prod_{\mathbb{N}} B_n / \bigoplus_{\mathbb{N}} B_n$ of A_∞.

The following fact was told to me by Wilhelm Winter[8].

Proposition A.1. *Let A be a separable[9] unital C^*-algebra of nuclear dimension one. Then there exist*

(i) a positive contraction $h \in A_\infty \cap A'$, and

(ii) sequences (C_n), (D_n) of C^*-subalgebras of A

such that:

1. *each C_n and each D_n is a quotient of a cone over a finite-dimensional C^*-algebra,*

2. *for all $a \in A$, $ha \in C_\infty$, $(1-h)a \in D_\infty$,*

Proof. Using [35, Theorem 3.2] (and that A is separable) there exists a sequence (ψ_n, ϕ_n, F_n) where:

[8] Professor Winter probably knows a better proof!

[9] Not really necessary, but the statement would be a little fiddlier otherwise.

(i) each F_n is a finite-dimensional C^*-algebra that decomposes as a direct sum $F_n = F_n^{(0)} \oplus F_n^{(1)}$;

(ii) each ψ_n is a ccp map $A \to F_n$ such that the induced diagonal map

$$\overline{\psi} : A \to F_\infty$$

is order zero;

(iii) each ϕ_n is a map $F_n \to A$ such that the restriction $\phi_n^{(i)}$ of ϕ_n to $F_n^{(i)}$ is ccp and order zero;

(iv) for each $a \in A$, $\phi_n \psi_n(a) \to a$ as $n \to \infty$.

Let $\overline{\phi} : F_\infty \to A_\infty$, and $\overline{\phi^{(i)}} : F_\infty^{(i)} \to A_\infty$ denote the induced maps, let $\kappa^{(i)} : F_\infty \to F_\infty^{(i)}$ denote the canonical quotient, and consider the composition

$$\theta^{(i)} := \overline{\phi^{(i)}} \circ \kappa^{(i)} \circ \overline{\psi} : A \to A_\infty.$$

Each $\theta^{(i)}$ is then ccp and order zero, and we have moreover that $\theta^{(0)} + \theta^{(1)} : A \to A_\infty$ agrees with the canonical diagonal inclusion.

Now, let $M_i := M(C^*(\theta^{(i)}(A)))$ be the multiplier algebra of the C^*-subalgebra $C^*(\theta^{(i)}(A))$ of A_∞ generated by $\theta^{(i)}(A)$. Using [34, Theorem 2.3] if we set $h_i := \theta^{(i)}(1)$, then h_i is a positive contraction in $C^*(\theta^{(i)}(A)) \cap A'$, and there exists a unital[10] $*$-homomorphism $\pi^{(i)} : A \to M_i \cap \{h_i\}'$ such that

$$\theta^{(i)}(a) = h_i \pi^{(i)}(a)$$

for all $a \in A$. As $1 = \theta^{(0)}(1) + \theta^{(1)}(1) = h_1 + h_2$, we will switch notation and write $h := h_1$, so $1 - h = h_2$, so for all $a \in A$,

$$a = h\pi^{(0)}(a) + (1-h)\pi^{(1)}(a). \tag{11}$$

Note in particular that h commutes with both $\theta^{(1)}(A)$ (as $h = h_1$ and h_1 commutes with this collection), and with $\theta^{(2)}(A)$ (as $1 - h = h_2$, and h_2 commutes with this collection). Hence h commutes with $\theta^{(1)}(A) + \theta^{(2)}(A) \supseteq A$, so in particular h is in $A_\infty \cap A'$.

[10]Unitality follows from the proof in the given reference, but does not appear explicitly in the statement.

Now, let us think of $\pi^{(i)} : A \to M_i$ as having image in the double dual $(A_\infty)^{**}$ by postcomposing with the canonical embedding $M_i \to (A_\infty)^{**}$. Let us replace $\pi^{(i)}$ with the map

$$a \mapsto \chi_{[0,1]\setminus\{i\}}(h)\pi^{(i)}(a) + \chi_{\{i\}}(h)a. \tag{12}$$

Then the equation in line (11) still holds for all $a \in A$. Let B be the unital C^*-algebra generated by h, A, $\pi^{(0)}(A)$ and $\pi^{(1)}(A)$, and note that h is central in B. For each $\lambda \in [0,1]$ in the spectrum of h in $C^*(h,1)$, let I_λ be the C^*-ideal in B generated by the corresponding maximal ideal in $C^*(h,1)$ (with $I_\lambda = B$ if λ is not in the spectrum of h). Then in B/I_λ, the equation in line (11) descends to

$$a = \lambda\pi^{(0)}(a) + (1-\lambda)\pi^{(1)}(a).$$

If $\lambda \in (0,1)$ and $a = u \in A$ is unitary, this writes the image of u in B/I_λ as a convex combination of two elements in the unit ball; as unitaries are always extreme points in the unit ball of a C^*-algebra [3, Theorem II.3.2.17], this is impossible unless $\pi^{(0)}(u) = \pi^{(1)}(u) = u$ modulo I_λ for all $\lambda \in (0,1)$. As the unitaries span any unital C^*-algebra [3, Proposition II.3.2.12], this forces $\pi^{(0)}(a) = \pi^{(1)}(a) = a$ modulo I_λ for all $a \in A$ and all $\lambda \in (0,1)$. On the other hand, if $\lambda = 0$, we clearly get $\pi^{(1)}(a) = a$ modulo I_0 for all $a \in A$, while $\pi^{(0)}(a) = a$ modulo I_0 follows from the replacement we made in line (12). Similarly, if $\lambda = 0$, we also get that $\pi^{(0)}(a) = a$ and $\pi^{(1)}(a) = a$ modulo I_1. Putting this together, we have that the postcomposition of either $\pi^{(0)}$ or $\pi^{(1)}$ with the natural diagonal $*$-homomorphism

$$\Phi : B \to \prod_{\lambda\in\text{spectrum}(h)} B/I_\lambda$$

agrees with the natural map $A \to \prod_{\lambda\in[0,1]} B/I_\lambda$ induced by the inclusion $A \to B$. However, as $C^*(h,1)$ is contained in the center of B, the map Φ is injective by [10, Theorem 7.4.2]. Hence we get that both $\pi^{(0)}$ and $\pi^{(1)}$ agree with the diagonal inclusion $A \to A_\infty$, and thus have the equations

$$\theta^{(0)}(a) = ha \quad \text{and} \quad \theta^{(1)}(a) = (1-h)a$$

for all $a \in A$.

To complete the proof, therefore, we need to find sequences (C_n) and (D_n) of C^*-subalgebras of A with the right properties. For each n and each $i \in \{0,1\}$, consider $\phi_n^{(i)} : F_n^{(i)} \to A$. As this is order zero, [34, Corollary

3.1] gives a $*$-homomorphism $\rho_n^{(i)} : C_0(0,1] \otimes F_n^{(i)} \to A$ such that $\phi_n^{(i)}(b) = \rho_n^{(i)}(x \otimes b)$ for all $b \in A$, where $x \in C_0(0,1]$ is the identity function. Set $C_n := \rho_n^{(0)}(C_0(0,1] \otimes F_n^{(0)})$ and $D_n = \rho_n^{(1)}(C_0(0,1] \otimes F_n^{(0)})$, which contain the images of $\phi_n^{(0)}$ and $\phi_n^{(1)}$ respectively. It is straightforward to check that (C_n) and (D_n) have the right properties. $\square$

The following corollary is reasonably straightforward by lifting the element $h \in A_\infty$ to a positive contraction $(h_n) \in \prod_n A$: we leave the details to the reader.

Corollary A.2. *Let A be a separable C^*-algebra of nuclear dimension one, and let $\mathcal{C}$ be the class of pairs (C, D) of C^*-subalgebras of A such that each of C and D is isomorphic to a quotient of a cone over a finite dimensional C^*-algebra. Then A decomposes over $\mathcal{C}$.* $\square$

B Finite dynamical complexity

In this appendix, we give examples of excisive decompositions coming from decompositions of groupoids as introduced in [16]. Our conventions on groupoids will be as in [16, Appendix A] and [26, Section 2.3].

The following is a slight variant of [16, Definition A.4].

Definition B.1. Let G be a locally compact, Hausdorff, étale groupoid, let H be an open subgroupoid of G, and let $\mathcal{C}$ be a set of open subgroupoids of G. We say that H is *decomposable* over $\mathcal{C}$ if for any open, relatively compact subset K of H there exists an open cover $H^{(0)} = U_0 \cup U_1$ of the unit space of H such that for each $i \in \{0, 1\}$ the subgroupoid of H generated by

$$\{h \in K \mid s(h) \in U_i\}$$

is contained in an element of $\mathcal{C}$.

The first technical result of this section is as follows. See Definitions 1.1 and 1.4 for terminology.

Proposition B.2. *Say G is a second countable, locally compact, Hausdorff étale groupoid that that decomposes over a set $\mathcal{D}$ of open subgroupoids of G. Then the reduced groupoid C^*-algebra $C_r^*(G)$ decomposes over the class of pairs*

$$\mathcal{C} := \{(C_r^*(H_1), C_r^*(H_2)) \mid H_1, H_2 \in \mathcal{D}\}.$$

Moreover, if every groupoid in $\mathcal{D}$ is clopen, then $\mathcal{C}$ is excisive.

The proof will proceed via some lemmas. First we give the existence of decompositions.

Lemma B.3. *Say G is a locally compact, Hausdorff étale groupoid that decomposes over a set $\mathcal{D}$ of subgroupoids of G in the sense of Definition B.1. Then the reduced groupoid C^*-algebra $C_r^*(G)$ decomposes over the class of pairs*

$$\mathcal{C} := \{(C_r^*(H_1), C_r^*(H_2)) \mid H_1, H_2 \in \mathcal{D}\}.$$

Proof. Let X be a finite-dimensional subspace of $C_r^*(G)$. Up to an approximation, we may assume that there is an open relatively compact subset K of G such that every element of X is an element of $C_c(G)$ supported in K. Using (a very slight variation on) [16, Lemma A.12], for any $\epsilon > 0$, there is an open cover $G^{(0)} = U_0 \cup U_1$ of the base space of G and a pair of continuous compactly supported functions $\{\phi_0, \phi_1 : G^{(0)} \to [0,1]\}$ with the following properties.

(i) each ϕ_i is supported in U_i;

(ii) for each $i \in \{0,1\}$, the set $\{k \in K \mid r(k) \in U_i\}$ generates an open subgroupoid of G that is contained in some element H_i of $\mathcal{D}$;

(iii) for each $x \in G^{(0)}$, $\phi_0(x) + \phi_1(x) = 1$ and for each $k \in K$, $\phi_0(r(k)) + \phi_1(r(k)) = 1$;

(iv) for any $k \in K$ and $i \in \{0,1\}$, $|\phi_i(s(k)) - \phi_i(r(k))| < \epsilon$.

We claim that for any $\delta > 0$, for ϵ suitably small, $(h, C, D) = (\phi_0, C_r^*(H_1), C_r^*(H_2))$ is a δ-decomposition.

Indeed, the fact that $\|[h,a] \leqslant \delta\|a\|$ for all $a \in X$ follows from condition (iv) above and [17, Lemma 8.20]. We have moreover that for any $a \in X$, $ha = \phi_0 a$, and this is supported in $\{k \in K \mid r(k) \in U_0\}$ by condition (i), whence is in $C_r^*(H_0)$ by condition (ii). On the other hand, $(1-h)a = \phi_1 a$ for any $a \in X$ by condition (iii), whence $(1-h)a$ is in $C_r^*(H_1)$ by the same argument. □

The next lemma is presumably well-known.

Lemma B.4. *Let G be a locally compact, Hausdorff, étale groupoid, and let $H \subseteq G$ be a clopen subgroupoid. Then the restriction map $E : C_c(G) \to C_c(H)$ extends to a conditional expectation $E : C_r^*(G) \to C_r^*(H)$.*

Proof. For $x \in H^{(0)}$, let $\pi_x : C_r^*(H) \to \mathcal{B}(\ell^2(H_x))$ be the associated regular representation defined by

$$(\pi_x(b)\xi)(h) := \sum_{k \in H_x} b(hk^{-1})\xi(h)$$

as in [26, Section 2.3.4]. Let $\xi, \eta \in \ell^2(H_x)$, and consider

$$\langle \xi, \pi_x(E(a))\eta \rangle_{\ell^2(H_x)} = \sum_{h,k \in H_x} E(a)(hk^{-1})\eta(k)\overline{\xi(h)} = \sum_{h,k \in G_x} a(hk^{-1})\tilde{\eta}(k)\overline{\tilde{\xi}(h)}$$

where $\tilde{\xi} \in \ell^2(G_x)$ is the function defined by extending ξ by zero on $G_x \backslash H_x$, and the second equality uses that H is a subgroupoid to deduce that if $h, k \in H$, then hk^{-1} is in H. Hence if $|pi_x^G$ is the corresponding representation of G on $\ell^2(G_x)$, we have

$$\langle \xi, \pi_x(E(a))\eta \rangle_{\ell^2(H_x)} = \langle \tilde{\xi}, \pi_x^G(a)\tilde{\eta} \rangle,$$

and so

$$\|E(a)\| = \sup_{\|\xi\|=\|\eta\|=1} |\langle \xi, \pi_x(E(a))\eta \rangle_{\ell^2(H_x)}| = \sup_{\|\xi\|=\|\eta\|=1} |\langle \tilde{\xi}, \pi_x^G(a)\tilde{\eta} \rangle| \leqslant \|a\|.$$

Hence E is contractive, and so in particular extends to an idempotent linear contraction $E : C_r^*(G) \to C_r^*(H)$. This extended map is necessarily a contraction by a classical theorem of Tomiyama: see for example [6, Theorem 1.5.10]. □

Lemma B.5. *Say G is a locally compact, Hausdorff, étale groupoid. Then the set of pairs of C^*-subalgebras of $C_r^*(G)$ of the form $(C_r^*(H_1), C_r^*(H_2))$ with $H_1, H_2 \subseteq G$ both open subgroupoids, and at least one of them also closed, is strongly excisive as in Definition 1.4.*

Proof. Say B is an arbitrary C^*-algebra, and consider $c \in C_r^*(H_1) \otimes B$ and $d \in C_r^*(H_2) \otimes B$. Say without loss of generality that H_2 is closed, and let $E : C_r^*(G) \to C_r^*(H_2)$ be the conditional expectation of Lemma B.4. As E is just defined on $C_c(G)$ by restriction of functions, it follows that E takes $C_r^*(H_1)$ into itself, and therefore into $C_r^*(H_1) \cap C_r^*(H_2)$. Hence by functoriality of tensor product maps, we see that $E \otimes \mathrm{id}$ restricted to $C_r^*(H_1) \otimes B$ is a map

$$E \otimes \mathrm{id} : C_r^*(H_1) \otimes B \to (C_r^*(H_1) \cap C_r^*(H_2)) \otimes B$$

and in particular $(E \otimes \mathrm{id})(d)$ is in $(C_r^*(H_1) \cap C_r^*(H_2)) \otimes B$. On the other hand, as $E \otimes \mathrm{id}$ is contractive (see for example [6, Theorem 3.5.3]) and takes $C_r^*(H_1)$ to itself, so we get that

$$\|c - (E \otimes \mathrm{id})(d)\| = \|(E \otimes \mathrm{id})(c - d)\| \leqslant \|c - d\|$$

and

$$\|d - (E \otimes \mathrm{id})(d)\| \leqslant \|c - d\| + \|c - (E \otimes \mathrm{id})(d)\| \leqslant 2\|c - d\|$$

so we are done. □

Proposition B.2 now follows directly from Lemmas B.3, B.4, and B.5.

We spend the rest of this appendix deriving some consequences of Proposition B.2.

Corollary B.6. *Say G is an ample second countable, locally compact, Hausdorff étale groupoid. Let $\mathcal{K}$ be the class of clopen subgroupoids of G, such that for any $H \in \mathcal{K}$, and any clopen subgroupoid K of $\mathcal{H}$, $C_r^*(K)$ satisfies the Künneth formula. Then $\mathcal{K}$ is closed under decomposability.*

Proof. Say H is a clopen subgroupoid of G that decomposes over $\mathcal{K}$. Then $C_r^*(H)$ strongly excisively decomposes over the class $\{(C_r^*(K_1), C_r^*(K_2)) \mid K_1, K_2 \in \mathcal{K}\}$ by Proposition B.2, and so $C_r^*(H)$ satisfies Künneth by Theorem 1.5. The same argument also applies to any clopen subgroupoid of H: indeed, any clopen subgroupoid of H is easily seen to also decompose over $\mathcal{K}$ (compare the proof of [16, Lemma 3.16]). □

Finally, we finish with an example that is closely related to the notion of finite dynamical complexity for groupoids introduced in [16, Definition A.4]

Definition B.7. Say G is an ample, locally compact, Hausdorff étale groupoid with finite dynamical complexity. Let $\mathcal{C}$ be the class of compact open subgroupoids of G, and let $\mathcal{D}$ be the smallest class of clopen subgroupoids of G containing $\mathcal{C}$ and closed under decomposability. Then G has *strong finite dynamical complexity* if G itself is contained in $\mathcal{D}$.

The following result is not new: groupoids as in the statement are amenable by [16, Theorem A.9], and therefore their C^*-algebras satisfy the UCT by a famous result of Tu [32, Proposition 10.7] (at least in the second countable case). Nonetheless, it seems interesting to give a relatively direct proof based on the internal structure of the C^*-algebra.

Theorem B.8. *Say G is a principal, locally compact, Hausdorff étale groupoid with strong finite dynamical complexity. Then $C_r^*(G)$ satisfies the Künneth formula.*

Proof. Let $\mathcal{K}$ be as in Corollary B.6, and let $\mathcal{C}$ be the class of compact open subgroupoids of $\mathcal{C}$. Then for any $H \in \mathcal{K}$, the reduced C^*-algebra $C_r^*(H)$ is principal and proper, so Morita equivalent to the continuous functions $C(H^{(0)}/H)$ on the orbit space by [23, Example 2.5 and Theorem 2.8] (the second countability assumptions in that paper are not necessary in the étale case [12]). Hence $C_r^*(H)$ satisfies the Künneth formula. As $\mathcal{C}$ is closed under taking clopen subgroupoids, $\mathcal{K}$ contains $\mathcal{C}$.

Hence if $\mathcal{D}$ is as in Definition B.7, then $\mathcal{K}$ contains $\mathcal{D}$ by Corollary B.6. However, strong finite dynamical complexity implies that G itself is in $\mathcal{D}$, so we are done. □

Example B.9. Let X be a bounded geometry metric space, and assume that X has finite decomposition complexity as introduced in [14] and studied in [15]. Then the associated coarse groupoid $G(X)$ has strong finite dynamical complexity by the proof of [16, Theorem A.4]. Hence the associated groupoid C^*-algebra $C_r^*(G(X))$, which canonically identifies with the uniform Roe algebra $C_u^*(X)$, satisfies the Künneth formula by Theorem B.8.

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