DYNAMIC ASYMPTOTIC DIMENSION AND MATUI'S HK CONJECTURE

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Abstract. We prove that the homology groups of a principal ample groupoid vanish in dimensions greater than the dynamic asymptotic dimension of the groupoid (as a side-effect of our methods, we also give a new model of groupoid homology in terms of the Tor groups of homological algebra, which might be of independent interest). As a consequence, the K-theory of the C^*-algebras associated with groupoids of finite dynamic asymptotic dimension can be computed from the homology of the underlying groupoid. In particular, principal ample groupoids with dynamic asymptotic dimension at most two and finitely generated second homology satisfy Matui's HK-conjecture.

We also construct explicit maps from the groupoid homology groups to the K-theory groups of their C^*-algebras in degrees zero and one, and investigate their properties.

Contents

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1. Introduction

Dynamic asymptotic dimension is a notion of dimension for actions of discrete groups on locally compact spaces, and more generally, for locally compact étale groupoids introduced by the last named author with Guentner and Yu in [24]. It is inspired by Gromov's theory of asymptotic dimension [22, Section 1.3]. At the same time it is strongly connected to other existing dimension theories for dynamical systems, for example the conditions introduced by Bartels, Lück and Reich [3] or Kerr's tower dimension [28].

The original article [24] focused on the fine structure of C^*-algebras associated with étale groupoids of finite dynamic asymptotic dimension, while later work by the same set of authors in [23] presented some consequences to K-theory and topology.

In the present work we aim to explore the implications of dynamic asymptotic dimension for groupoid homology and its relation to the K-theory of groupoid C^*-algebras. A homology theory for étale groupoids was introduced by Crainic and Moerdijk in [11]. More recently, groupoid homology attracted a considerable amount of interest from the topological dynamics and operator algebras communities following the work of Matui [34]. The main contribution of this article is the following:

Theorem A. Let G be a locally compact, Hausdorff, étale, principal, σ-compact, ample groupoid with dynamic asymptotic dimension at most d. Then $H_n(G) = 0$ for $n > d$ and $H_d(G)$ is torsion-free.

Our proof of Theorem A goes via a description of groupoid homology in terms of semi-simplicial spaces equipped with a G-action. As a byproduct this leads to a description of these homology groups in terms of the classical Tor groups of homological algebra, quite analogous to the well-known case of the homology of a discrete group. While this may be known to experts, it seems worthwhile recording it as we are not aware of its appearance in the literature.

Theorem B. Let G be a locally compact, Hausdorff, étale, ample groupoid with σ-compact base space. There is a canonical isomorphism $H_*(G) \cong \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$.

Our result allows us to draw some significant consequences to the following conjecture formulated by Matui in [36].

Conjecture (Matui)**.** *For a minimal, essentially principal, ample groupoid G there are isomorphisms*

$$K_i(C_r^*(G)) \cong \bigoplus_{n \geqslant 0} H_{2n+i}(G), \quad i = 0, 1$$

The conjecture has been confirmed in several interesting cases [17, 36, 39, 56] even beyond the originally postulated minimal setting. However, there are also counterexamples due to Scarparo [48], and Deeley [13]. Scarparo's counterexamples seem to be explained by the presence of torsion in isotropy groups, while Deeley's come from the presence of torsion phenomena in K-theory. It seems very interesting to see where exactly the conjecture holds, and to elucidate the obstructions that exist.

Combining Theorem A with a spectral sequence recently constructed by Proietti and Yamashita [44], we can confirm the HK-conjecture for a large class of low-dimensional groupoids:

Corollary C. Let G be a locally compact, Hausdorff, étale, principal, second countable, ample groupoid with dynamic asymptotic dimension at most 2. If $H_2(G)$ is finitely generated then the HK-conjecture holds for G, i.e.

$$K_0(C_r^*(G)) \cong H_0(G) \oplus H_2(G), \quad K_1(C_r^*(G)) \cong H_1(G).$$

Note that having finite dynamic asymptotic dimension forces all the isotropy groups to be locally finite. Consequently, it is natural to restrict our attention to the class of principal groupoids to avoid the trouble caused by torsion in the isotropy groups.

The second theme of this work is an attempt to make the HK-conjecture more explicit. It is well-known and easy to see that there is a canonical homomorphism

$$\mu_0 : H_0(G) \to K_0(C_r^*(G)).$$

In general, not much seems to be known about this map, and there are only partial results on the existence of maps in higher dimensions. To formulate our progress in this direction, for an ample groupoid G we denote by $[[G]]$ its topological full group. Moreover, Matui constructs in [34] an *index map* $I : [[G]] \to H_1(G)$.

Theorem D. Let G be a locally compact, Hausdorff, étale, ample groupoid. Then there exists a homomorphism

$$\mu_1 : H_1(G) \to K_1(C_r^*(G))$$

which factors the canonical map $[[G]] \to K_1(C_r^*(G))$ via the index map $I : [[G]] \to H_1(G)$.

If moreover G is principal, second countable, and has dynamic asymptotic dimension at most 2, then μ_0 and μ_1 induce the injection $H_0(G) \to K_0(C_r^*(G))$ and isomorphism $H_1(G) \to K_1(C_r^*(G))$ from Corollary C.

The construction of μ_1 is straightforward if the index map I is surjective, and under further structural assumptions on G, Matui was already able to prove this. Our construction of the map μ_1 is completely general: we in fact give two independent constructions, an elementary one based on ideas of Putnam [45], and a more sophisticated one based on the work of Proietti and Yamashita [44].

It was shown in [24] that the dynamic asymptotic dimension yields an upper bound for the nuclear dimension of reduced groupoid C^*-algebras. In particular, if G is a second countable, principal, minimal ample groupoid with finite dynamic asymptotic dimension, then $C_r^*(G)$ is classifiable.[1] Our result allows us to completely determine the classifying invariant (usually called the Elliott invariant and denoted $\mathrm{Ell}(\cdot)$) in the 1-dimensional case:

Corollary E. Let G be a locally compact, Hausdorff, étale, σ-compact, principal, ample groupoid with compact base space and with dynamic asymptotic dimension at most 1. Then

$$\mathrm{Ell}(C_r^*(G)) = (H_0(G), H_0(G)^+, [1_{G^{(0)}}], H_1(G), M(G), p).$$

In light of these results one is tempted to formulate a stronger version of the HK-conjecture in low dimensions, by asking that for ample principal groupoids with $H_n(G) = 0$ for $n \geqslant 2$ the canonical maps μ_i are isomorphisms. To see that this cannot be the case, we construct a counterexample using groupoids with topological property (T) introduced in [14]. The example is based on the construction of counterexamples to the Baum-Connes conjecture by Higson, Lafforgue and Skandalis [26] and work of Alekseev and Finn-Sell [1].

Theorem F. There exists a locally compact, Hausdorff, étale, second countable, principal, ample groupoid G with $H_n(G) = 0$ for all $n \geqslant 2$ such that $\mu_0 : H_0(G) \to K_0(C_r^*(G))$ is not surjective.

[1] This is due to Kirchberg and Phillips in the purely infinite case [30], [40], and due to many hands in the finite case, including Elliott, Gong, Lin, and Niu [20], [21], [15], and Tikuisis, White, and Winter [52] (see [9] for an alternative proof of classification in the finite case).

As an application of our results, we study a geometric class of examples. Given a metric space X with bounded geometry, Skandalis, Tu and Yu construct an ample groupoid $G(X)$ which encodes many coarse geometric properties of the underlying space X [49]. The following result adds to this list of connections. It might be known to experts but does not seem to appear in the literature so far (except for degree zero, which has been treated in [2]).

Theorem G. Let X be a bounded geometry metric space and $G(X)$ be the associated coarse groupoid. Then there is a canonical isomorphism

$$H_*(G(X)) \cong H_*^{\mathrm{uf}}(X)$$

between the groupoid homology of the coarse groupoid and the uniformly finite homology of X in the sense of Block and Weinberger [5].

As the dynamic asymptotic dimension of the coarse groupoid equals the asymptotic dimension of the underlying metric space (in the sense of Gromov), a combination of Theorems A (adapted to the non second countable case) and D yields the following purely geometric corollary.

Corollary H. Let X be a bounded geometry metric space.

(1) If $\mathrm{asdim}(X) \leqslant 2$ and $H_2^{\mathrm{uf}}(X)$ is free, or finitely generated, then
$$K_0(C_u^*(X)) \cong H_0^{\mathrm{uf}}(X) \oplus H_2^{\mathrm{uf}}(X), \quad K_1(C_u^*(X)) \cong H_1^{\mathrm{uf}}(X).$$
(2) If $\mathrm{asdim}(X) \leqslant 3$, $H_3^{\mathrm{uf}}(X)$ is free, or finitely generated, and X is non-amenable, then
$$K_0(C_u^*(X)) \cong H_2^{\mathrm{uf}}(X), \quad K_1(C_u^*(X)) \cong H_1^{\mathrm{uf}}(X) \oplus H_3^{\mathrm{uf}}(X).$$

The first part of this corollary applies for example if X is the fundamental group of a closed, orientable surface, and the second applies for example if X is the fundamental group of a closed, orientable, hyperbolic 3-manifold. The reader might compare this to [16, Theorem B, Theorem F, and Corollary H]: combining these implies related results after taking a "completed tensor product with $\mathbb{C}$" (in an appropriate sense) for certain spaces.

Outline of the paper. In Section 2 we give a new picture of Crainic-Moerdijk homology by defining a G-equivariant homology theory for an appropriate notion of semi-simplicial G-spaces. This is done in section 2.2, by describing the homology groups as the left derived functor of the coinvariants in the sense of classical homological algebra. The necessary background is given in section 2.1. It turns out that the groups $H_*(G)$ will be naturally isomorphic to the equivariant homology of a semi-simplicial G-space EG_*. We deduce Theorem B from this material.

The central piece of the vanishing result in Theorem A is tackled in Section 3. There we define a *colouring* of G, which will induce an appropriate cover of G. The *nerve* of this cover in the sense of section 3.3 is a semi-simplicial G-space, and that defines the homology of the colouring. The central idea is to define an anti-Čech sequence of G as a sequence of colourings with induced covers that are bigger and bigger, in the spirit of anti-Čech covers in coarse geometry, introduced by Roe (see [47, chapter 5]). We show that the inductive limit of an anti-Čech cover is well defined and that it converges to the Crainic-Moerdijk homology groups for principal σ-compact ample groupoids in theorem 3.29.

Section 4 is dedicated to the two constructions of the map

$$\mu_1 : H_1(G) \to K_1(C_r^*(G)),$$

from Theorem D, and a proof that these give the same result. We also deduce Corollary C (in a stronger form) and Corollary E.

Finally, in section 5 we interpret our results in various different settings, including coarse geometry (in particular, we deduce Theorem G and Corollary H) and Smale spaces, and present the negative results regarding a stronger form of the HK conjecture announced in Theorem F.

2. Models for groupoid homology

2.1. **The category of G-modules and the coinvariant functor.** Let us first fix our notations. A groupoid is a (small) category in which all arrows are invertible. We will write G and G^0 for the set of arrows and objects respectively; we usually also write G for the groupoid and assume the other data is given. We will call G^0 the base space. The range and source maps are denoted by $r, s : G \to G^0$, and the corresponding fibres by $G^x := r^{-1}(x)$ and $G_x := s^{-1}(x)$. A pair $(g, h) \in G \times G$ is composable if $s(g) = r(h)$, in which case the product is written gh. The identity at $x \in G^0$ is written $e_x \in G$. For subsets A, B of G we write $AB = \{gh \mid g \in A, h \in B, r(h) = s(g)\}$, and we write gA for $\{g\}A$.

We work exclusively with topological groupoids: G and G^0 carry locally compact Hausdorff topologies that are compatible, and all structure maps are continuous. A bisection is a subset B of G such that the restrictions $r|_B$ and $s|_B$ are homeomorphisms onto their images; a groupoid G is étale if the open bisections from a basis for its topology, and is ample if the compact open bisections form a basis for the topology.

We will be focusing throughout on (étale) ample groupoids, or equivalently étale groupoids with a totally disconnected base space. Examples include discrete groups and totally disconnected spaces, but also action groupoids of discrete groups acting on Cantor sets by homeomorphisms, coarse groupoids, and more examples associated to k-graphs and related combinatorial objects.

If G is an ample groupoid, let $\mathbb{Z}[G]$ denote the set of compactly supported continuous functions with integer values,

$$\mathbb{Z}[G] := C_c(G, \mathbb{Z}),$$

with the ring structure given by pointwise addition and convolution. This is a ring (in general without identity) with local units, i.e. for any $f_1, \dots, f_n \in \mathbb{Z}[G]$ there is an idempotent $e \in \mathbb{Z}[G]$ such that $ef_j = f_j e = f_j$ for all $j = 1, \dots, n$. The element e can always be picked in $\mathbb{Z}[G^0] \subseteq \mathbb{Z}[G]$ as the characteristic function on a compact open subset of G^0. Define the augmentation map $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}[G^0]$ by

$$\varepsilon(f)(x) = \sum_{g \in G_x} f(g).$$

We will denote the category of (left, non-degenerate[2]) $\mathbb{Z}[G]$-modules with $\mathbb{Z}[G]$-linear maps as morphisms by G-**mod**. We will often use the term G-module as shorthand for $\mathbb{Z}[G]$-module.

Definition 2.1. Let M be a $\mathbb{Z}[G]$-module. Define M_0 to be the submodule generated by the elements of the form

$$fm - \varepsilon(f)m, \quad f \in \mathbb{Z}[G],\ m \in M.$$

The group of *coinvariants* of M is the abelian group

$$M_G = M/M_0.$$

This naturally defines the coinvariant functor

$$\mathrm{Coinv} : G\text{-}\mathbf{mod} \to \mathbf{Ab}.$$

[2]If R is a ring with local units, a left R-module M is called *non-degenerate* whenever $RM = M$. All of our modules will be non-degenerate by assumption.

As it may make the above more conceptual, and as we will need it later, we give a different description of the coinvaraints functor. Define a right action of $\mathbb{Z}[G]$ on $\mathbb{Z}[G^0]$ by $a \cdot f := \varepsilon(af)$ for $a \in \mathbb{Z}[G^0]$ and $f \in \mathbb{Z}[G]$, i.e.

$$(a \cdot f)(x) = \sum_{g \in G_x} a(r(g))f(g) \quad \forall f \in \mathbb{Z}[G], a \in \mathbb{Z}[G^0], x \in G^0.$$

Lemma 2.2. *For any (left, non-degenerate) G-module M, there is a canonical isomorphism $M_G \cong \mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} M$.*

Proof. We leave it to the reader to check that the map

$$\mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} M \to M_G, \quad a \otimes m \mapsto am + M_0$$

is an isomorphism of abelian groups. The inverse is given as follows: for $m + M_0 \in M_G$ pick $a \in \mathbb{Z}[G^0] \subseteq \mathbb{Z}[G]$ such that $am = m$. The inverse map is given by $m + M_0 \mapsto a \otimes m$. □

We are now going to present a natural source of G-modules: spaces with a topological action of G.

Recall first that if $p : Y \to X$ and $q : Z \to X$ are continuous maps, the fibered product is the space

$$Y_p{\times}_q Z := \{(y, z) : p(y) = q(z)\}. \tag{1}$$

equipped with the subspace topology it inherits from $Y \times Z$.

Recall that a left G-space is the data of a topological space X together with:

- a continuous map $p : X \to G^0$, called the *anchor map*, and
- a continuous map $\alpha : G_s{\times}_p X \to X$, satisfying $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for every $(g, h, x) \in G_s{\times}_r G_s{\times}_p X$ and $\alpha(e_{p(x)}, x) = x$ for every $x \in X$.

A left G-space is étale if the anchor map is étale, i.e. a local homeomorphism. We will typically suppress α from the notation, for example writing 'gx' instead of '$\alpha(g, x)$'. A left G-space is free if $gx = x$ forces $g = e_{p(x)}$, and is proper if the map $\alpha \times pr_2 : G_s{\times}_p X \to X \times X$ is proper. The quotient of a G-space X is written $G\backslash X$, and is the quotient of X by the equivalence relation $x \sim gx$, equipped with the quotient topology. If the action of G on X is free and proper, then $G\backslash X$ is locally compact and Hausdorff, and the quotient map $q : X \to G\backslash X$ is étale.

Definition 2.3. Let $\mathbf{Top}_G$ be the category of left G-spaces which are locally compact, Hausdorff and étale, with morphisms being the G-equivariant étale maps.

We denote by $\mathbb{Z}[X]$ the abelian group of continuous compactly supported integer valued functions on X. It is a non-degenerate G-module with respect to the action defined by

$$(fa)(x) = \sum_{g \in G^{p(x)}} f(g)a(g^{-1}x) \quad \forall f \in \mathbb{Z}[G], a \in \mathbb{Z}[X], x \in X.$$

If $f : X \to Y$ is G-equivariant and étale, it induces a $\mathbb{Z}[G]$-linear map $f_* : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ by the formula

$$f_*(a)(y) := \sum_{x \in f^{-1}(y)} a(x) \quad \forall y \in Y.$$

The sum is finite since $a \in \mathbb{Z}[X]$ is compactly supported, and $f_*(a)$ is continuous as f is étale.

Lemma 2.4. *The assignments $X \mapsto \mathbb{Z}[X]$ and $f \mapsto f_*$ define a functor, called the functor of global sections,*

$$\mathbb{Z}[?] : \mathbf{Top}_G \to G\text{-}\mathbf{mod}.$$

The composition of the global section functor and the coinvariant functor is denoted by $\mathbb{Z}[?]_G$.

Lemma 2.5. *If X is a free and proper space in $\mathbf{Top}_G$ then there is an isomorphism of abelian groups*

$$\mathbb{Z}[X]_G \cong \mathbb{Z}[G\backslash X].$$

Proof. Denote by $[x] \in G\backslash X$ the class of $x \in X$, and define a $\mathbb{Z}[G]$-linear map $\tilde{\varepsilon} : \mathbb{Z}[X] \to \mathbb{Z}[G\backslash X]$ by

$$\tilde{\varepsilon}(f)([x]) = \sum_{g \in G_{p(x)}} f(g \cdot x) \quad \forall x \in X.$$

This obviously factors through $\mathbb{Z}[X]_G$, giving a map which we still denote by

$$\tilde{\varepsilon} : \mathbb{Z}[X]_G \to \mathbb{Z}[G\backslash X].$$

Let us build an inverse to $\tilde{\varepsilon}$. Since the action of G on X is proper, the quotient is locally compact Hausdorff, and hence a partition of unity argument shows that the family of functions $f \in \mathbb{Z}[G\backslash X]$ such that there exists a compact open subset $V \subseteq X$ with $supp(f) = q(V)$ and such that $q_{|V} : V \to q(V)$ is a homeomorphism, generates $\mathbb{Z}[G\backslash X]$ as an abelian group. Now given such a function f, we define $f_V := f \circ q_{|V} \in \mathbb{Z}[X]$ and our first goal is to show that the class of f_V in $\mathbb{Z}[X]_G$ does not depend on the choice of V. So let $V' \subseteq X$ be another compact open set with $supp(f) = q(V')$ and such that $q_{|V'} : V' \to q(V')$ is a homeomorphism. Then $q(V) = q(V')$ and hence every element $x \in V$ can be written as $x = gy$ for some $y \in V'$, $g \in G$. Using that G is étale and compactness of V we can decompose $V = \bigcup_i V_i$ and $V' = \bigcup_i V_i'$ such that there exist bisections $S_i \subseteq G$ implementing a homeomorphism $\alpha_i : V_i \to V_i'$. Hence doing another partition of unity argument, we may assume that the sets V and V' themselves are related in this way, i.e. there exists a bisection $S \subseteq G$ which induces a homeomorphism $\alpha_S : V \to V'$ by $\alpha_S(x) = hx$ where $h \in S$ is the unique element in $S \cap G^{p(x)}$. But then

$$f_V(x) = f(q_V(x)) = f(q_{V'}(\alpha_S(x))) = f_{V'}(\alpha_S(x)) \quad \forall x \in V.$$

This relation can be rewritten as $f_V = \chi_{S^{-1}} \cdot f_{V'}$, and hence we have $[f_V] = [f_{V'}]$ in $\mathbb{Z}[X]_G$ as desired. Let $\psi : \mathbb{Z}[G\backslash X] \to \mathbb{Z}[X]_G$ be the map given by $\psi(f) = [f_V]$. If $V \subseteq X$ is open such that $q_{|V} : V \to q(V)$ is a homeomorphism, then for a given $x \in X$ there is at most one $g \in G_{p(x)}$ such that $gx \in V$ since the action is free. It follows that for any function $f \in \mathbb{Z}[G\backslash X]$ supported in such a V we have

$$\tilde{\varepsilon}[f_V]([x]) = \sum_{g \in G_{p(x)}} f(q_V(gx)) = f([x])$$

and hence $\tilde{\varepsilon} \circ \psi = \mathrm{id}$.

Conversely, if $f \in \mathbb{Z}[X]$ is a function supported in a set of the form $S \cdot V$ for a bisection $S \subseteq G$ and $V \subseteq X$ is open such that $q_{|V}$ is a homeomorphism onto its image, then one easily checks that $\tilde{\varepsilon}[f]$ is supported in $q(V)$. Hence $[\psi(\tilde{\varepsilon}[f])] = [\tilde{\varepsilon}[f] \circ q_{|V}]$. Using freeness again, one checks that the latter class equals $[\chi_S \cdot f] = [\varepsilon(\chi_{S^{-1}}) \cdot f] = [f]$. Since functions f as above generate $\mathbb{Z}[X]$ we are done. □

2.2. **Semi-simplicial G-spaces and homology.** For n a nonnegative integer, denote by $[n]$ the interval $\{0, ..., n\}$. Recall (see for example [54, Chapter 8]) the definition of the semi-simplicial category Δ_s: its objects are the nonnegative integers, and $\mathrm{Hom}_{\Delta_s}(m, n)$ consists of the (injective) increasing maps $f : [m] \to [n]$. A semi-simplicial object in a category $\mathcal{C}$ is a contravariant functor from Δ_s to $\mathcal{C}$. The collection of all semi-simplicial objects in a category is itself a category, with morphisms given by natural transformations.

Let $\varepsilon_i^n : [n-1] \to [n]$ be the only increasing map whose image misses i. We will omit the superscript n if it does not cause confusion. Any increasing map $f : [m] \to [n]$ has a unique factorization $f = \varepsilon_{i_1}\varepsilon_{i_2}...\varepsilon_{i_k}$ with $0 \leqslant i_k \leqslant ... \leqslant i_1 \leqslant n$ (see Lemma 8.1.2 in [54]). Thus, any semi-simplicial object is the data, for all n, of an object C_n of $\mathcal{C}$, together with arrows $\varepsilon_i^n : C_n \to C_{n-1}$ in $\mathcal{C}$, called face maps, satisfying the (semi-)simplicial identities $\varepsilon_j^{n-1}\varepsilon_i^n = \varepsilon_i^{n-1}\varepsilon_{j-1}^n$ if $i < j$. Similarly, if C_* and D_* are semi-simplicial objects in $\mathcal{C}$, a morphism between them is a collection of morphisms $f_n : C_n \to D_n$ in $\mathcal{C}$ that are compatible with the face maps.

A semi-simplicial object in the category of locally compact Hausdorff topological spaces will be called a *semi-simplicial topological space*.

Definition 2.6. A *semi-simplicial G-space* is a semi-simplicial object in the category $\mathbf{Top}_G$.

As an example, define EG_* to be the semi-simplicial G-space

$$EG_n = G_r\times_r G_r\times_r ...{}_r\times_r G \quad \text{(n+1 times)}$$

with

- anchor map $p : EG_n \to G^0$ given by the common range of the tuple, $p(\gamma_0, ..., \gamma_n) = r(\gamma_0) = ... = r(\gamma_n)$ for $\gamma \in EG_n$,
- left action given by left multiplication by G on all factors,
- if $f : [m] \to [n]$, then $EG(f) : G_n \to G_m$ is defined by

$$(\gamma_0, ..., \gamma_n) \mapsto (\gamma_{f(0)}, ..., \gamma_{f(m)})$$

(note that the face maps are given by

$$\partial_i^{n+1} : (\gamma_0, ..., \gamma_n) \mapsto (\gamma_0, ..., \hat{\gamma}_i, ..., \gamma_n)$$

where the hat means that the entry is omitted).

One checks that the moment and the face maps are G-equivariant and étale.

On the other hand, define the *classifying space* of G, denoted by BG_*, to be the semi-simplicial topological space defined by

$$BG_n = \{(g_1, ..., g_n) \in G^n \mid r(g_i) = s(g_{i-1}) \text{ for all } i\}$$

with face maps

$$\epsilon_i^n : (g_1, ..., g_n) \mapsto \begin{cases} (g_2, ..., g_n) & \text{if } i = 0, \\ (g_1, ..., g_{i-1}, g_ig_{i+1}, ..., g_n) & \text{if } 1 \leqslant i < n, \\ (g_1, ..., g_{n-1}) & \text{if } i = n. \end{cases}$$

The G-action on EG is free and proper and hence $G\backslash EG_*$ is a semi-simplicial topological space. The maps $(g_1, ..., g_n) \mapsto [r(g_1), g_1, g_1g_2, \dots, g_1 \cdots g_n]$ and $[\gamma_0, \dots, \gamma_n] \mapsto (\gamma_0^{-1}\gamma_1, \dots, \gamma_{n-1}^{-1}\gamma_n)$ define maps of semi-simplicial topological spaces between BG_* and $G\backslash EG_*$ that are mutually inverse.

A semi-simplicial G-space $(X_*, \{\epsilon_i^*\}_i)$ naturally induces a chain complex of $\mathbb{Z}[G]$-modules $(\mathbb{Z}[X_*], \tilde{\partial}_*)$ by composing by the $\mathbb{Z}[?]$ functor, where the boundary maps are

$$\tilde{\partial}_n = \sum_{i=0}^{n}(-1)^i(\partial_i^n)_*$$

and thus a chain complex of abelian groups $(\mathbb{Z}[X_*]_G, \tilde{\partial}_*)$ by composing by the coinvariant functor.

Definition 2.7. Let $(X_*, \{\partial_i^*\}_i)$ be a semi-simplicial G-space. We define the equivariant homology group $H_*^G(X)$ to be the homology of the chain complex of abelian groups $(\mathbb{Z}[X_*]_G, \tilde{\partial}_*)$.

The relationship with the homology of Matui and Crainic-Moerdijk is now a consequence of Lemma 2.5. Namely, Matui introduced a chain complex of abelian groups to compute Crainic-Moerdijk homology groups in the special case of an ample groupoid. This chain complex is none other than $(\mathbb{Z}[BG_*], \tilde{\epsilon}_*)$. As EG_* is a free and proper semi-simplicial G-space with $BG_* \cong G\backslash EG_*$, this homology is isomorphic to $H^G_*(EG) \cong H_*(BG)$. In other words, we have proved the following.

Proposition 2.8. *Let G be an ample groupoid. The complex introduced by Matui to compute $H_*(G)$ identifies canonically with the homology of the complex $(\mathbb{Z}[EG_*]_G, \tilde{\partial}_*)$.* □

2.3. **Projective resolutions and Tor.** The aim of this section is to identify $H_*(G)$ with one of the standard objects studied in homological algebra. These results will not be used in the rest of the paper; we include them as they can be derived without too much difficulty from our other methods, and seem interesting.

Remark 2.9. If R is a non-unital ring, then free R-modules are not necessarily projective. However, for any idempotent $e \in R$ it is easily seen that $\mathrm{Hom}_R(Re, M) \cong eM$ naturally for any R-module M. If $N \twoheadrightarrow M$ is an epimorphism then elements in eM lift to eN, so it follows that Re is projective. Consequently, if R is a ring with enough idempotents, i.e. it contains a family $(e_i)_{i\in I}$ of mutually orthogonal idempotents such that $R \cong \bigoplus_{i\in I} Re_i \cong \bigoplus_{i\in I} e_i R$, then free R-modules are projective. More generally, a (non-degenerate) R-module is projective exactly when it is a direct summand of a free R-module.

We will restrict to ample groupoids G with σ-compact base space G^0. The main reason is that it implies that G^0 (resp. G) can be written as a disjoint union of compact open sets (resp. compact open bisections).[3] If $G^0 = \bigsqcup_{i\in I} U_i$ with U_i compact and open, then $\mathbb{Z}[G] \cong \bigoplus_i \mathbb{Z}[G]\chi_{U_i} \cong \bigoplus_i \chi_{U_i}\mathbb{Z}[G]$. By the above remark it follows that free G-modules are projective, and similarly for $\mathbb{Z}[G^0]$.

To state the main result of this subsection, note that G^0 admits left and right actions of G (with the anchor map being the identity) defined by

$$gx := r(g) \quad \text{and} \quad xg := s(g).$$

These actions make $\mathbb{Z}[G^0]$ into both a left and a right G-module. Thus the Tor groups $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ (see for example [54, Definition 2.6.4]) of homological algebra make sense.

Theorem 2.10. *Let G be an ample groupoid with σ-compact base space. There is a canonical isomorphism $H_*(G) \cong \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$.*

The rest of this section will be spent proving this theorem, which will proceed by a sequence of lemmas. To give the idea of the proof, recall (see for example [54, Definition 2.6.4 and Theorem 2.7.2]) that $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ can be defined by starting with an exact sequence

$$\cdots \xrightarrow{\partial} P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0$$

of $\mathbb{Z}[G]$-modules, where each P_n is projective. The group $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ is then by definition the n^{th} homology group of the complex

$$\cdots \xrightarrow{\mathrm{id}\otimes\partial} \mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} P_2 \xrightarrow{\mathrm{id}\otimes\partial} \mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} P_1 \xrightarrow{\mathrm{id}\otimes\partial} \mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} P_0$$

[3] In general, this fails for locally compact, Hausdorff, totally disconnected spaces, such as Counterexample 65 from [51] ($\mathbb{R}$ equipped with a rational sequence topology). One can also show that if X is this particular example, then $\mathbb{Z}[X]$ is not projective as a $\mathbb{Z}[X]$-module, so free $\mathbb{Z}[X]$-modules are not projective.

of abelian groups. We will prove Theorem 2.10 by showing that each $\mathbb{Z}[G]$-module $\mathbb{Z}[EG_n]$ is projective, and that we have an exact sequence

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1] \xrightarrow{\partial} \mathbb{Z}[EG_0] \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0$$

where the boundary maps are the alternating sums of the face maps. As

$$\mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} M = M_G$$

for any left non-degenerate G-module M by Lemma 2.2, Proposition 2.8 completes the proof.

We now embark on the details of the proof. Recall that G^0 is equipped with a left G-action defined by stipulating that the anchor map is the identity and defining the moment map by $gx := r(g)$, and that $\mathbb{Z}[G^0]$ is a left G-module with the induced structure. Define $\partial : \mathbb{Z}[EG_0] \to \mathbb{Z}[G^0]$ to be the map induced on functions by the étale map $r : G \to G^0$.

Lemma 2.11. *The sequence*

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1] \xrightarrow{\partial} \mathbb{Z}[EG_0] \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0\ .$$

is exact.

Proof. For $n \geqslant 0$, define

$$h : EG_n \to EG_{n+1}, \quad (g_0, \dots, g_n) \mapsto (r(g_0), g_0, \dots, g_n).$$

Then one computes on the spatial level that $\partial^i h = h\partial^{i-1}$ for $1 \leqslant i \leqslant n$, and $\partial^0 h$ is the identity map. Hence by functoriality

$$\begin{aligned}\partial h_* + h_*\partial &= \sum_{i=0}^{n}(-1)^i(\partial^i h)_* + \sum_{i=0}^{n-1}(-1)^i(h\partial^i)_* \\ &= (\partial^0 h)_* + \sum_{i=1}^{n}(-1)^i\big((\partial^i h)_* - (h\partial^{i-1})_*\big),\end{aligned}$$

and this is the identity. Define also $h : G^0 \to EG_0$ by $h(x) = x$. Then the map $rh : G^0 \to G^0$ is the identity, and so ∂h_* is the identity map on $\mathbb{Z}[G^0]$.

To summarise, h_* is a chain homotopy (as in for example [54, Definition 1.4.4]) between the identity map and the zero map. This implies that the complex has trivial homology (see for example [54, Lemma 1.4.5]), or, equivalently, is exact. □

For the next lemma, let us define a right action of $\mathbb{Z}[G^0]$ on $\mathbb{Z}[G]$ via

$$(f\phi)(g) := f(g)\phi(s(g)) \quad \forall f \in \mathbb{Z}[G], \phi \in \mathbb{Z}[G^0]$$

and similarly a left action of $\mathbb{Z}[G^0]$ on $\mathbb{Z}[EG_n]$ via $(\phi a)(g_0, ..., g_n) := \phi(r(g_0))a(g_0, ..., g_n)$. Note that this right action of $\mathbb{Z}[G^0]$ on $\mathbb{Z}[G]$ commutes with the canonical left action of $\mathbb{Z}[G]$ on itself: indeed, the action of $\mathbb{Z}[G^0]$ is just the action by right-multiplication of the submodule $\mathbb{Z}[G^0]$ of $\mathbb{Z}[G]$, so this commutativity statement is associativity of multiplication. For notational convenience, we define $EG_{-1} := G^0$.

Lemma 2.12. *With notation as above, for any $n \geqslant -1$ there is a canonical isomorphism of $\mathbb{Z}[G]$-modules.*

$$\mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} \mathbb{Z}[EG_n] \cong \mathbb{Z}[EG_{n+1}].$$

Proof. For $f \in \mathbb{Z}[G]$ and $a \in \mathbb{Z}[EG_n]$, we define

$$(f, a) : EG_{n+1} \to \mathbb{Z}, \quad (g_0, ..., g_{n+1}) \mapsto f(g_0)a(g_0^{-1}g_1, ..., g_0^{-1}g_{n+1}).$$

Note that (f, b) is in $\mathbb{Z}[EG_{n+1}]$ by the support conditions defining $\mathbb{Z}[G]$ and $\mathbb{Z}[EG_n]$. Note also that if $\phi \in \mathbb{Z}[G^0]$, then

$$(f\phi, a) : (g_0, ..., g_{n+1}) \mapsto f(g_0)\phi(s(g_0))a(g_0^{-1}g_1, ..., g_0^{-1}g_n)$$

and

$$(f, \phi a) : (g_0, ..., g_{n+1}) \mapsto f(g_0)\phi(r(g_0^{-1}g_1))a(g_0^{-1}g_1, ..., g_0^{-1}g_n);$$

these are the same however, as for any $(g_0, ..., g_{n+1}) \in EG_{n+1}$, $r(g_0^{-1}g_1) = s(g_0)$. Hence by the universal property of the balanced tensor product, we have a well-defined map

$$\mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} \mathbb{Z}[EG_n] \to \mathbb{Z}[EG_{n+1}].$$

We claim that this map is an isomorphism.

For injectivity, say we have an element $\sum_{i=1}^n f_i \otimes a_i$ that goes to zero. Splitting up the sum further, we may assume that each f_i is the characteristic function χ_{B_i} of a compact open bisection B_i such that $B_i \cap B_j = \varnothing$ for $i \neq j$. For each i, let ϕ_i be the characteristic function of $s(B_i)$. As $\chi_{B_i} = \chi_{B_i}\phi_i$, we have that $\chi_{B_i} \otimes a_i = \chi_{B_i} \otimes \phi_i a_i$, we may further assume that each a_i is supported in $\{(g_0, ..., g_n) \in EG_n \mid r(g_0) \in s(B_i)\}$. Now, we are assuming that $\sum_{i=1}^n (\chi_{B_i}, a_i) = 0$. As $B_i \cap B_j = \varnothing$ for $i \neq j$, the functions (χ_{B_i}, a_i) have disjoint supports, and therefore we have that $(\chi_{B_i}, a_i) = 0$ for each i. Assume for contradiction that $a_i \neq 0$, so there exists $(h_0, ..., h_n) \in EG_n$ with $a(h_0, ..., h_n) \neq 0$; our assumptions force $r(h_0) \in s(B_i)$. Let $g \in B_i$ be such that $s(g) = r(h_0)$. Then $(g, gh_0, ..., gh_n) \in EG_{n+1}$ and (χ_{B_i}, a) evaluates to $a(h_0, ..., h_n) \neq 0$ at this point, giving the contradiction and completing the proof of injectivity.

For surjectivity, as any element of $\mathbb{Z}[EG_{n+1}]$ is a finite $\mathbb{Z}$-linear combination of characteristic functions of subsets of the form $B_0 \times \cdots \times B_{n+1} \cap EG_{n+1}$ with each B_i a compact open bisection in G, it suffices to show that any such characteristic function is in B. Set $f \in \mathbb{Z}[G]$ to be the characteristic function of B_0, and $a \in \mathbb{Z}[EG_n]$ to be the characteristic function of $B_0^{-1}B_1 \times \cdots B_0^{-1}B_n \cap EG_n$. We leave it to the reader to check that $f \otimes a$ maps to the function we want. □

For the next lemma, we consider $\mathbb{Z}[G]$ as a left $\mathbb{Z}[G^0]$-module via the left-multiplication action induced by the inclusion $\mathbb{Z}[G^0] \subseteq \mathbb{Z}[G]$.

Lemma 2.13. *If G^0 is σ-compact, then, considered as a (left) module over $\mathbb{Z}[G^0]$, $\mathbb{Z}[G]$ is projective.*

In many interesting cases $\mathbb{Z}[G]$ is actually free over $\mathbb{Z}[G^0]$: for example, this happens for transformation groupoids associated to actions of discrete groups. However, freeness does not seem to be true in general.

Proof. For a compact open bisection U, let $\chi_U \in \mathbb{Z}[G]$ denote the characteristic function of that bisection. As G^0 is σ-compact we may choose a covering $G = \bigsqcup_{i \in I} U_i$ of G by disjoint compact open bisections, and for each i, let $\mathbb{Z}[G^0]\chi_{U_i}$ denote the $\mathbb{Z}[G^0]$-submodule of $\mathbb{Z}[G]$ generated by χ_{U_i}. Then as $\mathbb{Z}[G^0]$-modules, we have

$$\mathbb{Z}[G] \cong \bigoplus_{i \in I} \mathbb{Z}[G^0]\chi_{U_i}.$$

It thus suffices to prove that each $\mathbb{Z}[G^0]\chi_{U_i}$ is projective. For this, one checks that $\mathbb{Z}[G^0]\chi_{U_i}$ is isomorphic as a $\mathbb{Z}[G^0]$-module to $\mathbb{Z}[G^0]\chi_{r(U_i)}$, so projective by Remark 2.9, and we are done. □

Corollary 2.14. *If G^0 is σ-compact, the G-module $\mathbb{Z}[EG_n]$ is projective for $n \geqslant 0$.*

Proof. We proceed by induction on n. For $n = 0$, $\mathbb{Z}[EG_0]$ identifies with $\mathbb{Z}[G]$ as a left $\mathbb{Z}[G]$-module, so free, and thus projective by Remark 2.9. Now assume we have the result for $\mathbb{Z}[EG_n]$. By Remark 2.9, $\mathbb{Z}[EG_n]$ is a direct summand in a free $\mathbb{Z}[G]$-module, say $\bigoplus_J \mathbb{Z}[G]$. Hence, as $\mathbb{Z}[G^0]$-modules, $\mathbb{Z}[EG_n]$ is a direct summand in $\bigoplus_J \mathbb{Z}[G]$ which is projective by Lemma 2.13. Therefore, $\mathbb{Z}[EG_n]$ is projective as a $\mathbb{Z}[G^0]$-module. Let then N be a $\mathbb{Z}[G^0]$-module such that $\mathbb{Z}[EG_n]\oplus N$ is isomorphic as a $\mathbb{Z}[G^0]$-module to $\bigoplus_{i\in I} \mathbb{Z}[G^0]$ for some index set I. It follows that as $\mathbb{Z}[G]$-modules

$$\begin{aligned}\Big(\mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} \mathbb{Z}[EG_n]\Big) \oplus \Big(\mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} N\Big) &\cong \mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} (M \oplus N) \\ &\cong \mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} \Big(\bigoplus_{i\in I} \mathbb{Z}[G^0]\Big) \\ &\cong \bigoplus_{i\in I} \mathbb{Z}[G].\end{aligned}$$

Hence $\mathbb{Z}[G] \underset{\mathbb{Z}[G^0]}{\otimes} \mathbb{Z}[EG_n]$ is isomorphic to a direct summand of a free $\mathbb{Z}[G]$-module, so projective as a $\mathbb{Z}[G]$-module. Using Lemma 2.12, we are done. □

Proof of Theorem 2.10. Lemma 2.11 and Corollary 2.14 together imply that

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1] \xrightarrow{\partial} \mathbb{Z}[EG_0] \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0 \,.$$

is a resolution of $\mathbb{Z}[G^0]$ by projective modules. The groups $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ are therefore by definition the homology groups of the complex

$$\cdots \xrightarrow{\mathrm{id}\otimes\partial} \mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} \mathbb{Z}[EG_1] \xrightarrow{\mathrm{id}\otimes\partial} \mathbb{Z}[G^0] \underset{\mathbb{Z}[G]}{\otimes} \mathbb{Z}[EG_0] \longrightarrow 0 \,.$$

However, using Lemma 2.2, this is the same as the complex

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1]_G \xrightarrow{\partial} \mathbb{Z}[EG_0]_G \longrightarrow 0 \,,$$

and we have already seen that the homology of this is the same as the homology $H_*(G)$. □

3. Colourings and homology

The goal of this section is to use what we shall call a *colouring* of a groupoid G to produce an associated nerve space, and prove that the nerve is a semi-simplicial G-space. This lets us define the homology of a colouring in terms of the homology of the associated semi-simplicial G-space. Throughout most of the section we assume that G is an ample groupoid with compact base space G^0: the most important exception is the main result—Theorem 3.36—at the end, where we drop the compact base space assumption.

3.1. Colourings and nerves.

In this subsection we introduce colourings of a groupoid and the associated nerve spaces and homology.

Definition 3.1. A *colouring* for G is a finite ordered collection $\mathcal{C} = (G_0, ..., G_d)$ of compact open subgroupoids of G such that the collection $\{G_0^0, ..., G_d^0\}$ of unit spaces covers G^0.

Elements of the set $\{0, ..., d\}$ are called the *colours* of the colouring, and the colour of G_i is i.

We will associate a cover of G to a colouring. Let $\mathcal{P}(G)$ denote the collection of subsets of G.

Definition 3.2. Let $G_0, ..., G_d$ be a colouring of G. The *cover* associated to the colouring is

$$\mathcal{U} := \{(gG_i^{s(g)}, i) \in \mathcal{P}(G) \times \{0, ..., d\} \mid g \in G, G_i^{s(g)} \neq \varnothing\}.$$

Typically, we will just write U for an element of $\mathcal{U}$, and treat U as a (non-empty) subset of G. In particular, when considering intersections $U_0 \cap U_1$ for $U_0, U_1 \in \mathcal{U}$ we just mean the intersection of the corresponding subsets of G (ignoring the colour!). The precise definition however calls for pairs as we want each element of $\mathcal{U}$ to have a well-defined colour in $\{0, ..., d\}$. Note that an element $U \in \mathcal{U}$ could be equal to $(gG_i^{s(g)}, i)$ and $(hG_i^{s(h)}, i)$ for $g \neq h$ in G, i.e. representations of elements of $\mathcal{U}$ of the form $gG_i^{s(g)}$ need not be unique. Note also that the fact that $G_0^0, ..., G_d^0$ covers G^0 implies that $\mathcal{U}$ is a cover of G, but typically not by open sets: indeed, an element $gG_i^{s(g)}$ of $\mathcal{U}$ is contained in the single range fibre $G^{r(g)}$.

Definition 3.3. Let $G_0, ..., G_d$ be a colouring of G with associated cover $\mathcal{U}$. For $n \geqslant 0$, set

$$\mathcal{N}_n := \Big\{(U_0, ..., U_n) \in \mathcal{U}^{n+1} \;\Big|\; \bigcap_{i=0}^{n} U_i \neq \varnothing\Big\}.$$

The sequence $(\mathcal{N}_n)_{n=0}^{\infty}$ is denoted $\mathcal{N}_*$, and called the *nerve* of the colouring.

As noted before, the intersection in the definition above is to be interpreted as the intersection of the corresponding subsets of G (ignoring the colour). In particular we allow distinct U_i appearing in a tuple as above to have different colours.

Our next goal is to give $\mathcal{N}_*$ the structure of a semi-simplicial G-space.

Definition 3.4. For each n, the anchor map $r : \mathcal{N}_n \to G^0$ takes $(U_0, ..., U_n)$ to the unique $x \in G^0$ such that U_0 is a subset of[4] G^x. Let $G_s{\times_r}\mathcal{N}_n$ be the fibered product as in line (1) above, and define an action by

$$G_s{\times_r}\mathcal{N}_n \to \mathcal{N}_n, \quad (g, (U_0, ..., U_n)) \mapsto (gU_0, ..., gU_n).$$

Direct checks show this is a well-defined groupoid action of G on the set $\mathcal{N}_n$. Our next goal is to introduce a topology on $\mathcal{N}_n$, and prove that the action is continuous.

Definition 3.5. Let $i \in \{0, ..., d\}$ and let V be an open subset of G such that $G_i^{s(g)}$ is non-empty for all $g \in V$. Define

$$U_{V,i} := \{(gG_i^{s(g)}, i) \mid g \in V\},$$

and equip $\mathcal{N}_0$ with the topology generated by these sets. For each $n \geqslant 0$, equip $\mathcal{N}_0^{n+1}$ with the product topology, and give $\mathcal{N}_n \subseteq \mathcal{N}_0^{n+1}$ the subspace topology.

Lemma 3.6. *For each n, the topology on $\mathcal{N}_n$ is locally compact, Hausdorff, and totally disconnected. Moreover, the action defined in Definition 3.4 above is continuous.*

Proof. We first look at the case $n = 0$. Given a compact open bisection $V \subseteq G$ with $s(V) \subseteq G_i^0$ one readily verifies that the mapping $g \mapsto (gG_i^{s(g)}, i)$ defines a homeomorphism $V \to U_{V,i}$. In particular, the sets $U_{V,i}$ for such compact open bisections V are themselves compact open Hausdorff subsets of $\mathcal{N}_0$. Consequently, $\mathcal{N}_0$ is locally compact and locally Hausdorff. To check that it is Hausdorff, it suffices to check that if $(U_j)_j$ is a net in $\mathcal{N}_0$ that converges to both U and V, then $U = V$. Equivalently, say we have a net $(g_jG_{i_j}^{s(g_j)}, i_j)_j$ in $\mathcal{N}_0$ such that there are h_j in G with $g_jG_{i_j}^{s(g_j)} = h_jG_{i_j}^{s(h_j)}$ for all j, and so that $g_j \to g$ and $h_j \to h$. First note that

[4] and therefore all the U_j are subsets of

since the net converges, i_j will be eventually constant so we may as well assume that $i_j = i$ for all j; we need to check that $gG_i^{s(g)} = hG_i^{s(h)}$.

Say then that gk is in $gG_i^{s(g)}$ with $k \in G_i^{s(g)}$. By symmetry, it suffices to check that gk is in $hG_i^{s(h)}$. As G_i is open in G, it is étale, whence for all suitably large j, we can find $k_j \in G_i^{s(g_j)}$ such that $k_j \to k$. As $g_j G_i^{s(g_j)} = h_j G_i^{s(h_j)}$ for all j, we can also find $l_j \in G_i^{s(h_j)}$ for all j so that $g_j k_j = h_j l_j$. Using that G_i is compact, we may pass to a subnet and so assume that (l_j) converges to some $l \in G_i$, which is necessarily in $G_i^{s(h)}$. Hence $gk = \lim g_j k_j = \lim h_j l_j = hl$, and so gk is in $hG_i^{s(h)}$ as required.

The first paragraph of the proof implies that $\mathcal{N}_0$ admits a basis of compact open subsets and hence it is totally disconnected.

Continuity of the action follows on observing that if $(g_j, h_j G_i^{s(h)})$ is a convergent net in $G_s \times_r \mathcal{N}_0$, then $(g_j h_j G_i^{s(g_j h)})$ is a convergent net in $\mathcal{N}_0$ by continuity of the multiplication in G.

We now look at the case of general n. The facts that $\mathcal{N}_n$ is Hausdorff and totally disconnected, as well as the continuity of the G-action, all follow directly from the corresponding properties for $\mathcal{N}_0$.

We claim that $\mathcal{N}_n$ is closed in $\mathcal{N}_0^{n+1}$; as closed subsets of locally compact spaces are locally compact, this will suffice to complete the proof. To check closedness, for each $j \in \{0, ..., n\}$, say $(g_j^k)_{k \in K}$ is a net such that $g_j^k \to g_j$ as $k \to \infty$, and such that $(g_0^k G_i^{s(g_0^k)}, ..., g_n^k G_i^{s(g_n^k)})$ is in $\mathcal{N}_n$ for all k. We need to show that $(g_0 G_i^{s(g_0)}, ..., g_n G_i^{s(g_n)})$ is also in $\mathcal{N}_n$. Indeed, as $(g_0^k G_i^{s(g_0^k)}, ..., g_n^k G_i^{s(g_n^k)})$ is in $\mathcal{N}_n$, there exist $h_0^k, ..., h_n^k \in G_i$ such that

$$g_0^k h_0^k = g_1^k h_1^k = \cdots = g_n^k h_n^k.$$

As G_i is compact, we may assume that each net $(h_j^k)_{k \in K}$ converges to some h_j in G_i. Hence

$$g_0 h_0 = g_1 h_1 = \cdots = g_n h_n$$

is a point in $g_0 G_i^{s(g_0)} \cap \cdots \cap g_n G_i^{s(g_n)}$, which is thus non-empty. Hence

$$(g_0 G_i^{s(g_0)}, ..., g_n G_i^{s(g_n)})$$

is in $\mathcal{N}_n$ as required. □

We thus have shown that each of the spaces $\mathcal{N}_n$ is in the category $\mathbf{Top}_G$ of Definition 2.3. To show that $\mathcal{N}_*$ is a semi-simplicial G-space, it remains to build the face maps and show that they are equivariant and étale.

Definition 3.7. For each $n \geqslant 1$ and each $j \in \{0, ..., n\}$, define the j^{th} *face map* to be the function

$$\partial_j^n : \mathcal{N}_n \to \mathcal{N}_{n-1}, \quad (U_0, ..., U_n) \mapsto (U_0, ..., \widehat{U_j}, ..., U_n),$$

where the hat "$\widehat{\cdot}$" means to omit the corresponding element.

Lemma 3.8. *For each j, ∂_j as in Definition 3.7 is equivariant and étale, and moreover*

$$\partial_j^{n-1} \partial_i^n = \partial_i^{n-1} \partial_{j-1}^n \text{ if } i < j.$$

Proof. Let $(U_0, ..., U_n)$ be a point in $\mathcal{N}_n$, and write $U_k = g_k G_{i_k}^{s(g_k)}$ for each $k \in \{0, ..., n\}$. For each k, let B_k be an open bisection containing g_k. Then the set

$$W := \left\{ (h_0 G_{i_0}^{s(h_0)}, ..., h_n G_{i_n}^{s(h_n)}) \;\middle|\; h_k \in B_k \text{ for each } k \text{ and } \bigcap_{k=0}^{n} h_k G_{i_k}^{s(h_k)} \neq \varnothing \right\}$$

is an open neighbourhood of $(U_0, ..., U_n)$ in $\mathcal{N}_n$. We claim that ∂^j restricts to a homeomorphism on W. Indeed, let W_j be the image of W under ∂^j. Then an inverse is defined by sending a point $(V_0, ..., V_{n-1})$ in W_j to the point $(V_0, ..., hG_{i_j}^{s(h)}, ..., V_{n-1})$ in W, where $hG_{i_j}^{s(h)}$ occurs in the j^{th} entry, and where h is the unique point in B_j so that $r(h) = p(V_0, ..., V_{n-1})$.

The G-equivariance and the claimed relations between the face maps are straightforward. □

Definition 3.9. Let $\mathcal{C}$ be a colouring of G. The *homology of the colouring*, denoted $H_*(\mathcal{C})$, is the homology $H_*^G(\mathcal{N}_*)$ of the semi-simplicial G-space $\mathcal{N}_*$ as in Definition 2.7.

The homology groups $H_*(\mathcal{C})$ depend strongly on the colouring. For example, $\mathcal{C}$ could just consist of a partition of G^0 by compact open subsets, in which case one can check that the groups $H_n(\mathcal{C})$ are zero for $n > 0$. We will, however, eventually show that an appropriate limit of the homologies $H_*(\mathcal{C})$ as the colourings vary recovers the Cranic-Moerdijk-Matui homology $H_*(G)$ for principal and σ-compact G.

3.2. **Homology vanishing.** Our goal in this subsection is to show that if

$$\mathcal{C} = \{G_0, ..., G_d\}$$

is a colouring of G, then $H_n(\mathcal{C}) = 0$ for $n > d$. We will actually establish something a little more precise than this, as it will be useful later. The computations in this section are inspired by classical results in sheaf cohomology: see for example [19, Section 3.8]. More specifically, the precise formulas we use are adapted from [50, Tag 01FG].

Throughout this subsection, we fix an ample groupoid G with compact unit space G^0, a colouring $\mathcal{C} = \{G_0, ..., G_d\}$ as in Definition 3.1, and associated nerve space $\mathcal{N}_*$ as in Definition 3.3.

Lemma 3.10. *Let $\mathcal{U}$ be the cover associated to the colouring $\mathcal{C}$ as in Definition 3.2. Then any two elements of $\mathcal{U}$ that are the same colour and intersect non-trivially are the same. In particular, if $(U_0, ..., U_n)$ is a point of some $\mathcal{N}_n$, then any two elements of the same colour are actually the same.*

Proof. Say $gG_i^{s(g)} \cap hG_i^{s(h)} \neq \varnothing$; we need to show that $gG_i^{s(g)} = hG_i^{s(h)}$. Indeed, there are $k_g \in G_i^{s(g)}$ and $k_h \in G_i^{s(h)}$ with $gk_g = hk_h$. It follows that $h^{-1}g = k_h k_g^{-1}$, so $h^{-1}g$ is in G_i, as G_i is a subgroupoid. Hence whenever gk is in $gG_i^{s(g)}$, we have that $gk = h(h^{-1}gk)$ is also in $hG_i^{s(h)}$, and so $gG_i^{s(g)} \subseteq hG_i^{s(h)}$. Hence by symmetry, $gG_i^{s(g)} = hG_i^{s(h)}$. □

Definition 3.11. For each n, define the *colour map*

$$c : \mathcal{N}_n \to \{0, ..., d\}^{n+1}, \quad U \mapsto (\text{colour of } U_0, ..., \text{colour of } U_n).$$

We leave it to the reader to check that c is continuous and invariant under the action of G.

Throughout, we identify the symmetric group S_{n+1} with the permutations of $\{0, ..., n\}$. Fix now $\sigma \in S_{n+1}$ and for each $a \in \{0, ..., n\}$ define a new permutation σ_a as follows. Let σ_0 be the identity. For each $a \in \{1, ..., n\}$, let σ_a be the unique permutation of $\{0, ..., n\}$ such that:

(i) the restriction of σ_a to $\{0, ..., a-1\}$ agrees with the restriction of σ to this set;
(ii) the restriction of σ_a to $\{a, ..., n\}$ is the unique order-preserving bijection

$$\{a, ..., n\} \to \{0, ..., n\} \backslash \{\sigma(0), ..., \sigma(a-1)\}.$$

For a point $x = (i_0, ..., i_n) \in \{0, ..., d\}^{n+1}$, let $\sigma^x \in S_{n+1}$ be the unique permutation determined by the conditions below:

(i) $i_{\sigma^x(0)} \leqslant i_{\sigma^x(1)} \leqslant \cdots \leqslant i_{\sigma^x(n)}$;
(ii) σ^x is order-preserving when restricted to each subset S of $\{0, ..., n\}$ such that the elements $\{i_j \mid j \in S\}$ all have the same colour (i.e. so that the set $\{i_j \mid j \in S\}$ consists of a single element of $\{0, ..., d\}$).

Definition 3.12. For each n, let $c : \mathcal{N}_n \to \{0, ..., d\}^{n+1}$ be the colour map of Definition 3.11 and define

$$\mathcal{N}_n^{>} := c^{-1}\big(\{(i_0, ..., i_n) \mid i_0 < \cdots < i_n\}\big).$$

Similarly, define

$$\mathcal{N}_n^{\geqslant} := c^{-1}\big(\{(i_0, ..., i_n) \mid i_0 \leqslant \cdots \leqslant i_n\}\big).$$

We note that each $\mathcal{N}_*^{>}$ and $\mathcal{N}_*^{\geqslant}$ is a semi-simplicial G-space with the restricted structures from $\mathcal{N}_*$: indeed, each $\mathcal{N}_n^{>}$ and $\mathcal{N}_n^{\geqslant}$ is a closed, open and G-invariant subset of $\mathcal{N}_n$ as c is continuous and G-invariant, and the face maps of Definition 3.8 clearly restrict to maps $\mathcal{N}_n^{>} \to \mathcal{N}_n^{>}$ and $\mathcal{N}_n^{\geqslant} \to \mathcal{N}_n^{\geqslant}$.

For each $a \in \{0, ..., n\}$ define now $h_a : \mathcal{N}_n \to \mathcal{N}_{n+1}$ by the formula

$$h_a(U) := (U_{\sigma_a^{c(U)}(0)}, ..., U_{\sigma_a^{c(U)}(a-1)}, U_{\sigma^{c(U)}(a)}, U_{\sigma_a^{c(U)}(a)}, U_{\sigma_a^{c(U)}(a+1)}, ..., U_{\sigma_a^{c(U)}(n)})$$

(in words, we use $\sigma_a^{c(U)}$ to rearrange the order of the components of U, but also insert $U_{\sigma^{c(U)}(a)}$ into the a^{th} position).

It is not too difficult to see that each h_a is an equivariant étale map, and so induces a pushforward map $(h_a)_* : \mathbb{Z}[\mathcal{N}_n] \to \mathbb{Z}[\mathcal{N}_{n+1}]$ of $\mathbb{Z}[G]$-modules. We define

$$h : \mathbb{Z}[\mathcal{N}_n] \to \mathbb{Z}[\mathcal{N}_{n+1}]$$

by stipulating that for each $x \in \{0, ..., d\}^{n+1}$, its restriction to each subset $\mathbb{Z}[c^{-1}(x)]$ equals

$$\sum_{a=0}^{n} (-1)^a \text{sign}(\sigma_a^x)(h_a)_*.$$

On the other hand, define

$$p : \mathbb{Z}[\mathcal{N}_n] \to \mathbb{Z}[\mathcal{N}_n^{\geqslant}]$$

by stipulating that for each $x \in \{0, ..., d\}^{n+1}$, its restriction to each subset $\mathbb{Z}[c^{-1}(x)]$ equals $\text{sign}(\sigma^x)\sigma_*^x$. Finally, let

$$i : \mathbb{Z}[\mathcal{N}_*^{\geqslant}] \to \mathbb{Z}[\mathcal{N}_*]$$

be the canonical inclusion.

Lemma 3.13. *Let ∂ be the boundary map on $\mathbb{Z}[\mathcal{N}_*]$. Then*

$$\partial h + h\partial = \text{identity} - i \circ p.$$

Proof. We look at the restriction of ∂h to $\mathbb{Z}[c^{-1}(x)]$ for some x; it suffices to prove the given identity for such restrictions. For notational simplicity, let $\sigma := \sigma^x$. We have then for this restriction that

$$\partial h = \sum_{i=0}^{n+1} \sum_{a=0}^{n} (-1)^{i+a} \text{sign}(\sigma_a) \partial_*^i (h_a)_*$$

We split the terms into three types.

(i) Terms of the form $\partial_*^i (h_a)_*$ where $i < a$.
(ii) Terms of the form $\partial_*^{i+1} (h_a)_*$ where $i \geqslant a$ and $\sigma_a(i) \neq \sigma(a)$.
(iii) Terms of the form $\partial_*^a (h_a)_*$, or of the form $\partial_*^{i+1} (h_a)_*$ where $\sigma_a(i) = \sigma(a)$.

We look at each type in turn.

(i) We leave it to the reader to compute that as maps on the spatial level, $\partial^i h_a = h_{a-1}\partial^{\sigma_a(i)}$. Moreover, $(\partial^i h_a)_*$ occurs in the sum defining ∂h with sign $(-1)^i(-1)^a\text{sign}(\sigma_a)$, and $(h_{a-1}\partial^{\sigma_a(i)})_*$ occurs in the sum defining $h\partial$ with the sign $(-1)^{a-1}(-1)^{\sigma_a(i)}\text{sign}(\sigma'_{a-1})$, where σ'_{a-1} is the permutation defined as the composition

$$\begin{array}{ccccc} \{0,...,n-1\} & \xrightarrow{f} & \{0,...,\widehat{i},...,n\} & \xrightarrow{\sigma_a} & \{0,...,\widehat{\sigma_a(i)},...,n\} \\ & & & & \downarrow g \\ & & & & \{0,...,n-1\} \end{array}$$

with f and g the unique order preserving bijections. One can compute that $\text{sign}(\sigma_a) = (-1)^{\sigma_a(i)-i}\text{sign}(\sigma'_{a-1})$, essentially as σ_a can be built from the same transpostitions as used to construct σ'_a (conjugated by f and g) together with a cycle of length $|\sigma_a(i) - i| + 1$, which has sign $(-1)^{\sigma_a(i)-i}$. In conclusion, the term

$$(-1)^i(-1)^a\text{sign}(\sigma_a)(\partial^i h_a)_*$$

appearing in the sum defining ∂h is matched by the term

$$\begin{aligned}(-1)^{a-1}(-1)^{\sigma_a(i)}&\text{sign}(\sigma'_{a-1})(h_{a-1}\partial^{\sigma_a(i)})_* \\ &= (-1)^{a-1}(-1)^i\text{sign}(\sigma_a)(h_{a-1}\partial^{\sigma_a(i)})_*\end{aligned}$$

appearing in the sum defining $h\partial$; as these precisely match other than having opposing signs, they cancel.

(ii) We compute that as maps on the spatial level $\partial^{i+1}h_a = h_a\partial^{\sigma_a(i)}$. Moreover, $(\partial^{i+1}h_a)_*$ occurs in the sum defining ∂h with sign $(-1)^{i+1}(-1)^a\text{sign}(\sigma_a)$, and $(h_a\partial^{\sigma_a(i)})_*$ occurs in the sum defining $h\partial$ with the sign $(-1)^a(-1)^{\sigma_a(i)}\text{sign}(\sigma'_a)$, where σ'_a is the permutation defined as the composition

$$\begin{array}{ccccc} \{0,...,n-1\} & \xrightarrow{f} & \{0,...,\widehat{i},...,n\} & \xrightarrow{\sigma_a} & \{0,...,\widehat{\sigma_a(i)},...,n\} \\ & & & & \downarrow g \\ & & & & \{0,...,n-1\} \end{array}$$

with f and g the unique order preserving bijections. Much as in case (i), we have that $\text{sign}(\sigma_a) = (-1)^{\sigma_a(i)-i}\text{sign}(\sigma'_{a-1})$, and can conclude that the term

$$(-1)^{i+1}(-1)^a\text{sign}(\sigma_a)(\partial^{i+1}h_a)_*$$

appearing in the sum defining ∂h is matched by the term

$$(-1)^a(-1)^{\sigma_a(i)}\text{sign}(\sigma'_a)(h_a\partial^{\sigma_a(i)})_* = (-1)^a(-1)^i\text{sign}(\sigma_a)h_a\partial^{\sigma_a(i)})_*$$

appearing in the sum defining $h\partial$; as these precisely match other than having opposite signs, they cancel.

At this point, one can check that we have canceled *all* the terms appearing in the sum defining $h\partial$ using terms from ∂h of types (i) and (ii). It remains to consider terms of type (iii), and show that the sum of these equals

$$\text{identity} - i \circ p$$

as claimed. For each a, let i_a be the index such that $\sigma_a(i_a) = \sigma(a)$. The totality of terms of type (iii) looks like

$$\begin{aligned}&(\partial^0 h_0)_* \\ &+ \sum_{a=0}^{n-1} \big((-1)^{i_a+1}(-1)^a \mathrm{sign}(\sigma_a)(\partial^{i_a+1}h_a)_* + (-1)^{a+1}(-1)^{a+1}\mathrm{sign}(\sigma_{a+1})(\partial^{a+1}h_{a+1})_*\big) \\ &\quad + (-1)^n(-1)^n\mathrm{sign}(\sigma_n)(\partial^{i_n}h_n)_*.\end{aligned}$$

The first term is the identity, and the last term is $i \circ p$ (note that $\sigma_n = \sigma$). Hence it suffices to show that each term

$$(-1)^{i_a+1}(-1)^a \mathrm{sign}(\sigma_a)(\partial^{i_a+1}h_a)_* + (-1)^{a+1}(-1)^{a+1}\mathrm{sign}(\sigma_{a+1})(\partial^{a+1}h_{a+1})_*$$

in the sum in the middle is zero. Now, one computes on the spatial level that $\partial^{a+1}h_{a+1} = \partial^{i_a+1}h_a$ (this uses Lemma 3.10 to conclude that two coordinates with the same colour are actually the same), and so it suffices to prove that

$$(-1)^{i_a+1}(-1)^a \mathrm{sign}(\sigma_a) + (-1)^{a+1}(-1)^{a+1}\mathrm{sign}(\sigma_{a+1}) = 0,$$

or having simplified slightly, that $(-1)^{i_a+a+1}\mathrm{sign}(\sigma_a) + \mathrm{sign}(\sigma_{a+1}) = 0$. Indeed, one checks that σ_{a+1} differs from σ_a by a cycle moving the element in the i_a^{th} position to the a^{th} (and keeping all other elements in the same order), and such a cycle has sign $(-1)^{a-i_a}$. Hence $\mathrm{sign}(\sigma_a) = \mathrm{sign}(\sigma_{a+1})(-1)^{i_a-a}$ and we are done. □

Corollary 3.14. *The natural inclusion $i : \mathbb{Z}[\mathcal{N}_*^{\geqslant}] \to \mathbb{Z}[\mathcal{N}_*]$ is a chain homotopy equivalence.*

Proof. Lemma 3.13 implies in particular that the map $i \circ p$ is a chain map; as i is an injective chain map, this implies that $p : \mathbb{Z}[\mathcal{N}_n] \to \mathbb{Z}[\mathcal{N}_n^{\geqslant}]$ is a chain map too. Lemma 3.13 implies that $i \circ p$ is chain-homotopic to the identity, while $p \circ i$ just is the identity. Hence p provides an inverse to i on the level of chain homotopies. □

Our next goal is to show that the natural inclusion $j : \mathbb{Z}[\mathcal{N}_*^{>}] \to \mathbb{Z}[\mathcal{N}_*^{\geqslant}]$ is again a chain homotopy equivalence. For each $a \in \{0, ..., n-1\}$, let us write D_a^n for the subset of $\{0, ..., d\}^{n+1}$ consisting of those tuples $(i_0, ..., i_n)$ such that $i_0 < i_1 < \cdots i_a = i_{a+1} \leqslant i_{a+2} \leqslant \cdots \leqslant i_n$. For $x \in D_a^n$, define $k^x : \mathbb{Z}[c^{-1}(x)] \to \mathbb{Z}[\mathcal{N}_{n+1}]$ by

$$k^x : (U_0, ..., U_n) \mapsto (U_0, U_1, ..., U_a, U_a, U_{a+1}, U_{a+2}, ..., U_n).$$

Now define $k : \mathbb{Z}[\mathcal{N}_n^{\geqslant}] \to \mathbb{Z}[\mathcal{N}_{n+1}^{\geqslant}]$ by stipulating that for each $x \in \{0, ..., d\}^{n+1}$ the restriction of k to $\mathbb{Z}[c^{-1}(x)]$ is given by $(-1)^a k_*^x$ if x is in D_a^n for some $a \in \{0, ..., n-1\}$, and by zero otherwise. On the other hand, let

$$q : \mathbb{Z}[\mathcal{N}_n^{\geqslant}] \to \mathbb{Z}[\mathcal{N}_n^{>}]$$

be the natural projection that acts as the identity on each $\mathbb{Z}[c^{-1}(x)]$ with $c^{-1}(x) \subseteq \mathcal{N}_*^{>}$, and as 0 otherwise.

Lemma 3.15. *Let ∂ be the boundary map on $\mathbb{Z}[\mathcal{N}_*^{\geqslant}]$ and let $j : \mathbb{Z}[\mathcal{N}_*^{>}] \to \mathbb{Z}[\mathcal{N}_*^{\geqslant}]$ be the canonical inclusion. Then*

$$\partial k + k\partial = \mathrm{identity} - j \circ q.$$

Proof. It suffices to check the formula for each restriction to a submodule of the form $\mathbb{Z}[c^{-1}(x)]$.

Say first that $x = (i_0, ..., i_n)$ satisfies $i_0 < \cdots < i_n$. Then $j \circ q$ acts as the identity on $\mathbb{Z}[c^{-1}(x)]$, whence the right hand side is zero. Note that k restricts to zero on $\mathbb{Z}[c^{-1}(x)]$, whence ∂k is zero. On the other hand, the image of $\mathbb{Z}[c^{-1}(x)]$ is contained in a direct sum of subgroups of the form $\mathbb{Z}[c^{-1}(y)]$ where $y = (j_0, ..., j_{n-1}) \in \{0, ..., d\}^n$ satisfies $j_0 < \cdots < j_{n-1}$. Hence $k\partial = 0$ too, and we are done with this case.

Say then that $x = (i_0, ..., i_n)$ does not satisfy $i_0 < \cdots < i_n$. Then x is in D^n_a for some a. We then compute using the assumption that x is in D^n_a that $\partial^i k^x$ is the identity map for $i \in \{a, a+1, a+2\}$, and therefore that

$$\begin{aligned}
\partial k &= \sum_{i=0}^{n+1} (-1)^i \partial^i_* k \\
&= \sum_{i=0}^{a-1} (-1)^a (-1)^i \partial^i_* k^x_* + (-1)^a \big((-1)^a + (-1)^{a+1} + (-1)^{a+2}\big)\text{identity} \\
&\qquad\qquad + \sum_{i=a+3}^{n+1} (-1)^a (-1)^i \partial^i_* k^x_* \\
&= \sum_{i=0}^{a-1} (-1)^a (-1)^i \partial^i_* k^x_* + \text{identity} + \sum_{i=a+3}^{n+1} (-1)^a (-1)^i \partial^i_* k^x_*. \qquad (2)
\end{aligned}$$

Looking instead at $k\partial$, note first that as x is in D_a, we have that $\partial^a = \partial^{a+1}$ when restricted to $c^{-1}(x)$, and so

$$k\partial = k\Big(\sum_{i=0}^{n} (-1)^i \partial^i_*\Big) = \sum_{i=0}^{a-1} (-1)^i k\partial^i_* + \sum_{i=a+2}^{n} (-1)^i k\partial^i_*. \qquad (3)$$

Now, for $y \in \{0, ..., d\}^{n+1}$, let $\partial^i y$ denote y with the i^{th} component removed. Then for $0 \leqslant i \leqslant a-1$, $\partial^i x \in D^{n-1}_{a-1}$, and so $k\partial^*_i = (-1)^{a-1} k^y_* \partial^i_*$. It is not difficult to prove that for such y, $k^y_* \partial^i_* = \partial^*_i k^x_*$. On the other hand, for $a+2 \leqslant i \leqslant n$, $\partial^i_* x$ is in D^{n-1}_a, and so $k\partial^i_* = (-1)^a k^y_* \partial^i_*$; one computes moreover that for such i, $k^y_* \partial^i_* = \partial^{i+1}_* k^x_*$. Putting this discussion together with the formula in line (3) gives that

$$\begin{aligned}
k\partial &= \sum_{i=0}^{a-1} (-1)^i (-1)^{a-1} \partial^i_* k^x_* + \sum_{i=a+2}^{n} (-1)^i (-1)^a \partial^{i+1}_* k^x_* \\
&= \sum_{i=0}^{a-1} (-1)^i (-1)^{a-1} \partial^i_* k^x_* + \sum_{i=a+3}^{n+1} (-1)^{i-1} (-1)^a \partial^i_* k^x_*.
\end{aligned}$$

Comparing this with the formula in line (2), it follows that $k\partial + \partial k$ restricts to the identity on this summand $\mathbb{Z}[c^{-1}(x)]$. On the other hand, the same is true for the right hand side 'identity $- j \circ q$', so we are done. □

Corollary 3.16. *The inclusion $j : \mathbb{Z}[\mathcal{N}^>_*] \to \mathbb{Z}[\mathcal{N}^{\geqslant}_*]$ is a chain homotopy equivalence.*

Proof. Lemma 3.15 implies that the natural projection q is an inverse on the level of chain homotopies. □

Theorem 3.17. *Let $\mathcal{N}_*$ be a nerve complex built from a colouring $\mathcal{C} = (G_0, ..., G_d)$. Then the canonical inclusion $\mathcal{N}^>_* \to \mathcal{N}_*$ induces an isomorphism $H_*(\mathcal{C}) \cong H(\mathcal{N}^>_*)$. In particular, $H_n(\mathcal{C}) = 0$ for $n > d$.*

Proof. Corollaries 3.14 and 3.16 imply that the inclusion $\mathbb{Z}[\mathcal{N}^>_*] \to \mathbb{Z}[\mathcal{N}_*]$ is a chain homotopy equivalence. Functoriality of taking coinvariants then implies that the naturally induced map $\mathbb{Z}[\mathcal{N}^>_*]_G \to \mathbb{Z}[\mathcal{N}_*]_G$ is a chain homotopy equivalence, so in particular induces an isomorphism on homology. The remaining statement follows as $\mathcal{N}^>_n$ is empty for $n > d$. □

3.3. **Maps between nerves and G.** In this subsection, we build maps between our nerve spaces $\mathcal{N}_0$ and G that induce maps between the higher nerve spaces $\mathcal{N}_n$ and the spaces EG_n we used to define groupoid homology. This will allow us to compare the homology of colourings of G to the homology of G.

Definition 3.18. Let $G_0, ..., G_d$ be a colouring of G as in Definition 3.1, and let K be a subset of G. We say that the colouring $G_0, ..., G_d$ is:

(i) K-*bounded* if every G_i is contained in K;
(ii) K-*Lebesgue* if for every $x \in G^0$, there exists G_i such that $G^x \cap K$ is contained in G_i.

For $n \geqslant 0$, recall that $EG_n = \{(g_0, ..., g_n) \in G^{n+1} \mid r(g_0) = ... = r(g_n)\}$, equipped with the subspace topology that it inherits from G^{n+1}. For $n \geqslant 1$ and any subset K of G, let EG_n^K denote the subspace of EG_n consisting of those tuples $(g_0, ..., g_n)$ such that $g_i^{-1} g_j \in K$ for all i, j. Let EG_0^K be just $EG_0 = G$, whatever K is.

Lemma 3.19. *Say K is a compact open subset of G, and that $G_0, ..., G_d$ is a K-Lebesgue colouring of G with associated nerve $\mathcal{N}_*$. Then there exists an equivariant étale map $\Phi_0 : G \to \mathcal{N}_0$ such that:*

(i) $\Phi_0(g) \supseteq gK$ for all $g \in G$;
(ii) for all n, the map

$$\Phi_n : EG_n^K \to \mathcal{N}_n, \quad (g_0, ..., g_n) \mapsto (\Phi_0(g_0), ..., \Phi_0(g_n))$$

is a well-defined, equivariant local homeomorphism.

Proof. For each $i \in \{0, ..., d\}$, define $V_i := \{x \in G^0 \mid G^x \cap K \subseteq G_i\}$. We claim that each V_i is open. Indeed, let $x \in V_i$. As the set $G^x \cap K$ is compact and discrete, it is finite. Write $g_1, ..., g_n$ for the elements of this set, which are all in G_i by assumption that x is in V_i. For each $j \in \{1, ..., n\}$, let $W_j \subseteq G_i$ be a compact open bisection containing g_j. We may assume the W_j are disjoint by shrinking them if necessary. Define $W := \bigcap_{j=1}^n r(W_j)$, which is a compact open neighbourhood of x. We may write the compact open set $r^{-1}(W) \cap K$ as a finite disjoint union of compact open bisections of the form $W_1 \cap r^{-1}(W), ..., W_n \cap r^{-1}(W), B_1, ..., B_m$. Note that no B_j can intersect $G^x \cap K$, whence none of the sets $r(B_j)$ can contain x. Define

$$V := W \setminus \Big(\bigcup_{j=1}^m B_j \Big).$$

This is an open set containing x. Moreover, $r^{-1}(V)$ is contained in $W_1 \cup \cdots \cup W_n$, and therefore in G_i. Hence V is an open neighborhood of x contained in V_i, so V_i is open as claimed.

Note now that $V_0, ..., V_d$ covers G^0 by the assumption that the underlying colouring is K-Lebesgue. As G^0 has a basis of compact open sets and is compact, there is a finite cover, say $\mathcal{U}$ of G^0 consisting of disjoint compact open sets, and such that each $U \in \mathcal{U}$ is contained in some V_i. Define E_i to be the union of those $U \in \mathcal{U}$ such that i is the smallest element of $\{0, ..., d\}$ with U contained in V_i. Then the sets $E_0, ..., E_d$ are a partition of G^0 by compact open subsets, and each E_i is contained in V_i.

Now, for each $g \in G$, let $i(g) \in \{0, ..., d\}$ be the unique i such that $s(g)$ is in E_i; as the partition $G^0 = \bigsqcup_{i=0}^n E_d$ is into clopen sets, the map $i : G \to \{0, ...d\}$ this defines is continuous. Define

$$\Phi_0 : G \to \mathcal{N}_0, \quad g \mapsto g G_{i(g)}^{s(g)}.$$

We claim this has the right properties. First, note that for $g \in G$, $G^{s(g)} \cap K \subseteq G_{i(g)}$ by definition of $i(g)$ and the cover $V_0, ..., V_d$. Hence

$$gK = g(G^{s(g)} \cap K) \subseteq gG_{i(g)} = gG_{i(g)}^{s(g)} = \Phi_0(g).$$

We now show that Φ_0 is étale. Continuity of the restriction of Φ_0 to each set $s^{-1}(E_i)$ follows from the definition of the topology of $\mathcal{N}_0$, and continuity of Φ_0 on all of G follows from this as the sets $s^{-1}(E_0), ..., s^{-1}(E_d)$ are a closed partition of G. Let now $g \in G$, and let B be a clopen bisection containing g such that the map $i : G \to \{0, ..., d\}$ is constant on B (such exists as i is continuous). Let $C = \Phi_0(B)$. Then the map $C \to B$ defined by sending U to the unique element of $B \cap r(U)$ is a well-defined continuous inverse to the restriction $\Phi_0|_B$, completing the proof that Φ_0 is étale.

Equivariance of Φ_0 follows as if $s(g) = r(h)$, then $s(gh) = s(h)$ and $i(gh) = i(h)$, whence

$$\Phi_0(gh) = ghG_{i(gh)}^{s(gh)} = ghG_{i(h)}^{s(h)} = g\Phi_0(h).$$

To see that Φ_n is well-defined, note that if $(g_0, ..., g_n) \in EG_n^K$, then $g_i^{-1}g_j$ is in K for all i, j. Hence in particular g_0 is in $g_iK \subseteq \Phi_0(g_i)$ for each i, and so g_0 is in $\Phi_0(g_0) \cap \cdots \cap \Phi_0(g_n)$, and so this set is non-empty. Hence $(\Phi_0(g_0), ..., \Phi_0(g_n))$ is a well-defined element of $\mathcal{N}_n$. Equivariance of Φ_n and the fact that it is étale are straightforward from the corresponding properties for Φ_0, so we are done. □

Lemma 3.20. *Assume that G is principal, and K is a compact open subset of G that contains G^0. Let $G_0, ..., G_d$ be a K-bounded colouring with associated nerve $\mathcal{N}_*$. Then there exists an equivariant étale map $\Psi_0 : \mathcal{N}_0 \to G$ with the following properties:*

(i) $\Psi_0(U) \in U$ for all $U \in \mathcal{N}_0$;
(ii) for all n, the map

$$\Psi_n : \mathcal{N}_n \to EG_n^{KK^{-1}}, \quad (U_0, ..., U_n) \mapsto (\Psi_0(U_0), ..., \Psi_0(U_n))$$

is a well-defined, equivariant local homeomorphism.

To prove this, we need an ancillary lemma, which is based on the following structural result from [18, Lemma 3.4].

Lemma 3.21. *Let H be a compact, ample, principal groupoid. Then there are $m \in \mathbb{N}$ and*

(i) disjoint clopen subgroupoids $H_1, ..., H_m$ of H,
(ii) clopen subsets $X_1, ..., X_m$ of H^0 (equipped with the induced, i.e. trivial, groupoid structure), and
(iii) finite pair groupoids $P_1, ..., P_m$,

such that H identifies with the disjoint union $H = \bigsqcup_{i=k}^m H_k$ as a topological groupoid, and such that each H_k is isomorphic as a topological groupoid to $X_k \times P_k$. □

Corollary 3.22. *Let H be a compact, ample, principal groupoid, and let H^0/H be the quotient space of H^0 by the equivalence relation induced by H: precisely $x \sim y$ if there is $h \in H$ with $s(h) = x$ and $r(h) = y$. Then H^0/H is Hausdorff and there are étale maps $\sigma : H^0/H \to H^0$ and $\tau : H^0 \to H$ such that:*

(i) σ splits the quotient map $\pi : H^0 \to H^0/H$ (so in particular, π is étale);
(ii) $r \circ \tau =$ identity and and $s \circ \tau = \sigma \circ \pi$.

Proof. Assume first that $H = X \times P$, where X is a compact trivial groupoid and P is the pair groupoid on some finite set $\{0, ..., n\}$. Then H^0/H identifies homeomorphically with X (so in particular is Hausdorff) via the map

$$H^0 \to X, \quad (x, (i, i)) \mapsto x.$$

Making this identification, we may define $\sigma(x) = (x,(0,0))$ and $\tau(x,(i,i)) = (x,(i,0))$. These maps have the right properties when $H = X \times P$.

In the general case, Lemma 3.21 gives a decomposition of H into groupoids of the form $X \times P$ as above, and we may build σ and τ on each separately using the method above. □

Proof of Lemma 3.20. Let π_i, σ_i and τ_i be as in Corollary 3.22 for $H = G_i$. Define

$$\Psi_0 : \mathcal{N}_0 \to G, \quad gG_i^{s(g)} \mapsto g\tau_i(s(g)).$$

We first check that this is well-defined. Indeed, if $hG_i^{s(h)} = gG_i^{s(g)}$, then $r(h) = r(g)$ and $h^{-1}g \in G_i$. Hence

$$\pi_i(s(g)) = \pi_i(s(h^{-1}g)) = \pi_i(r(h^{-1}g)) = \pi_i(s(h)),$$

and so $\sigma_i(\pi_i(s(g)))$ and $\sigma_i(\pi_i(s(h)))$ are the same. As $\tau_i(x)$ has source $\sigma_i(\pi_i(x))$ for all $x \in G_i^{(0)}$, this implies that both $\tau_i(s(g))$ and $\tau_i(s(h))$ have the same source. As moreover g and h have the same range, the elements $g\tau_i(s(g))$ and $h\tau_i(s(h))$ of G have the same source and range and are therefore the same as G is principal. Having seen that Ψ_0 is well-defined, equivariance of Ψ_0 is straightforward. The fact that $\Psi_0(U) \in U$ for all $U \in \mathcal{N}_0$ follows as if we write $U = gG_i^{s(g)}$, then $\Psi_0(U) = g\tau_i(s(g))$, and $g\tau_i(s(g))$ is in $G_i^{s(g)}$, as $\tau_i(s(g))$ is in $G_i^{s(g)}$.

To see that Ψ_0 is étale, let $gG_i^{s(g)}$ be an element of $\mathcal{N}_0$, and and let B be a clopen bisection of g in G such that the set $\{hG_i^{s(h)} \mid h \in B\}$ is a clopen neighbourhood of $gG_i^{s(g)}$ in $\mathcal{N}_0$; in particular, this implies that $s(h) \in G_i^0$ for all $h \in B$. Using that both s and τ_i are étale, we have that $s(\tau_i(B))$ is open, and therefore that $\Psi_0(B)$ is open. We claim that the map

$$\kappa : \Psi_0(B) \to \mathcal{N}_0, \quad h \mapsto hG_i^{s(h)}$$

is a local inverse to Ψ_0; as it is continuous, this will suffice to complete the proof. Indeed, for any $h \in B$,

$$\kappa(\Psi_0(hG_i^{s(h)})) = h\tau_i(s(h))G_i^{s(\tau_i(s(h)))} = hG_i^{s(h)}.$$

On the other hand, for $h \in \Psi_0(B)$, as h is in the image of $\Psi_0(B)$, we have that $s(h)$ is in the image of σ_i, and therefore that $\sigma_i(\pi_i(s(h))) = s(h)$, and so $\tau_i(s(h)) = s(h)$. Hence

$$\Psi_0(\kappa(h)) = h\tau_i(s(h)) = h$$

and we are done with showing that Ψ_0 is étale.

To see that Ψ_n is well-defined, we need to check that if $(U_0, ..., U_n)$ is in $\mathcal{N}_n$, then $(\Psi_0(U_0), ..., \Psi_0(U_n))$ is in $EG_n^{KK^{-1}}$. Write $g_j := \Psi_0(U_j)$ for notational simplicity, so g_j is in U_j by the properties of Ψ_0. Let h be an element of $U_0 \cap \cdots \cap U_n$. Then for all j, the fact that the colouring is K-bounded implies that $g_j^{-1}h$ is in K for each j. Hence for any i, j, $g_i^{-1}g_j = g^{-1}hh^{-1}g_j \in KK^{-1}$, completing the proof that Ψ_n is well-defined. The facts that Ψ_n is étale and equivariant follow from the corresponding properties for Ψ_0, so we are done. □

3.4. **Anti-Čech homology.** In this subsection, we show that the Crainic-Moerdijk-Matui homology groups $H_*(G)$ can be realised by a direct limit of homology groups of appropriate colourings.

The key definition is as follows.

Definition 3.23. An *anti-Čech sequence* for G consists of the following data:

(i) a sequence $\mathcal{C}_m := (G_0^{(m)}, ..., G_{d_m}^{(m)})_{m=0}^{\infty}$ of colourings of G with associated sequence of nerves $\mathcal{N}_*^{(m)}$;

(ii) for each m a morphism $\iota^{(m)} : \mathcal{N}_*^{(m-1)} \to \mathcal{N}_*^{(m)}$ of semi-simplicial G-spaces such that for all $U \in \mathcal{N}_0^{(m-1)}$ we have that $\iota^{(m)}(U) \supseteq U$, and moreover so that for any compact open subset K of G, there exists m_K such that for all $m \geqslant m_K$ and all $U \in \mathcal{N}_0^{(m-1)}$, we have that $\iota^{(m)}(U) \supseteq UK$.

Definition 3.24. Let $\mathcal{A} = (\mathcal{C}_m)_{m=1}^{\infty}$ be an anti-Čech sequence for G, with associated sequence of morphisms $\iota^{(m)} : \mathcal{N}_*^{(m-1)} \to \mathcal{N}_*^{(m)}$. We define the *homology* of $\mathcal{A}$, denoted $H_*(\mathcal{A})$, to be the corresponding direct limit of the sequence of maps $(\iota_*^{(m)} : H_*(\mathcal{C}_{m-1}) \to H_*(\mathcal{C}_m))_{m=1}^{\infty}$.

Anti-Čech sequences always exist under the assumptions that G is principal and σ-compact. This follows from the next three lemmas.

Lemma 3.25. *Assume that G is principal, and that K is a compact open subset of G that contains G^0 and that satisfies $K = K^{-1}$. Let $G_0, ..., G_d$ and $H_0, ..., H_e$ be colourings of G with associated nerves $\mathcal{N}_*$ and $\mathcal{M}_*$ respectively. Assume moreover that the colouring $G_0, ..., G_d$ is K-bounded, and that the colouring $H_0, ..., H_e$ is K^3-Lebesgue.*

Then there exists an equivariant étale map $\iota_0 : \mathcal{N}_0 \to \mathcal{M}_0$ with the following properties:

(i) $\iota_0(U) \supseteq UK$ for all $U \in \mathcal{N}_0$;
(ii) for all n, the map

$$\iota_n : \mathcal{N}_n \to \mathcal{M}_n, \quad (U_0, ..., U_n) \mapsto (\iota_0(U_0), ..., \iota_0(U_n))$$

is, well-defined, equivariant, and étale.

Proof. Using Lemma 3.20, there is an equivariant étale map $\Psi_0 : \mathcal{N}_0 \to G$ such that $\Psi_0(U) \in U$ for all $U \in \mathcal{N}_0$, and such that for all n, the prescription

$$\Psi_n : \mathcal{N}_n \to EG_{n+1}^{K^3}, \quad (U_0, ..., U_n) \mapsto (\Psi_0(U_0), ..., \Psi_0(U_n))$$

gives gives a well-defined equivariant étale map (Lemma 3.20 has KK^{-1} in place of K^3, but note that our assumptions imply that K^3 contains KK^{-1}). Using Lemma 3.19, there is an equivariant local homeomorphism $\Phi_0 : G \to \mathcal{M}_0$ such that $gK^3 \subseteq \Phi_0(g)$ for all $g \in G$, and so that the prescription

$$\Phi_n : EG_n^{K^3} \to \mathcal{N}_n, \quad (g_0, ..., g_n) \mapsto (\Phi_0(g_0), ..., \Phi_0(g_n))$$

is a well-defined equivariant local homeomorphism.

Define then $\iota_0 := \Phi_0 \circ \Psi_0$, and note that $\iota_n := \Phi_n \circ \Psi_n$ for all n. Each ι_n is then a well-defined equivariant étale map. Moreover, fix $U \in \mathcal{N}_0$ and write $U = gG_i^{s(g)}$. As $\Psi_0(U) \in U$, we may write $\Psi_0(U) = gh$ for some $h \in G_i^{s(g)}$. Then h^{-1} is in G_i, and so in K as each G_i is a subset of K. Hence for an arbitrary element gk of $gG_i^{s(g)}$ with $k \in G_i \subseteq K^{-1}$, we have that $gk = ghh^{-1}k \in \Psi_0(U)K^2$. As gk was an arbitrary element of U, this gives that $U \subseteq \Psi_0(U)K^2$, and so $UK \subseteq \Psi_0(U)K^3$. Using the properties of Ψ_0 and Φ_0, we thus get that

$$UK \subseteq \Psi_0(U)K^3 \subseteq \Phi_0(\Psi_0(U)) = \iota_0(U)$$

and are done. □

Lemma 3.26. *Assume that G is principal and that K is a compact open subset of G that contains G^0. Then for any $x \in G^0$ there is a compact open subset B_0 of G^0 containing x and open bisections $B_1, ..., B_n$ such that:*

(i) $r^{-1}(B_0) \cap K = \bigsqcup_{i=0}^{n} B_i$;
(ii) for each i, $r|_{B_i} : B_i \to B_0$ is a homeomorphism;
(iii) for each $i \neq j$, $s(B_i) \cap s(B_j) = \varnothing$.

Proof. Say the elements of $r^{-1}(x) \cap K$ are $g_0 = x, g_1, ..., g_n$. As G is principal, we have $s(g_i) \neq s(g_j)$ for all $i \neq j$. Hence for each g_i, we may choose a clopen bisection D_i containing g_i such that $s(D_i) \cap s(D_j) = \varnothing$ for $i \neq j$, so that $r|_{D_i}$ is a homeomorphism, and so that D_0 is contained in G^0.

Set $C_0 := \bigcap_{i=0}^{n} r(B_i)$, which is a clopen set containing x, and for each i, set $C_i := B_i \cap r^{-1}(C_0)$, which is a clopen set containing g_i. The set $r^{-1}(C_0) \cap K$ is compact. We may thus write it as

$$\bigsqcup_{i=0}^{n} C_i \sqcup \bigsqcup_{j=0}^{m} E_j,$$

where each E_j is a clopen bisection such that $r|_{E_j}$ is a homeomorphism, and so that each E_j does not intersect $r^{-1}(x_0)$. Set

$$B_0 := C_0 \setminus \bigcup_{j=0}^{m} r(E_j),$$

and set $B_i := r^{-1}(B_0) \cap C_i$. These sets have the right properties. □

Lemma 3.27. *Let G be principal, and assume that K is a compact open subset of G that contains G^0. Then there exists a K-Lebesgue colouring $G_0, ..., G_d$ for G.*

Proof. Fix $x \in X$, and let $B_0, ..., B_n$ be a collection of sets with the properties in Lemma 3.26. For each i, let $\rho_i : B_0 \to B_i$ be the inverse of $r|_{B_i}$. Let $P = \{0, ..., n\}^2$ be the pair groupoid on the set $\{0, ..., n\}$, and define

$$f : B_0 \times P \to G, \quad (x, (i, j)) \mapsto \rho_i(x)\rho_j(x)^{-1}.$$

It is not difficult to check that f is a homeomorphism onto its image, which is a compact open subgroupoid of G. Write G_x for the image of f. Moreover, by construction we have that for every $y \in G^0_x$, the set $r^{-1}(y) \cap K$ is contained in G_x.

The collection $\{G^0_x \mid x \in G^0\}$ is an open cover of G^0, and thus has a finite subcover. Let $G_0, ..., G_d$ be the collection of compact open subgroupoids of G whose base spaces appear in this subcover. This collection has the right properties. □

Corollary 3.28. *For any σ-compact principal G with compact base space, an anti-Čech sequence exists.*

Proof. As G is σ-compact, there is a sequence $L_0 \subseteq L_1 \subseteq \cdots$ of compact open subsets of G such that each L_n equals L_n^{-1} and contains G^0, and such that any compact subset of G is eventually contained in all of the L_n. Set $K_0 = L_0$. Lemma 3.27 implies that there is a K_0^3-Lebesgue collection of compact open subgroupoids of G, say $G_0, ..., G_d$, with associated nerve space $\mathcal{N}_*^{(0)}$. As this collection is finite, there exists some compact open subset M_0 of G that contains all of $G_0, ..., G_d$, and that satisfies $M_0 = M_0^{-1}$. Set $K_1 := K_0 \cup L_1 \cup M_0$. Lemma 3.27 gives a new colouring that is K_1^3-Lebesgue with associated nerve $\mathcal{N}_*^{(1)}$, and Lemma 3.25 gives a morphism of semi-simplicial étale G-spaces $\iota^{(1)} : \mathcal{N}_*^{(0)} \to \mathcal{N}_*^{(1)}$ with the properties there. Now let M_1 be a compact open subset of G such that $M_1 = M_1^{-1}$, and that contains all the groupoids from this new colouring. Set $K_2 := K_1 \cup L_2 \cup M_1$ and use Lemma 3.27 to build a K_2^3-Lebesgue covering, and Lemma 3.25 to build a map $\iota^{(2)}$ from $\mathcal{N}_*^{(1)}$ to the associated nerve $\mathcal{N}_*^{(2)}$ with the properties in that lemma. Iterating this process builds an anti-Čech sequence as desired. □

Our main goal in this subsection is to prove the following theorem.

Theorem 3.29. *Let G be principal and σ-compact, with compact base space. Let $\mathcal{A}$ be an anti-Čech sequence for G. Then the homology groups $H_*(\mathcal{A})$ and $H_*(G)$ are isomorphic.*

The proof will proceed by some lemmas. First, we need a definition.

Definition 3.30. Let C_* be a semi-simplicial G-space and D_* be either the nerve of a colouring $\mathcal{N}_*$, or one of the EG^L_* for some compact open subset L of G. Let $\alpha, \beta : C_* \to D_*$ be two morphisms of semi-simplicial G-spaces. Then α, β are *close* if there exists a compact open subset K of G, either:

(i) D_* is a nerve $\mathcal{N}_*$, and for all $x \in C_0$ there exists $g \in G$ such that $\alpha(x)$ and $\beta(x)$ are both subsets of gK;
(ii) D_* is of for the form EG^L_*, and for all $x \in C_0$, $\alpha(x)^{-1}\beta(x)$ is in K.

Lemma 3.31. *Let $(\mathcal{N}^{(m)}_*, \iota^{(m)})$ be an anti-Čech sequence, and let $\alpha, \beta : C_* \to \mathcal{N}^{(m)}_*$ be close morphisms for some m. Let*

$$\iota^{(\infty)} : H_*(\mathcal{N}^{(m)}) \to \varinjlim H_*(\mathcal{N}^{(l)})$$

be the natural map to the direct limit, i.e. to the homology of the anti-Čech sequence. Then the compositions

$$\iota^{(\infty)} \circ \alpha_* : H_*(C) \to \varinjlim H_*(\mathcal{N}^{(l)}) \quad \textit{and} \quad \iota^{(\infty)} \circ \beta_* : H_*(C) \to \varinjlim H_*(\mathcal{N}^{(l)})$$

are the same.

Proof. Let K be as in the definition of closeness for α and β, and assume also that K is so large that the colouring underlying $\mathcal{N}^{(m)}$ is K-bounded. Let $l \geqslant m$ be large enough so that if $\iota : \mathcal{N}^{(m)} \to \mathcal{N}^{(l)}$ is the composition of the morphisms in the definition of the anti-Čech sequence, then for all $U \in \mathcal{N}^{(m)}$, we have that $\iota(U) \supseteq UK^2$ (such an l exists by definition of an anti-Čech sequence). It will suffice to show that $\iota \circ \alpha$ and $\iota \circ \beta$ induce the same map $H_*(C) \to H_*(\mathcal{N}^{(l)})$.

Let now x be a point in C_n for some n. For each $j \in \{0, ..., n\}$ let $\pi_j : C_n \to C_0$ be the map corresponding under the semi-simplicial structure to the map $\{0\} \to \{0, ..., n\}$ that sends 0 to j (see Section 2.2 for notation). Define $x_j := \pi_j(x)$. We claim that the intersection

$$\bigcap_{j=0}^{n} \iota(\alpha(x_j)) \cap \bigcap_{j=0}^{n} \iota(\beta(x_j)) \tag{4}$$

is non-empty. Indeed, $(\alpha(x_0), ..., \alpha(x_n))$ is a point of $\mathcal{N}^{(l)}_n$, whence there is some g_α in the intersection $\bigcap_{j=0}^n \alpha(x_j)$, and similarly for g_β with β replacing α. As α and β are close with respect to K, we have that $\alpha(x_0)$ and $\beta(x_0)$ are both subsets of gK for some $g \in G$, whence there are k_α and k_β in K such that $g_\alpha = gk_\alpha$ and $g_\beta = gk_\beta$. Hence $g_\alpha = g_\beta k_\beta k_\alpha^{-1}$, so in particular g_α is in $g_\beta K^2$. Now, by choice of ι, $\iota(\beta(x_j)) \supseteq \beta(x_j)K^2$ for all j, whence g_α is in $\iota(\beta(x_j))$ for all j. Moreover, $g_\alpha \in \alpha(x_j) \subseteq \iota(\alpha(x_j))$ for all j, so g_α is a point in the claimed intersection.

For each n and each $i \in \{0, ..., n\}$, we define a map

$$h^i : C_n \to \mathcal{N}^{(l)}_{n+1}$$

by the formula

$$x \mapsto (\iota \circ \alpha(\pi_0(x)), ..., \iota \circ \alpha(\pi_i(x)), \iota \circ \beta(\pi_i(x)), ..., \iota \circ \beta(\pi_n(x))),$$

which is well-defined by the claim. It is moreover an equivariant local homeomorphism as ι, α and β all have these properties. Hence h^i induces a map $h^i_* : \mathbb{Z}[C_n] \to \mathbb{Z}[\mathcal{N}^{(l)}_{n+1}]$ in the usual way. We define

$$h := \sum_{i=0}^{n} (-1)^i h^i_* : \mathbb{Z}[C_n] \to \mathbb{Z}[\mathcal{N}^{(l)}_{n+1}].$$

Direct checks show that h (and the map induced on coinvariants by h) is a chain homotopy between the maps induced by $\iota \circ \alpha$ and $\iota \circ \beta$. Hence $\iota \circ \alpha$ and $\iota \circ \beta$ indeed induce the same map on homology as claimed. □

For the next lemma, let $K_0 \subseteq K_1 \subseteq K_2 ...$ be a sequence of compact open subsets of G, all of which contain G^0, and whose union is G. We then get a sequence $(EG_*^{K_m})_{m=0}^{\infty}$ of spaces. Note moreover that the corresponding limit $\varinjlim H_*(G^{K_n})$ canonically identifies with $H_*(G)$: indeed, this follows directly from the observation that for each n, EG_n is the increasing union of the $EG_n^{K_m}$, and the fact that taking homology groups commutes with direct limits.

Lemma 3.32. *Let $(G_*^{K_m})_{m=0}^{\infty}$ be a sequence of spaces associated to a nested sequence of compact open subsets of G as above. Let $\alpha, \beta : C_* \to G_*^{(K_m)}$ be close morphisms for some m. Let*

$$\kappa : H_*(G^{K_m}) \to \varinjlim H_*(G^{K_m})$$

be the natural map to the direct limit, i.e. to the homology $H_*(G)$ of G. Then the compositions

$$\kappa \circ \alpha_* : H_*(C) \to H_*(G) \quad \text{and} \quad \kappa \circ \beta_* : H_*(C) \to H_*(G)$$

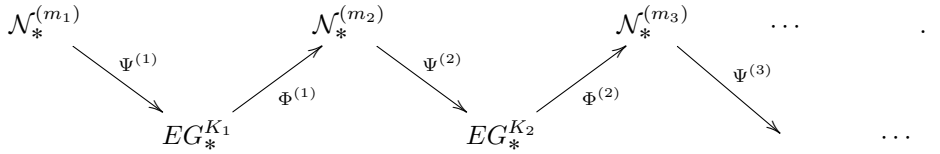
are the same.

Proof. The proof is very similar to that of Lemma 3.31. We leave the details to the reader. □

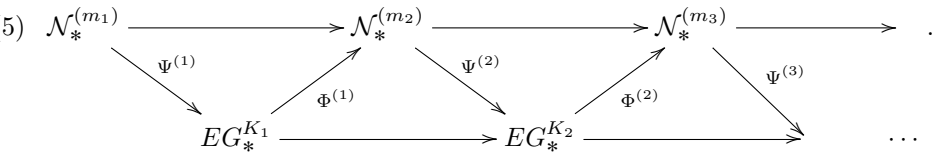
Proof of Theorem 3.29. Let $\mathcal{A}$ be the given anti-Čech sequence with associated nerves and morphisms $\iota^{(m)} : \mathcal{N}_*^{(m-1)} \to \mathcal{N}^{(m)}$.

Let $m_1 = 1$. Then the colouring underlying $\mathcal{N}_*^{(1)}$ is K-bounded for some compact open subset K of G, which we may assume contains G^0, and that satisfies $K = K^{-1}$. Set $K_1 := K^2$. Then Lemma 3.20 gives a morphism $\Psi^{(1)} : \mathcal{N}_*^{(m_1)} \to G_*^{K_1}$. On the other hand, by definition of an anti-Čech sequence there is $m_2 > m_1$ such that $\mathcal{N}^{(m_2)}$ is K_1-Lebesgue, whence Lemma 3.19 gives a morphism $\Phi^{(1)} : G_*^{K_1} \to \mathcal{N}^{(m_2)}$. Continuing, the colouring underlying $\mathcal{N}^{(m_2)}$ is K-bounded for some compact open subset K of G that we may assume contains K_1 and satisfies $K = K^{-1}$. Set $K_2 := K^2$, so Lemma 3.20 gives a morphism $\Phi^{(2)} : \mathcal{N}_*^{(m_2)} \to G_*^{K_2}$.

Continuing in this way, we get sequences $1 = m_1 < m_2 < m_3 < \cdots$ of natural numbers and $K_1 \subseteq K_2 \subseteq \cdots$ of compact open subsets of G together with morphisms



We may fill in horizontal arrows in the diagram: on the top row, these should be appropriate compositions of the morphisms $\iota^{(m)}$ coming from the definition of an anti-Čech sequence, while on the bottom row they should be induced by the canonical inclusions $EG_n^{K_k} \to EG_n^{K_{k+1}}$ coming from the fact that $K_k \subseteq K_{k+1}$ for all k. We thus get a (non-commutative!) diagram

(5)


Notice that the limit of the horizontal maps in the top row is $H_*(\mathcal{A})$. Moreover, the definition of an anti-Čech sequence and the construction of the sequence (K_k) forces $G = \bigcup K_k$, so the limit of the horizontal maps on the bottom row is $H_*(G)$.

Now, consider the compositions

$$H_*(EG^{K_k}) \xrightarrow{\Phi_*^{(k)}} H_*(\mathcal{N}^{(m_k)}) \longrightarrow H_*(\mathcal{A}) \ . \tag{6}$$

where the second arrow is the canonical one that exists by definition of the direct limit. Any two morphisms into any $\mathcal{N}^{(m_k)}$ are close using that all the colourings are bounded. It follows therefore from Lemma 3.31 that for any k, the diagram

$$\begin{array}{ccc} H_*(\mathcal{A}) & = & H_*(\mathcal{A}) \\ \uparrow & & \uparrow \\ H_*(EG^{K_k}) & \longrightarrow & H_*(EG^{K_{k+1}}) \end{array}$$

commutes; here the vertical maps are the ones in line (6), and the bottom horizontal line is induced by the canonical inclusion $EG_*^{K_k} \to EG_*^{K_{k+1}}$. Taking the limit in k of the maps in line (6), we thus get a well-defined homomorphism

$$\Phi : H_*(G) \to H_*(\mathcal{A}).$$

Precisely analogously, using Lemma 3.32 in place of Lemma 3.31, we get a homomorphism

$$\Psi : H_*(\mathcal{A}) \to H_*(G).$$

We claim that Φ and Ψ are mutually inverse, which will complete the proof. Indeed, for any k, the triangles

$$\begin{array}{ccc} \mathcal{N}_*^{(m_k)} & \longrightarrow & \mathcal{N}_*^{(m_{k+1})} \\ & {\scriptstyle \Psi^{(k)}} \searrow \quad \nearrow {\scriptstyle \Phi^{(k)}} & \\ & EG_*^{K_k} & \end{array}$$

and

$$\begin{array}{ccc} & \mathcal{N}_*^{(m_{k+1})} & \\ & {\scriptstyle \Phi^{(k)}} \nearrow \quad \searrow {\scriptstyle \Psi^{(k+1)}} & \\ EG_*^{K_k} & \longrightarrow & EG_*^{K_{k+1}} \end{array}$$

appearing in line (5) commute up to closeness, whence the claim follows directly from Lemmas 3.31 and 3.32, so we are done. □

3.5. **Dynamic asymptotic dimension.** In this subsection, we show that the homology of an ample σ-compact principal groupoid vanishes above its dynamic asymptotic dimension, and also that the top-dimensional homology group is torsion free.

The following definition is [24, Definition 5.1].

Definition 3.33. Let $d \in \mathbb{N}$. A (locally compact, Hausdorff, étale) groupoid has *dynamic asymptotic dimension at most d* if for any relatively compact open subset K of G there are open subsets $U_0, ..., U_d$ of G^0 that cover $r(K) \cup s(K)$ and such that for each i, the set $\{g \in K \mid s(g), r(g) \in U_i\}$ is contained in a relatively compact subgroupoid of G.

We record the some basic facts about products of subsets of a groupoid. See for example [24, Lemma 5.2] for a proof.

Lemma 3.34. *Let G be an étale groupoid, and H and K are subsets of G. Then if H and K are open (respectively, compact, or relatively compact), the product HK is open (respectively, compact, or relatively compact).*

Moreover, if K is open then the subgroupoid generated by K is also open. □

Here is our key use of dynamic asymptotic dimension.

Lemma 3.35. *Let G be an ample groupoid with compact base space which has dynamic asymptotic dimension at most d. Then for any compact open subset $K \subseteq G$ there exists a K-Lebesgue colouring of G with at most $d+1$ elements.*

Proof. Say G has dynamic asymptotic dimension at most d, and let $K \subseteq G$ be compact and open. As making K larger only makes the problem more difficult, we may assume that $K = K^{-1}$, and that K contains G^0. Note that K^3 is still compact and open by Lemma 3.34. The definition of dynamic asymptotic dimension at most d therefore gives an open cover $U_0, ..., U_d$ of G^0 such that for each i, the subgroupoid generated by $\{g \in K^3 \mid r(g), s(g) \in U_i\}$ is relatively compact. As G^0 is compact and as its topology has a basis of compact open sets, there is a cover of G^0 consisting of compact open sets $V_0, ..., V_d$ such that each V_i is contained in U_i. For each i, let $W_i = s(r^{-1}(V_i) \cap K)$. Note that $r^{-1}(V_i) \cap K$ is compact and open, whence W_i is also compact and open as s is an open continuous map. Moreover, W_i contains V_i as K contains G^0. Let now G_i be the subgroupoid of G generated by $\{g \in K \mid r(g), s(g) \in W_i\}$. We claim that $G_0, ..., G_d$ is a K-Lebesgue colouring of G.

Indeed, each G_i is open by Lemma 3.34, as it is generated by an open set. For compactness, we first claim that G_i has compact closure. Indeed, say $k_1...k_n$ is an element of G_i, where each k_j is in $\{g \in K \mid s(g), r(g) \in W_i\}$. Then by definition of W_i, for each $j \in \{1, ..., n\}$ there is an element h_j of K such that $h_{j-1}k_jh_j^{-1}$ has range and source in U_i. Hence for each $j \in \{1, ..., n\}$, $h_{j-1}k_jh_j^{-1}$ is in $\{g \in K^3 \mid r(g), s(g) \in U_i\}$. Let H_i be the subgroupoid of G generated by $\{g \in K^3 \mid r(g), s(g) \in U_i\}$, so H_i has compact closure by choice of the cover $U_0, ..., U_d$. Moreover,

$$k_1 \cdots k_n = h_0^{-1}\Big(\prod_{j=1}^{n} h_{j-1}k_jh_j^{-1}\Big)h_n \in KH_iK.$$

We now have that G_i is contained in KH_iK. However, KH_iK is relatively compact by Lemma 3.34, so we see that G_i is also relatively compact.

We next claim that there is $N \in \mathbb{N}$ such that any element of G_i can be written as a product of at most N elements of $\{g \in K \mid s(g), r(g) \in W_i\}$. Indeed, from [24, Lemma 8.10] there exists $N \in \mathbb{N}$ such that each range fibre G_i^x has at most N elements. Now, let $g = k_1 \cdots k_n$ be an element of G, where each k_j is in K. Consider the 'path' $k_1, k_1k_2, \ldots, k_1 \cdots k_n$. If this path contains any repetitions, we may shorten it by just omitting all the elements in between. Hence the length n of this path can be assumed to be at most the number of elements of $G^{r(g)}$, which is N as claimed.

To complete the proof that G_i is compact, it now suffices to show that it is closed. For this, let (g_j) be a net of elements of G_i. Using the previous claim, we may write $g_j = k_1^j...k_N^j$, where each k_j^i is in $\{g \in K \mid r(g), s(g) \in W_i\}$ (possibly identity elements). Note that the latter set is compact, whence up to passing to subnets, we may assume that each net (k_i^j) converges to some $k_i \in \{g \in K \mid r(g), s(g) \in W_i\}$. Hence $g = k_1...k_N$ is in G_i, so G_i is indeed compact.

Finally, we check that the colouring is K-Lebesgue. Note that for any $x \in G^0$, x is contained in V_i for some i, whence $s(r^{-1}(x) \cap K)$ is contained in W_i by definition of W_i. Hence G_i contains $r^{-1}(x) \cap K$, giving the K-Lebesgue condition. □

Theorem 3.36. *Let G be a σ-compact ample groupoid. If G is principal and has dynamic asymptotic dimension at most d, then $H_n(G) = 0$ for $n > d$, and $H_d(G)$ is torsion-free.*

Proof. We first prove the result under the additional assumption that G^0 is compact. In that case Lemma 3.35 lets us build an anti-Čech sequence $\mathcal{A}$ for G consisting of colourings $(\mathcal{C}^{(n)})$ each of which only has $d+1$ colours. As $H_*(\mathcal{A}) = \lim_{n\to\infty} H_*(\mathcal{C}^{(n)})$ and as Theorem 3.17 implies that $H_n(\mathcal{C}^{(m)}) = 0$ for all m and all $n > d$, we see that $H_n(\mathcal{A}) = 0$ for all $n > d$. Theorem 3.29 now gives the vanishing result.

For the claim that $H_d(G)$ is torsion-free result, keep notation as above, and let us write $\mathcal{N}_*^{(n)}$ for the nerve space associated to $\mathcal{C}^{(n)}$. Then we have

$$\begin{aligned} H_d(\mathcal{A}) &= \lim_{n\to\infty} H_d(\mathcal{C}^{(n)}) = \lim_{n\to\infty} H_d(\mathbb{Z}[\mathcal{N}_*^{(n)}]_G) = \lim_{n\to\infty} H_d(\mathbb{Z}[(\mathcal{N}_*^{(n)})^{>}]_G) \\ &= H_d(\lim_{n\to\infty} \mathbb{Z}[(\mathcal{N}_*^{(n)})^{>}]_G), \end{aligned}$$

where the first equality is just by definition of $H_d(\mathcal{A})$, the second is by definition of $H_d(\mathcal{C}^{(n)})$, the third is Theorem 3.17, and the fourth follows as taking homology commutes with direct limits. Now, using Lemma 2.5, $\mathbb{Z}[(\mathcal{N}_d^{(n)})^{>}]_G$ naturally identifies with $\mathbb{Z}[G\backslash(\mathcal{N}_d^{(n)})^{>}]$. Each $\mathbb{Z}[G\backslash(\mathcal{N}_d^{(n)})^{>}]$ is clearly torsion-free[5]. Hence the limit $\lim_{n\to\infty} \mathbb{Z}[G\backslash(\mathcal{N}_d^{(n)})^{>}]$ is torsion-free. On the other hand, $\lim_{n\to\infty} \mathbb{Z}[(\mathcal{N}_m^{(n)})^{>}] = 0$ for $m > d$. Hence $H_d(\mathcal{A})$ identifies with a subgroup of $\lim_{n\to\infty} \mathbb{Z}[G\backslash(\mathcal{N}_d^{(n)})^{>}]$, and so is itself torsion-free.

Let us now assume that G^0 is only locally compact. In this case use σ-compactness to write G^0 as an increasing union $G^0 = \bigcup_{m\in\mathbb{N}} K_m$ of compact open subsets. Let $G_m := \{g \in G \mid s(g), r(g) \in K_m\}$ denote the restriction of G to K_m. Note that each G_m is a σ-compact principal ample groupoid in its own right and it is straightforward to check from the definition (see [24, Definition 5.1]) that the dynamic asymptotic dimension of each G_m is dominated by the dynamic asymptotic dimension of the ambient groupoid G. Then $G = \bigcup_{m\in\mathbb{N}} G_m$ and the inclusion maps $G_m \hookrightarrow G$ induce isomorphisms $H_n(G) \cong \lim_{m\to\infty} H_n(G_m)$ for each n (see for example [17, Proposition 4.7]). Applying the compact unit space case from above for each G_m and passing to the limit, we obtain both the vanishing and the torsion-freeness results for G. □

4. The one-dimensional comparison map and the HK-conjecture

In this section our first goal is to construct a canonical comparison map

$$\mu_1 : H_1(G) \to K_1(C_r^*(G))$$

for an arbitrary ample groupoid G, extending earlier constructions of Matui under additional restrictions on the structure of G (see [34, Corollary 7.15] and [35, Theorem 5.2]). In fact we will provide two different constructions. The first approach is based on relative homology and K-theory groups, and is quite explicit and elementary; this is carried out in Subsection 4.1. The second (suggested by a referee) has its origins in the triangulated category perspective on groupoid equivariant Kasparov theory as recently exploited by Proietti and Yamashita [44, Corollary 4.4]; it is carried out in Subsection 4.2. The two approaches turn out to be equivalent, but this requires some fairly lengthy computations; these are carried out in Subsection 4.3.

[5]In fact, it is even free: it can be regarded as a commutative ring (with pointwise multiplication) that is generated by idempotents, and hence its underlying group is free abelian by [4, Theorem 1.1].

In the remaining Subsection 4.4, we apply these results and our vanishing results from Theorem 3.36 to deduce a general theorem on validity of the HK conjecture for low-dimensional groupoids. The result is actually more precise than this: it shows not only that the HK conjecture is true, but identifies the isomorphisms with the comparison maps we have constructed.

4.1. **Comparison maps from relative K-theory.** In this subsection, we give an elementary construction of a map $\mu_1 : H_1(G) \to K_1(C_r^*(G))$ that works for any ample groupoid. This is based on two ingredients: a suitable description of the K-theory of the mapping cone of the inclusion $\iota : C_0(G^0) \hookrightarrow C_r^*(G)$, and a relative version of groupoid homology with respect to an open subgroupoid.

Let us start with our description of the K_0-group of a mapping cone, which is inspired by results of Putnam in [45]. Recently, Haslehurst has independently developed a similar description to ours [25]. Neither Putnam's nor Haslehurst's constructions are quite the same as ours, but the latter in particular has substantial overlap. We will thus omit some details below that can be filled in using techniques from Haslehurst's paper [25, Section 3]; full details can also be found in the first version of the current paper on the arXiv.

For a C^*-algebra (or Hilbert module) B, define $IB := C([0,1], B)$, and recall that the *mapping cone* of a $*$-homomorphism $\phi : A \to B$ is

$$C(\phi) := \{(a_0, b) \in A \oplus IB \mid \phi(a_0) = b(0),\ b(1) = 0\}. \tag{7}$$

Definition 4.1. Given a $*$-homomorphism of C^*-algebras $\phi\colon A \to B$, we define the *relative K-theory groups* with respect to ϕ by

$$K_*(A, B; \phi) := K_*(C(\phi)). \tag{8}$$

The first coordinate projection $e^0 : A \oplus IB \to A$ restricts to a surjection $C(\phi) \to A$ with kernel $\Sigma B := C_0((0,1), B)$. This induces a six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_1(B) & \longrightarrow & K_0(A, B; \phi) & \longrightarrow & K_0(A) \\ {\scriptstyle \phi_1}\uparrow & & & & \downarrow{\scriptstyle \phi_0} \\ K_1(A) & \longleftarrow & K_1(A, B; \phi) & \longleftarrow & K_0(B). \end{array} \tag{9}$$

Our first aim is to give a different picture of the relative K_0-group. In what follows, when $\phi\colon A \to B$ is a $*$-homomorphism, we abuse notation by letting ϕ also denote the induced homomorphism $M_n(A) \to M_n(B)$ for $n \in \mathbb{N}$.

Definition 4.2. Let $\phi\colon A \to B$ be a $*$-homomorphism between C^*-algebras. For $n \in \mathbb{N}$, let $V_n(A, B; \phi)$ denote the set of triples (p, v, q) where $p, q \in M_n(A)$ are projections and $v \in M_n(B)$ satisfies $vv^* = \phi(p)$, and $v^*v = \phi(q)$. For each n, regard $V_n(A, B; \phi)$ as a subset of $V_{n+1}(A, B; \phi)$ via the usual top-left corner inclusion, and define

$$V_\infty(A, B; \phi) := \bigcup_{n \in \mathbb{N}} V_n(A, B; \phi),$$

equipped with the inductive limit topology. Let $\sim_{\mathrm{h}}$ denote homotopy in $V_\infty(A, B; \phi)$,[6] and let $\approx$ be the equivalence relation on $V_\infty(A, B; \phi)$ defined as follows: $(p, v, q) \approx (p', v', q')$ exactly when there are projections $r, r' \in M_n(A)$ such that

$$(p, v, q) \oplus (r, \phi(r), r) \sim_{\mathrm{h}} (p', v', q') \oplus (r', \phi(r'), r'). \tag{10}$$

[6] A homotopy in $V_\infty(A, B; \phi)$ is automatically contained in $V_n(A, B; \phi)$ for some $n \in \mathbb{N}$, so one can also take this as the definition of homotopy.

In the special case where ϕ is an inclusion of a C^*-subalgebra, we simply write $V_\infty(A,B)$ instead of $V_\infty(A,B;\phi)$. Moreover, if $(p,v,q) \in V_\infty(A,B)$ then $p = vv^*$ and $q = v^*v$. Hence we will simply denote the elements of $V_\infty(A,B)$ by v.

We equip $V_\infty(A,B;\phi)/\approx$ with the block sum operation defined by

$$[p_1,v_1,q_1] \oplus [p_2,v_2,q_2] := \left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}, \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \right];$$

it is straightforward to check that this makes $V_\infty(A,B;\phi)/\approx$ into a well-defined monoid with identity $[0,0,0]$.

Our goal is to construct an isomorphism $\eta : K_*(A,B;\phi) \to V_\infty(A,B;\phi)/\approx$ when ϕ is non-degenerate, and A has an approximate unit of projections; we will apply our construction to the canonical $*$-homomorphism $\iota : C_0(G^0) \to C_r^*(G)$ for an ample groupoid G, so these assumptions are satisfied. For this, we need an ancillary construction.

Let $\phi\colon A \to B$ be a $*$-homomorphism and define the *double* of ϕ to be

$$D(\phi) := \{(a_0,b,a_1) \in A \oplus IB \oplus A \mid \phi(a_0) = b(0) \text{ and } \phi(a_1) = b(1)\}. \tag{11}$$

With $C(\phi)$ as in line (7), we obtain a short exact sequence

$$0 \longrightarrow C(\phi) \xrightarrow{\ \iota\ } D(\phi) \xrightarrow{\ e^1\ } A \longrightarrow 0 \tag{12}$$

with morphisms defined by $\iota : (a_0,b) \mapsto (a_0,b,0)$ and $e^1 : (a_0,b,a_1) \mapsto a_1$. Let $\mathsf{c}\colon B \to IB$ denote the constant $*$-homomorphism. Then the short exact sequence in line (12) is split by the map $s : a \mapsto (a, \mathsf{c}(\phi(a)), a)$. Hence the composition

$$K_*(A,B;\phi) \xrightarrow{\iota_*} K_*(D(\phi)) \twoheadrightarrow K_*(D(\phi))/s_*(K_*(A)) \tag{13}$$

is an isomorphism.

We let $U_n(A)$ denote the unitary group of $M_n(A)$ for a unital C^*-algebra A, and let $V(A)$ denote the the semigroup of equivalence classes of projections in $\bigcup_{n\in\mathbb{N}} M_n(A)$, so that $K_0(A)$ is the Grothendieck group of $V(A)$ whenever A is unital, or more generally, when A has an approximate unit consisting of projections.

Assume now that A is a C^*-algebra with an approximate unit consisting of projections, and let $\phi : A \to B$ be a non-degenerate $*$-homomorphism; note then that B and so $D(\phi)$ also have approximate units consisting of projections, so in particular $K_0(D(\phi))$ is the Grothendieck group of $V(D(\phi))$. Let $(p_0,\mathsf{p},p_1) \in M_n(D(\phi))$ be a projection representing a K-theory class. Using for example [27, Corollary 4.1.8], there is a continuous path of unitaries $(u_t)_{t\in[0,1]}$ in $M_n(B)$ such that

$$\mathsf{p}_t = u_t^*\mathsf{p}_0u_t = u_t^*\phi(p_0)u_t, \quad \text{for } t \in [0,1] \quad \text{and} \quad u_0 = 1.$$

We define $\eta : V(D(\phi)) \to V_\infty(A,B;\phi)$ by the formula

$$\eta(p_0,\mathsf{p},p_1) := (p_0, \phi(p_0)u_1, p_1). \tag{14}$$

Here then is our promised picture of relative K-theory. Due to similarities to Haslehurst's methods [25], we do not give a proof here: the interested reader can find a complete proof in the original arXiv version of this paper.

Theorem 4.3. *Let $\phi : A \to B$ be a non-degenerate $*$-homomorphism, where A has an approximate unit consisting of projections.*

With notation as in line (13), the map η from line (14) above descends to a well-defined monoid isomorphism

$$\eta : \frac{K_0(D(\phi))}{s_0(K_0(A))} \to \frac{V_\infty(A,B;\phi)}{\approx}.$$

In particular, $V_\infty(A,B;\phi)/\approx$ is an abelian group that is isomorphic to $K_0(A,B;\phi)$.

Moreover, the six term exact sequence from line (9) identifies with an exact sequence

$$K_1(A) \xrightarrow{\phi_1} K_1(B) \to \frac{V_\infty(A,B;\phi)}{\approx} \to K_0(A) \xrightarrow{\phi_0} K_0(B)$$

where the map $K_1(B) \to \frac{V_\infty(A,B;\phi)}{\approx}$ *is given by* $[u]_1 \mapsto [p \otimes 1_{M_n}, u, p \otimes 1_{M_n}]$ *for* $u \in U_n(\phi(p)B\phi(p))$ *(where* $p \in A$ *is a projection); and* $\frac{V_\infty(A,B;\phi)}{\approx} \to K_0(A)$ *is given by* $[p,v,q] \mapsto [p]_0 - [q]_0$. □

Remark 4.4. We note the following basic properties of cycles (p_1, v_1, q_1) and (p_2, v_2, q_2) in $V_\infty(A, B; \phi)$; these can be justified using the same sort of rotation homotopies that establish the analogous properties for the classical K_0 and K_1 groups.

(i) If $p_1p_2 = q_1q_2 = 0$, then in $V_\infty(A, B; \phi)/\approx$ we have that

$$[p_1, v_1, q_1] + [p_2, v_2, q_2] = [p_1 + q_1, v_1 + v_2, p_2 + q_2].$$

(ii) If $p_2 = q_1$, then

$$[p_1, v_1, q_1] + [p_2, v_2, q_2] = [p_1, v_1v_2, p_2].$$

Let us now turn to the second ingredient in the construction of the comparison map $\mu_1 : H_1(G) \to K_1(C_r^*(G))$: relative groupoid homology. To define it, we return to Matui's picture of groupoid homology as it allows for an elementary description. For an open subgroupoid $H \subseteq G$ the canonical inclusion induces for each $n \geqslant 0$ a short exact sequence of abelian groups

$$0 \to \mathbb{Z}[H^{(n)}] \to \mathbb{Z}[G^{(n)}] \to \mathbb{Z}[G^{(n)}]/\mathbb{Z}[H^{(n)}] \to 0. \tag{15}$$

Let $\mathbb{Z}_n[G, H]$ denote the quotient group. One easily checks that $\mathbb{Z}[H^{(n)}]$ is invariant under the boundary maps ∂_n for $\mathbb{Z}[G^{(n)}]$, and hence we obtain induced maps ∂'_n turning $(\mathbb{Z}[G, H], \partial'_n)$ into a chain complex such that the sequence (15) is an exact sequence of chain complexes.

Definition 4.5. Let G be an ample groupoid. The *relative homology* of G with respect to an open subgroupoid H is defined as the homology of the chain complex $(\mathbb{Z}[G, H], \partial'_n)$, i.e.

$$H_n(G, H) := \ker(\partial'_n)/\mathrm{im}(\partial'_{n+1}).$$

From the short exact sequence (15) we obtain a long exact sequence of homology groups

$$\cdots \to H_1(H) \to H_1(G) \to H_1(G, H) \to H_0(H) \to H_0(G) \to H_0(G, H)$$

Important for us is the special case where $H = G^0$: there one easily checks that $H_0(G, G^0) = 0$, and $H_1(G, G^0) \cong \mathbb{Z}[G]/\mathrm{im}(\partial_2)$. We are now ready to put everything together and construct the map $\mu_1 : H_1(G) \to K_1(C_r^*(G))$.

Lemma 4.6. *Let G be an ample groupoid. Then there exists a well-defined homomorphism*

$$\tilde{\rho} : \mathbb{Z}[G] \to V_\infty(C_0(G^0), C_r^*(G))/\approx$$

determined by $\tilde{\rho}(1_W) = [1_W^*]$ *and* $\tilde{\rho}(-1_W) = [1_W]$ *for every compact open bisection* $W \subseteq G$.

Moreover, $im(\partial_2) \subseteq \ker(\tilde{\rho})$ *and hence* $\tilde{\rho}$ *factors through a well-defined homomorphism*

$$\rho : H_1(G, G^0) \cong \mathbb{Z}[G]/im(\partial_2) \to V_\infty(C(G^0), C_r^*(G))/\approx .$$

Proof. As G is an ample groupoid, every function $f \in \mathbb{Z}[G]$ can be written (non-uniquely) as a linear combination $f = \sum \lambda_i 1_{W_i}$ where $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ and $W_1, \dots, W_n$ are pairwise disjoint compact open bisections of G. We have to show that the resulting class of the partial isometry

$$\bigoplus_{\{i|\lambda_i \geqslant 0\}} \lambda_i 1_{W_i} \oplus \bigoplus_{\{i|\lambda_i<0\}} |\lambda_i| 1_{W_i^{-1}}$$

in $V_\infty(C_0(G^0), C_r^*(G))/\approx$ only depends on f and not on the particular choice of the bisections W_i. Let us first note the following: If W is a compact open bisection of G such that $W = U \sqcup V$ for compact open subsets $U, V \subseteq W$, then $1_W = 1_U + 1_V \approx 1_V \oplus 1_U$ by Remark 4.4, part (i).

With this in mind we can establish the lemma: Let $f \in \mathbb{Z}[G]$. We may assume that $f \geqslant 0$. Suppose we have $f = \sum \lambda_i 1_{U_i} = \sum \mu_j 1_{V_j}$ for $\lambda_i, \mu_j \in \mathbb{N}$ and two families $(U_i)_i$ and $(V_j)_j$ of pairwise disjoint compact open bisections of G. Since G is ample we may choose a common refinement of these two families by compact open bisections $(W_k)_k$. Let η_k be equal to λ_i if $W_k \subseteq U_i$ and equal to μ_j if $W_k \subseteq V_j$. Then we have

$$\bigoplus_k \eta_k 1_{W_k} \approx \bigoplus_j \bigoplus_{\{k|W_k \subseteq V_j\}} \mu_j 1_{W_k} \approx \bigoplus_j \mu_j \bigoplus_{\{k|W_k \subseteq V_j\}} 1_{W_k} \approx \bigoplus_j \mu_j 1_{V_j}.$$

Similarly, we obtain $\bigoplus_k \eta_k 1_{W_k} \approx \bigoplus_i \lambda_i 1_{U_i}$, from which we conclude that $\tilde{\rho}$ is indeed well-defined. It is a group homomorphism by construction.

For the second part let U and V be compact open bisections of G such that $s(U) = r(V)$. Then we have $\tilde{\rho}(\partial_2(1_{(U\times V)\cap G^{(2)}})) = \tilde{\rho}(1_U - 1_{UV} + 1_V) = 0$ since $1_U \oplus 1_V \approx 1_U 1_V = 1_{UV}$ by Remark 4.4, part (ii). □

Theorem 4.7. *Let G be an ample groupoid. Then there exists a canonical homomorphism*

$$\mu_1 : H_1(G) \to K_1(C_r^*(G)).$$

Proof. Let $\iota : C_0(G^0) \to C_r^*(G)$ denote the canonical inclusion, which is non-degenerate. As G is ample, $C_0(G^0)$ contains an approximate identity of projections. By Theorem 4.3 and Lemma 4.6 we obtain a commutative diagram with exact rows (16)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(G) & \longrightarrow & H_1(G, G^0) & \longrightarrow & H_0(G^0) & \longrightarrow & H_0(G) \\ & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mu_0} \\ 0 & \longrightarrow & K_1(C_r^*(G)) & \longrightarrow & V_\infty(C_0(G^0), C_r^*(G))/\approx & \longrightarrow & K_0(C_0(G^0)) & \xrightarrow{\iota_0} & K_0(C_r^*(G)) \end{array},$$

where the top row is part of the long exact sequence for the pair (G, G^0). By exactness, there exists a unique homomorphism $\mu_1 : H_1(G) \to K_1(C_r^*(G))$ filling in the dashed arrow such that the diagram commutes. □

4.2. **Comparison maps from the ABC spectral sequence.** In this subsection we recall the spectral sequence constructed by Proietti and Yamashita in [44, Corollary 4.4] (which is in turn based on the ABC spectral sequence of Meyer [37]), and show that this gives rise to canonical comparison maps $\mu_k : H_k(G) \to K_k(C_r^*(G))$ for $k \in \{0, 1\}$. This approach to the construction of comparison maps was suggested by the referee.

Throughout this section, G is a second countable ample groupoid. We will need to work extensively with G-C^*-algebras and the G-equivariant Kasparov category KK^G: see [31] for background. We will also need to treat KK^G as a triangulated category as described in [44, Section A.4]. For background on the material we will need about triangulated categories and homological ideals and functors, see [38].

Let $\mathcal{I}$ be the homological ideal (see [38, Definition 2.20 and Remark 2.21]) in KK^G defined as the kernel of the restriction functor $F := \mathrm{Res}^G_{G^0} : KK^G \to KK^{G^0}$. If we define also $E := \mathrm{Ind}^G_{G^0} : KK^{G^0} \to KK^G$ (see [7, Section 2.1] for a detailed treatment of this), then (E, F) form an adjoint pair as established in [6, Section 6] (see also [7, Theorem 2.3]). Define $L := E \circ F : KK^G \to KK^G$, and for an object B of KK^G, let $\epsilon_B \in KK^G(L(B), B)$ be the counit of adjunction, which is computed explicitly in [7, Theorem 2.3].

Now, as a consequence of [44, Proposition 3.1] we see there is an (even) $\mathcal{I}$-projective resolution in the sense of [44, Definition 2.14] of the object $A = C_0(G^0)$ in KK^G

$$A \xleftarrow[\delta_0]{} P_0 \xleftarrow[\delta_1]{} P_1 \xleftarrow[\delta_2]{} P_2 \xleftarrow[\delta_3]{} \cdots, \tag{17}$$

where for each $p \in \mathbb{N}$, $P_p := L^{p+1}(A)$ and

$$\delta_p := \sum_{i=0}^{p} (-1)^i L^i(\epsilon_{L^{n-i}(A)}). \tag{18}$$

Moreover, one computes from the explicit description of $\mathrm{Ind}^G_{G^0}$ in [7, Section 2.1] that $L^{p+1}(A)$ identifies canonically with $C_0(G^{(p+1)})$, where we recall that

$$G^{(p)} := \{(g_1, ..., g_p) \in G^p \mid s(g_i) = r(g_{i+1}) \text{ for all } i \in \{1, ..., n-1\}\}$$

and we write $G^{(0)}$ and G^0 interchangeably. Here and throughout, we equip $G^{(p)}$ with the left G-action induced by

$$g : (h_1, h_2, ..., h_p) \mapsto (gh_1, h_2, ..., h_p) \tag{19}$$

and $C_0(G^{(p)})$ with the corresponding G-action. We will have need of the following definition and lemma multiple times.

Definition 4.8. Let X and Y be locally compact spaces and let $\mathrm{p} : Y \to X$ be an étale map. Equip $C_c(Y)$ with the $C_0(X)$-valued inner product defined[7] by

$$\langle \xi, \eta \rangle(x) := \sum_{y \in \mathrm{p}^{-1}(\{x\})} \overline{\xi(y)} \eta(y)$$

The corresponding completion is a Hilbert $C_0(X)$-module that we will denote $L^2(Y, p)$.

Let now A be a G-C^*-algebra, and let r^*A be the pullback along $r : G \to G^0$ (see for example [29, 2.7(d)]). The reduced crossed product of A by G, denoted $A \rtimes_r G$, is defined to be a certain completion of $C_c(G) \cdot r^*A$, where the latter is equipped with a natural $*$-algebra structure: we refer the reader to [29, Section 3] for more details. For an element $f \in C_c(G) \cdot r^*A$ and $h \in G$, we write $f(h)$ for the corresponding element of $(r^*A)_h = A_{r(h)}$. In the special case $A = C_0(G^{(p)})$ with the left G-action from line (19), we see that $f(h)$ identifies with an element of $C_0(G^{r(h)}{}_s \times_r G^{(p-1)})$. The next lemma is well-known and will be used by us several times below: it follows from direct checks based on the formulas we give that we leave to the reader.

Lemma 4.9. *For each $p \geqslant 2$, let $\mathrm{pr} : G^{(p)} \to G^{(p-1)}$ be the map $(g_1, ..., g_p) \mapsto (g_2, ..., g_p)$, and let $\mathrm{pr} : G^{(1)} \to G^{(0)} = G^0$ be the map $g \mapsto s(g)$. For each $p \geqslant 1$, $f \in C_c(G) \cdot r^*C_0(G^{(p)})$, $\xi \in L^2(G^{(p)}, \mathrm{pr})$, and $g \in G$ define*

$$(\kappa_p(f)\xi)(g_1, ..., g_p) := \sum_{h \in G^{r(g_1)}} f(h)(g_1, ..., g_p)\xi(h^{-1}g_1, g_2, ..., g_p).$$

[7] The function $\langle \xi, \eta \rangle$ is well-defined as the sum is finite; it is continuous as p is étale; and it is compactly supported as it is supported in $p(\mathrm{supp}(\xi) \cap \mathrm{supp}(\eta))$.

This extends to an injective representation

$$\kappa_p : C_0(G^{(p)}) \rtimes_r G \to \mathcal{L}(L^2(G^{(p)}, \mathrm{pr}))$$

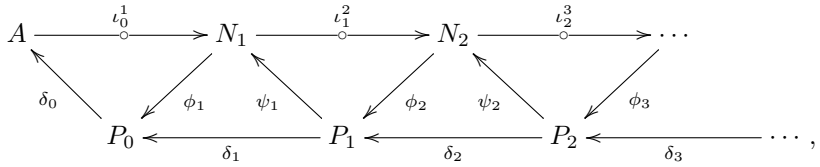
whose image is exactly $\mathrm{K}(L^2(G^{(p)}, \mathrm{pr}))$.

In particular, the Kasparov class

$$[\kappa_p, L^2(G^{(p)}, \mathrm{pr}), 0] \in KK(C_0(G^{(p)}) \rtimes_r G, C_0(G^{(p-1)}))$$

is an isomorphism, induced by a canonical Morita equivalence $C_0(G^{(p)}) \rtimes_r G \overset{M}{\sim} C_0(G^{(p-1)})$. □

We will need to work with a phantom tower in the sense of [37, Definition 3.1] associated to the $\mathcal{I}$-projective resolution in line (17) above. This is an augmentation of that resolution to a diagram of the form

(20)


satisfying certain conditions: we refer the reader to [37] for more details. Note that as in [37] we are using circled arrows for morphisms of degree one and plain arrows for morphisms of degree zero, but the circles are not in the same place as those of [37, Definition 3.1]. This is because (following [44]), we are working with *even* $\mathcal{I}$-projective resolutions, whereas in [37, Definition 3.1], Meyer works with *odd* resolutions, where the maps δ_p in line (17) have degree one (compare [44, Definition 2.15]). Analogously to the explanation below [44, Definition 2.15], it is straightforward to switch between these two degree conventions for phantom towers: if $\Sigma : KK^G \to KK^G$ is the suspension automorphism, and we are given a phantom tower associated to an odd $\mathcal{I}$-projective resolution with objects N_p and P_p as in [37, Definition 3.1], replacing these by $\Sigma^{-p}N_p$ and $\Sigma^{-p}P_p$ yields a phantom tower associated to the corresponding even resolution as in diagram (20) above.

Now, following [37, Lemma 3.2], the $\mathcal{I}$-projective resolution in line (17) can be augmented in an essentially unique way to a phantom tower as in line (20). Let $J : KK^G \to \mathrm{Ab}^{\mathbb{Z}/2}$ be the functor from KK^G to the category of $\mathbb{Z}/2$-graded abelian groups defined by $J := K_* \circ j_G$, where $j_G : KK^G \to KK$ is descent (see for example [7, Section 1.3]), and $K_* : KK \to \mathrm{Ab}^{\mathbb{Z}/2}$ is K-theory; note that on objects, $J(A) := K_*(A \rtimes_r G)$. This is a stable homological functor (see [38, Definitions 2.12 and 2.14]), so we may use it to build the ABC spectral sequence of [37, Sections 4 and 5] based on the phantom tower from line (20). If G is amenable and has torsion-free isotropy groups, [44, Corollary 4.4] shows that the ABC spectral sequence for $A = C_0(G^0)$ is a convergent spectral sequence

$$E^r_{p,q} \Rightarrow K_{p+q}(C^*_r(G)). \tag{21}$$

We call this the *Proietti-Yamashita (or PY) spectral sequence.*

As we will need to do explicit computations, let us give more details about how the general construction of the ABC spectral sequence specialises to our case, following [37, Sections 4 and 5]. For each $p, q \in \mathbb{N}$ as in [37, page 189], define[8]

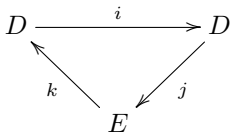
$$E^1_{p,q} := K_q(P_p \rtimes_r G) \cong \begin{cases} K_0(C_0(G^{(p)})) & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$$

[8]Note that the conventions of [37, Sections 4 and 5] would use "K_{p+q}" where we have "K_q" in the definitions of E^1_{pq} and of D^1_{pq}; the difference comes from our choice of using even projective resolutions - which necessitates introducing p-fold desuspensions - as explained above

where the isomorphism uses the canonical Morita equivalence $P_p \rtimes_r G = C_0(G^{(p+1)}) \rtimes_r G \overset{M}{\sim} C_0(G^{(p)})$ from Lemma 4.9. Define also

$$D^1_{p,q} := K_q(N_{p+1} \rtimes_r G).$$

Continuing following [37, page 189], there is an exact couple (see for example [54, Section 5.9] for background)



with morphisms $i_{pq} : D^1_{p,q} \to D^1_{p+1,q-1}$ defined by $i_{pq} := J_q(\iota^{p+2}_{p+1})$, $j_{pq} : D^1_{p,q} \to E^1_{p,q}$ defined by $j_{pq} := J_q(\phi_p)$, and $k_{pq} : E^1_{p,q} \to D^1_{p-1,q}$ defined by $k_{pq} := J_q(\psi_p)$. As in [44, Corollary 4.4], one computes that the second page E^2 has entries given by

$$E^2_{p,q} \cong \begin{cases} H_p(G) & q \text{ even} \\ 0 & q \text{ odd} \end{cases}.$$

With notation as in [37, page 172], [37, Theorem 5.1] implies that the filtration on $K_{p+q}(C^*_r(G))$ that corresponds to the convergence in line (21) above is given by $(J : \mathcal{I}^k(A))_{k\in\mathbb{N}}$; moreover, using the discussion on [37, pages 176-177], we see that for each $k \geqslant 1$

$$J : \mathcal{I}^k(A) = \text{Kernel}\big(J(\iota^k_{k-1} \circ \cdots \circ \iota^1_0) : K_*(A \rtimes_r G) \to K_*(N_k \rtimes_r G)\big), \tag{22}$$

while $J : \mathcal{I}^0(A) = 0$ by definition.

Now, following the proof of [37, Proposition 4.1] and using that in our case the spectral sequence converges, for each r, p, q with $0 \leqslant p \leqslant r$, the map[9]

$$k^{(r+1)}_{pq} = J_q(\psi_p) : E^{r+1}_{p,q} \to D^{r+1}_{p-1,q} \tag{23}$$

induced by k on the $(r+1)^{th}$ page has image equal to

$$J_q(\iota^p_{p-1} \circ \cdots \circ \iota^1_0)\big(J_q : \mathcal{I}^{p+1}(A)\big) \cong \frac{J_q : \mathcal{I}^{p+1}(A)}{J_q : \mathcal{I}^p(A)} \cong E^\infty_{p,q} \tag{24}$$

(for $p = 0$, $\iota^p_{p-1} \circ \cdots \circ \iota^1_0$ should be interpreted as the identity map on A).

Now, taking $p = 0$, $q = 0$, and $r = 2$ we get a canonical map

$$\mu_0 : H_0(G) = E^2_{0,0} \to \frac{J_0 : \mathcal{I}^1(C_0(G^0))}{J_0 : \mathcal{I}^0(C_0(G^0))}, \tag{25}$$

which is induced exactly by $J(\delta_0)$. Note however that by definition $J_0 : \mathcal{I}^0(C_0(G^0)) = 0$, and by definition of J, $J_0 : \mathcal{I}^1(C_0(G^0))$ is a subgroup of $K_0(C^*_r(G))$. Hence we may identify μ_0 with the map

$$\mu_0 : H_0(G) \to K_0(C^*_r(G))$$

induced by $\delta_0 \in KK^G(C_0(G), C_0(G^0))$. Let us record this more explicitly in a lemma.

Lemma 4.10. *Let $\delta_0 \in KK^G(C_0(G), C_0(G^0)) = KK^G(L(C_0(G^0)), C_0(G^0))$ be the co-unit of adjunction. Let $[X_G] \in KK(C_0(G^0), C_0(G) \rtimes_r G)$ be the KK class arising from the canonical Morita equivalence $C_0(G^0) \overset{M}{\sim} C_0(G) \rtimes_r G$ of Lemma 4.9. Then there is a canonical homomorphism*

$$\mu_0 : H_0(G) \to K_0(C^*_r(G))$$

[9] As mentioned by Meyer on [37, page 193], this map is functorially determined by A, and does not depend on any of the choices of projective resolution or phantom tower involved.

defined by taking the composition

$$J_0(\delta_0) \circ K_0([X_G]) : K_0(C_0(G^0)) \to K_0(C_r^*(G)),$$

then identifying $K_0(C_0(G^0)) = \mathbb{Z}[G^0]$*, and noting that this map descends to the quotient* $H_0(G)$ *of* $\mathbb{Z}[G^0]$*.* □

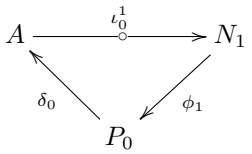
On the other hand, taking $p = 1$, $q = 0$, and $r = 2$ in lines (23) and (24), we get a canonical map

$$\mu_1 : H_1(G) = E^2_{1,0} \to \frac{J_1 : \mathcal{I}^2(C_0(G^0))}{J_1 : \mathcal{I}^1(C_0(G^0))} \tag{26}$$

induced by $J(\psi_1)$ and taking image in $J_1(\iota_0^1)\big(J_1(A)\big) \subseteq J_0(N_1)$, which is isomorphic to the right hand side in line (26) above. Note that by definition (compare line (22)), the group $J_1 : \mathcal{I}^1(C^0(G^0))$ is the kernel of the map

$$J_1(\iota_0^1) : J_1(A) \to J_0(N_1).$$

However, thanks to exactness of the first triangle

(27)


from line (20) and the fact that J is homological, we have an exact sequence

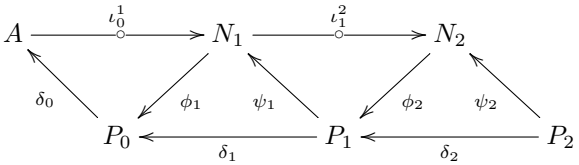
$$J_1(P_0) \xrightarrow{J(\psi_0)} J_1(A) \xrightarrow{J(\iota_0^1)} J_0(N_1)\ .$$

The group $J_1(P_0)$ is by definition equal to $K_1(C_0(G) \rtimes_r G)$, which is isomorphic to $K_1(C_0(G^0))$ by the Morita equivalence from Lemma 4.9, so is zero as G is ample. Hence by exactness, $J_1(\iota_0^1)$ is injective, and so $J_1 : \mathcal{I}^1(C_0(G^0)) = 0$. Putting this information into line (26), together with the fact that $J_1 : \mathcal{I}^2(C^0(G^0))$ is naturally a subgroup of $K_1(C_r^*(G))$, we get a canonical map

$$\mu_1 : H_1(G) \to K_1(C_r^*(G))$$

induced by $\psi_1 \in KK^G(C_0(G^{(2)}), N_1)$. Let us again record this more explicitly as a lemma.

Lemma 4.11. *Let* $\psi_1 \in KK^G(C_0(G^{(2)}), N_1) = KK^G(P_1, N_1)$ *be the morphism from the following part of a phantom tower for* $A = C_0(G^0)$

(28)


Let $J_1(\iota_0^1)|^{Im(J_1(\iota_0^1))}$ *be the corestriction of* $J(\iota_0^1)$ *to an isomorphism* $J_1(A) \to Image(J_1(\iota_0^1)) \subseteq J_0(N_1)$*. Let* $[X_{G^{(2)}}] \in KK(C_0(G), C_0(G^{(2)}) \rtimes_r G)$ *be the* KK*-isomorphism arising from the canonical Morita equivalence* $C_0(G) \overset{M}{\sim} C_0(G^{(2)}) \rtimes_r G$ *of Lemma 4.9.*

Then there is a canonical homomorphism

$$\mu_1 : H_1(G) \to K_1(C_r^*(G))$$

defined by taking the composition[10]

$$\left(J_1(\iota_0^1)|^{Im(J_1(\iota_0^1))}\right)^{-1} \circ J_0(\psi_1) \circ K_0([X_{G^{(2)}}]) : Ker(J_0(\delta_1)) \to K_1(C_r^*(G)),$$

and noting that this restriction passes to the quotient by the image of $J_0(\delta_2) : K_0(C_0(G^{(2)}) \rtimes_r G) \to K_0(C_0(G) \rtimes_r G)$[11] . □

Remark 4.12. In the context of the HK conjecture, it is natural to ask if there are "higher" maps $\mu_k : H_k(G) \to K_k(C_r^*(G))$ arising from the Proietti-Yamashita spectral sequence in a canonical way. This does not seem clear, even for $k = 2$: here lines (23) and (24) would give a map

$$\mu_2 : H_2(G) = E^2_{2,0} \to \frac{J_0 : \mathcal{I}^3(C_0(G^0))}{J_0 : \mathcal{I}^2(C_0(G^0))}.$$

For $k = 0$ and $k = 1$, the denominator on the right hand side turned out to be the zero group; for $k = 2$, this is no longer clear, so the natural map arising from the spectral sequence could a priori take its image in a proper quotient of $K_0(C_r^*(G))$, not in $K_0(C_r^*(G))$ itself. For $k = 3$, the situation is similar: one has

$$\mu_3 : H_3(G) = E^3_{3,0} \to \frac{J_1 : \mathcal{I}^4(C_0(G^0))}{J_1 : \mathcal{I}^3(C_0(G^0))}$$

and this map a priori takes values in a quotient of $K_1(C_r^*(G))$ (note that we need to increase to $r = 3$ so the condition "$0 \leqslant p \leqslant r$" needed to apply line (23) is satisfied; as the differentials on the E^2 page are all zero for degree reasons, we still have an identification $H_3(G) = E^3_{3,0}$ however). For $k = 4$, the situation seems worse: one has

$$\mu_4 : E^4_{4,0} \to \frac{J_0 : \mathcal{I}^5(C_0(G^0))}{J_0 : \mathcal{I}^4(C_0(G^0))}$$

but $E^4_{4,0}$ could in principle be a proper subquotient of $H_4(G)$, so a priori one only gets a map from a subquotient of $H_4(G)$ to a subquotient of $K_0(C_r^*(G))$; the situation is similar to this for all $k \geqslant 4$.

The recent principal counterexamples to the HK conjecture of Deeley [13] suggest that the a priori obstructions to the existence of the higher comparison maps discussed above really do pertain; however, we did not yet attempt the relevant computations.

4.3. **Identification of the comparison maps.** Our goal in this subsection is to identify the map $\mu_0 : H_0(G) \to K_0(C_r^*(G))$ of Lemma 4.10 with the comparison map constructed by Matui, and to identify the map $\mu_1 : H_1(G) \to K_1(C_r^*(G))$ of Lemma 4.11 with the explicitly constructed comparison map of Theorem 4.7. Much of what follows is essentially routine "book-keeping" computations; however, as some of it is of quite an involved nature, we thought it was worthwhile to record the details.

The special cases $L^2(G, r)$ and $L^2(G, s)$ of Definition 4.8 will be particularly important for us: we introduce the shorthand $\mathrm{K}_r := \mathrm{K}(L^2(G, r))$ and $\mathrm{K}_s := \mathrm{K}(L^2(G, s))$ for the compact operators on these modules. These C^*-algebras are equipped with the canonical G-actions coming from the left action of G on itself for $L^2(G, r)$, and the right action of G on itself for $L^2(G, s)$. We also write $M_r : C_0(G) \to \mathrm{K}_r$ and $M_s : C_0(G) \to \mathrm{K}_s$; the former is equivariant, while the latter is only equivariant if we consider $C_0(G)$ as a G-C^*-algebra via the right action

[10]Note that the restriction to the kernel $\mathrm{Ker}(J_0(\delta_1))$ of $J(\delta_1)$ is needed to ensure that the image of $J_0(\psi_1) \circ K_*([X_{G^{(2)}}])$ is contained in the image of $J_1(\iota_0^1)$; we mention this fact explicitly as it is slightly buried in the spectral sequence machinery.

[11]This last holds as commutativity of the rightmost triangle in diagram (28) and exactness of the second triangle from the right imply that $J(\psi_1) \circ J(\delta_2) = J(\psi_1) \circ J(\phi_2) \circ J(\psi_2) = 0 \circ J(\psi_2) = 0$.

of G on itself (however, we will never do this: $C_0(G)$ *always* either has the left G-action in what follows, or the trivial action if we have passed through descent).

With notation as in Lemma 4.10, μ_0 is the map induced on $H_0(G)$ by

$$J_0(\delta_0) \circ K_0([X_G]) = K_0([X_G] \otimes j_G(\delta_0)), \tag{29}$$

where $\delta_0 \in KK^G(C_0(G), C_0(G^0))$ equals the counit of adjunction $\epsilon_{C_0(G^0)}$ for the adjoint pair (E, F) discussed at the start of Subsection 4.2. Using the description of this counit in [7, Theorem 2.3], one computes that

$$\delta_0 = [M_r, L^2(G, r), 0]. \tag{30}$$

For the statement of the next lemma, let $\iota : C_0(G^0) \to C_r^*(G)$ denote the canonical inclusion. Recall also (compare [7, Section 1.3]) that the descent of a Hilbert G-A-module E is defined to be $E \rtimes_r G := E \otimes_A A \rtimes_r G$. To have concrete formulas to work with, let $\mathcal{E}$ denote the upper-semicontinuous field of Hilbert modules over G^0 associated with E. Then $E \rtimes_r G$ can alternatively be constructed as the completion of the vector space of compactly supported continuous sections $\Gamma_c(G; r^*\mathcal{E})$ with respect to the $A \rtimes_r G$-valued inner product

$$\langle \xi, \xi' \rangle(g) := \sum_{h \in G_{r(g)}} \alpha_{h^{-1}}\big(\langle \xi(h), \xi'(hg) \rangle_{A_{r(h)}}\big),$$

for $\xi, \xi' \in \Gamma_c(G; r^*\mathcal{E})$ and $g \in G$. Moreover, if $\pi : B \to \mathcal{L}(E)$ is a G-equivariant representation of B, then for $f \in \Gamma_c(G; r^*\mathcal{B})$, $\xi \in \Gamma_c(G; r^*\mathcal{E})$, and $g \in G$, the formula

$$(f \cdot \xi)(g) := \sum_{h \in G^{r(g)}} \pi_{r(h)}(f(h)) W_h\big(\xi(h^{-1}g)\big)$$

defines a representation $\pi \rtimes_r G : B \rtimes_r G \to \mathcal{L}(E \rtimes_r G)$.

The next lemma follows from direct (if somewhat lengthy) computations that we leave to the reader.

Lemma 4.13. *For $\xi \in L^2(G, s)$, $f \in C_c(G)$, $g \in G$, and $h \in G^{r(g)}$ define*

$$(\omega(\xi \otimes f)(g))(h) := \xi(h) f(h^{-1}g).$$

Then ω extends to a unitary isomorphism of Hilbert $C_r^*(G)$-modules

$$\omega : L^2(G, s) \otimes_\iota C_r^*(G) \cong L^2(G, r) \rtimes_r G.$$

Moreover, for κ_1 as in Lemma 4.9, ω satisfies

$$\omega(\kappa_1(a) \otimes 1_{C_r^*(G)})\omega^* = (M_r \rtimes_r G)(a)$$

for all $a \in C_0(G) \rtimes_r G$. In particular, there is a $*$-homomorphism $\beta : \mathrm{K}_s \to \mathrm{K}_r \rtimes_r G$ defined by

$$\beta : \mathrm{K}_s \to \mathrm{K}_r \rtimes_r G, \quad k \mapsto \omega(k \otimes 1_{C_r^*(G)})\omega^*.$$

Finally, $\omega^*(\mathrm{K}_r \rtimes_r G)\omega = \mathrm{K}(L^2(G, s) \otimes_\iota C_r^*(G))$, and so the element $[L^2(G, s) \otimes_\iota C_r^*(G), ad_{\omega^*} \circ M_r \rtimes G, 0] \in KK(\mathrm{K}_r \rtimes_r G, C_r^*(G))$ is an isomorphism in KK, induced by a Morita equivalence bimodule. □

We are now ready to show that the description of μ_0 arising from the spectral sequence agrees with the classical map $H_0(G) \to K_0(C_r^*(G))$ introduced by Matui.

Proposition 4.14. *The map $\mu_0 : H_0(G) \to K_0(C_r^*(G))$ of Lemma 4.10 is the map induced by the canonical inclusion $\iota : C_0(G^0) \to C_r^*(G)$ on the level of K_0-groups.*

Proof. Lemma 4.13 implies that $j_G(\delta_0)$ is represented by the class

$$[\kappa_1 \otimes 1_{C_r^*(G)}, L^2(G, s) \otimes_\iota C_r^*(G), 0] \in KK(C_0(G) \rtimes_r G, C_r^*(G)),$$

(where we have used the isomorphism $\mathrm{K}_s \cong C_0(G) \rtimes_r G$, which is a special case of Lemma 4.9). Lemma 4.9 implies that $[X_G]$ is represented by the opposite of

the Morita equivalence Kasparov cycle $(\kappa_1, L^2(G,s), 0)$. Hence $[X_G] \otimes j_G(\delta_0)$ is represented by the tensor product of these, which one computes directly is the cycle $(\iota, C_r^*(G), 0)$ in $KK(C_0(G^0), C_r^*(G))$. The image of this under the K-theory functor is indeed the map induced by ι, so we are done. □

We now move on to μ_1. Let us start by more explicitly describing the exact triangle

$$\begin{array}{ccc} C_0(G^0) & \xrightarrow{\iota_0^1} & N_1 \\ & {\scriptstyle \delta_0} \nwarrow \quad \swarrow {\scriptstyle \phi_1} & \\ & C_0(G) & \end{array} \tag{31}$$

from line (27) above. Note that the class $[\mathrm{id}, L^2(G,r), 0] \in KK^G(\mathrm{K}_r, C_0(G^0))$ is an isomorphism (as it arises from an equivariant Morita equivalence) and from the formula in line (30) for δ_0 we clearly have

$$[M_r, \mathrm{K}_r, 0] \otimes [\mathrm{id}, L^2(G,r), 0] = \delta_0,$$

so up to replacing $C_0(G^0)$ by K_r using the isomorphism $[\mathrm{id}, L^2(G,r), 0] \in KK^G(\mathrm{K}_r, C_0(G^0))$, we may replace the exact triangle in line (31) above with

$$\begin{array}{ccc} \mathrm{K}_r & \xrightarrow{\iota_0^1} & N_1 \\ & {\scriptstyle M_r} \nwarrow \quad \swarrow {\scriptstyle \phi_1} & \\ & C_0(G) & \end{array} \tag{32}$$

where $M_r \in KK^G(C_0(G), \mathrm{K}_r)$ is the class corresponding to the $*$-homomorphism $M_r : C_0(G) \to \mathrm{K}_r$, and we have abused notation slightly by keeping the same label for the top horizontal map. The remaining parts of the diagram can be described explicitly in terms of the mapping cone of M_r; we recall this next.

Now, by definition of the exact triangles in KK^G (see [44, Appendix A.4]), in diagram (32) we may take $N_1 = C(M_r)$, and ι_0^1 and ϕ_1 are then the KK-classes given by the left and right maps respectively in the canonical short exact sequence

$$0 \longrightarrow \Sigma \mathrm{K}_r \xrightarrow{\iota} C(M_r) \xrightarrow{e^0} C_0(G) \longrightarrow 0\,.$$

Hence we may replace the first part of the phantom tower in line (28) above with the diagram

$$\begin{array}{ccccc} \mathrm{K}_r & & \xrightarrow{\iota} & & C(M_r) \\ & {\scriptstyle M_r} \nwarrow & & \swarrow {\scriptstyle e^0} \quad \nwarrow {\scriptstyle \psi_1} & \\ & C_0(G) & \xleftarrow[\delta_1]{} & C_0(G^{(2)}). & \end{array}$$

According to the proof of [37, Lemma 3.2], the morphism ψ_1 appearing above is unique, subject to the condition that the right hand triangle commutes (and the various other conditions defining a phantom tower, which are satisfied by the diagram above). Our next task towards computing μ_1 is to give an explicit description of ψ_1; for this, it will be helpful to see why $M_r \circ \delta_1 = 0$ (note that this is indeed the case, by definition of an $\mathcal{I}$-projective resolution).

We first find an explicit representation for $\delta_1 \in KK^G(C_0(G^{(2)}), C_0(G))$. Let $\mathrm{pr}_1 : G_s \times_r G \to G$ be the projection onto the first factor, and define

$$F_0 := L^2(G_s \times_r G, \mathrm{pr}_1).$$

Similarly, let $\mathrm{pr}_2 : G_r \times_r G \to G$ be the projection on the second factor and define

$$F_1 := L^2(G_r \times_r G, \mathrm{pr}_2).$$

Equip F_0 with the G-action defined by the left translation action of G on the first factor, and F_1 with the G-action defined by the diagonal left action of G; note that both F_0 and F_1 are then G-$C_0(G)$ Hilbert modules, where the action on $C_0(G)$ is (as usual) induced from the left action of G on itself. Define a representation

$$\pi_0 : C_0(G^{(2)}) \to \mathcal{L}(F_0)$$

by pointwise multiplication, and a representation

$$\pi_1 : C_0(G^{(2)}) \to \mathcal{L}(F_1)$$

by

$$(\pi_1(f)\xi)(g,h) := f(g, g^{-1}h)\xi(g,h);$$

both of these representations are equivariant and take values in the compact operators on the corresponding modules. Using the explicit description of the induction functor from [7, Section 2.1] and of the co-unit of adjunction from [7, Theorem 2.3], one computes that the elements $L(\epsilon_{C_0(G^0)})$ and $\epsilon_{L(C_0(G^0))}$ of $KK^G(C_0(G^{(2)}), C_0(G))$ satisfy

$$\epsilon_{L(C_0(G^0))} = [\pi_0, F_0, 0]$$

and

$$L(\epsilon_{C_0(G^0)}) = [\pi_1, F_1, 0];$$

from the formula in line (18), we thus have that

$$\delta_1 = [\pi_0, F_0, 0] - [\pi_1, F_1, 0]. \tag{33}$$

On the other hand, from the fact that line (17) is a projective resolution, we have that $\delta_1 \otimes \delta_0 = 0$; as we are identifying δ_0 with $[M_r, \mathrm{K}_r, 0] \in KK^G(C_0(G), \mathrm{K}_r)$, we therefore see that

$$[\pi_0 \otimes 1_{\mathrm{K}_r}, F_0 \otimes_{M_r} \mathrm{K}_r, 0] = [\pi_1 \otimes 1_{\mathrm{K}_r}, F_1 \otimes_{M_r} \mathrm{K}_r, 0]$$

in $KK^G(C_0(G^{(2)}), \mathrm{K}_r)$. In order to construct ψ_1 and also its image under the descent morphism, we need to make this identity explicit; this is our next task.

We may identify $L^2(G,s) \otimes_\iota C_r^*(G)$ with the completion of $C_c(G_s \times_r G)$ for the inner product valued in the dense subset $C_c(G)$ of $C_r^*(G)$ defined by

$$\langle \xi, \eta \rangle_s(g) := \sum_{h \in G^{r(g)}} \sum_{k \in G_{s(h)}} \overline{\xi(k, h^{-1})}\eta(k, h^{-1}g).$$

Similarly, $L^2(G,r) \otimes_\iota C_r^*(G)$ identifies with the completion of $C_c(G_r \times_r G)$ for the inner product defined by

$$\langle \xi, \eta \rangle_r(g) := \sum_{h \in G^{r(g)}} \sum_{k \in G^{s(h)}} \overline{\xi(k, h^{-1})}\eta(k, h^{-1}g).$$

Given these descriptions, direct checks show that the map $G_r \times_r G \to G_s \times_r G$, $(g,h) \mapsto (g, g^{-1}h)$ induces a unitary $C_r^*(G)$-module isomorphism

$$w : L^2(G,s) \otimes_\iota C_r^*(G) \to L^2(G,r) \otimes_\iota C_r^*(G).$$

Lemma 4.15. *There is a unitary, equivariant, isomorphism*

$$u : F_0 \otimes_{M_r} \mathrm{K}_r \to F_1 \otimes_{M_r} \mathrm{K}_r$$

such that $u(\pi_0 \otimes 1_{\mathrm{K}_r})u^* = \pi_1 \otimes 1_{\mathrm{K}_r}$.

Moreover if $L^2(G^{(2)}, \mathrm{pr})$ and κ_2 are as in Lemma 4.9, then there is a commutative diagram

$$\begin{array}{ccc} \big(F_0 \otimes_{M_r} \mathrm{K}_r\big) \rtimes_r G \otimes_{id} L^2(G,r) \rtimes_r G & \xrightarrow{j_G(u)\otimes 1} & \big(F_1 \otimes_{M_r} \mathrm{K}_r\big) \rtimes_r G \otimes_{id} L^2(G,r) \rtimes_r G \\ \downarrow \Theta_0 & & \downarrow \Theta_1 \\ L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_s} L^2(G,s) \otimes_\iota C_r^*(G) & \xrightarrow{1\otimes w} & L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_r} L^2(G,r) \otimes_\iota C_r^*(G) \end{array}$$

where all the maps are unitary isomorphisms. The maps Θ_0 and Θ_1 satisfy

$$\Theta_0(\pi_0 \rtimes_r G)\Theta_0^* = \kappa_2 \otimes 1_{\mathrm{K}_s \otimes_\iota C_r^*(G)}$$

and

$$\Theta_1(\pi_1 \rtimes_r G)\Theta_1^* = \kappa_2 \otimes 1_{\mathrm{K}_r \otimes_\iota C_r^*(G)}$$

respectively. The map $w : L^2(G,s) \otimes_\iota C_r^*(G) \to L^2(G,r) \otimes_\iota C_r^*(G)$ is that defined above, and satisfies

$$w(M_s \otimes 1_{C_r^*(G)})w^* = M_r \otimes 1_{C_r^*(G)}.$$

Proof. We begin by defining a unitary isomorphism of Hilbert $C_0(G^0)$-modules

$$U : F_0 \otimes_{M_r} L^2(G,r) \to F_1 \otimes_{M_r} L^2(G,r)$$

such that $U(\pi_0 \otimes 1)U^* = \pi_1 \otimes 1$ as follows: a straightforward computation gives identifications

$$F_0 \otimes_{M_r} L^2(G,r) \cong L^2(G^{(2)}, r \circ \mathrm{pr}_1), \quad F_1 \otimes_{M_r} L^2(G,r) \cong L^2(G_r \times_r G, r \circ \mathrm{pr}_2). \tag{34}$$

It is then easy to see that the map $G_r \times_r G \to G^{(2)}$, $(g,h) \mapsto (g, g^{-1}h)$ gives rise to the desired equivalence.

Now recall, that K_r splits as a tensor product $L^2(G,r) \otimes_{C_0(G^0)} L^2(G,r)^{\mathrm{op}}$. After making this identification we can define u as

$$F_0 \otimes_{M_r} L^2(G,r) \otimes_{C_0(G^0)} L^2(G,r)^{\mathrm{op}} \overset{U\otimes 1}{\to} F_1 \otimes_{M_r} L^2(G,r) \otimes_{C_0(G^0)} L^2(G,r)^{\mathrm{op}}.$$

We now move on to the commutative diagram. Notice that $(F_0 \otimes_{M_r} \mathrm{K}_r) \rtimes_r G \otimes (L^2(G,r) \rtimes_r G) \cong (F_0 \otimes_{M_r} L^2(G,r)) \rtimes_r G$, and similarly $(F_1 \otimes_{M_r} \mathrm{K}_r) \rtimes_r G \otimes (L^2(G,r) \rtimes_r G) \cong (F_1 \otimes_{M_r} L^2(G,r)) \rtimes_r G$. Applying these identifications to the top row of the diagram in the lemma and using identifications similar to the ones in line (34) in the bottom row (and slightly abusing notation by still denoting the maps Θ_0 and Θ_1) shows that it will be enough to exhibit a commutative diagram of the form

$$\begin{array}{ccc} \big(F_0 \otimes_{M_r} L^2(G,r)\big) \rtimes_r G & \xrightarrow{U \rtimes_r G} & \big(F_1 \otimes_{M_r} L^2(G,r)\big) \rtimes_r G \\ \downarrow \Theta_0 & & \downarrow \Theta_1 \\ L^2(G^{(2)}, s \circ \mathrm{pr}) \otimes_\iota C_r^*(G) & \xrightarrow{W} & L^2(G^{(2)}, r \circ \mathrm{pr}) \otimes_\iota C_r^*(G) \end{array}$$

where W is induced by the map $G^{(2)} \times_{s\circ\mathrm{pr},r} G \to G^{(2)} \times_{r\circ\mathrm{pr},r} G$, $(h_1, h_2, g) \mapsto (h_1, h_2, h_2^{-1}g)$. Similar to the discussion just prior to the present lemma, the modules involved in the diagram all have a canonical dense subspace of compactly supported functions defined on a suitable fibred product of $G^{(2)}$ or $G_r \times_r G$ with G. Hence it will be enough to describe a "dual" commutative diagram of homeomorphisms

$$\begin{array}{ccc} G^{(2)}_{r\circ\mathrm{pr}_1}\times_r G & \xleftarrow[\natural]{} & (G_r\times_r G)_{r\circ\mathrm{pr}_2}\times_r G \\ {\scriptstyle\theta_0}\uparrow & & \uparrow{\scriptstyle\theta_1} \\ G^{(2)}_{s\circ\mathrm{pr}_2}\times_r G & \xleftarrow[\omega]{} & G^{(2)}_{r\circ\mathrm{pr}_2}\times_r G \end{array}$$

The map $\natural$ inducing $U\rtimes_r G$ is given by

$$\natural(h_1,h_2,g)=(h_1,h_1^{-1}h_2,g)$$

and the map ω inducing W is given by

$$\omega(h_1,h_2,g)=(h_1,h_2,h_2^{-1}g).$$

If we set $\theta_0(h_1,h_2,g):=(h_1,h_2,h_1h_2g)$ and $\theta_1(h_1,h_2,g):=(h_1,h_1^{-1}h_2,h_1g)$, then the diagram commutes. It follows that the induced diagram of Hilbert modules commutes. Moreover, direct checks show that Θ_0 intertwines the representations $\pi_0\rtimes_r G$ and $\kappa_2\otimes 1_{L^2(G,s)\otimes_\iota C^*_r(G)}$ and Θ_1 intertwines $\pi_1\rtimes_r G$ and $\kappa_2\otimes 1_{L^2(G,r)\otimes_\iota C^*_r(G)}$. □

Using this lemma, we may finally compute an explicit formula for both ψ_1 and its descent $j_G(\psi_1)$. Define

$$F_{\mathrm{K}}:=I\big((F_0\otimes_{M_r}\mathrm{K}_r)\oplus(F_1\otimes_{M_r}\mathrm{K}_r)\big)$$

with the $I\mathrm{K}_r$-valued inner product given by adding the componentwise inner products pointwise for each $t\in[0,1]$. Define F to be the collection of all triples (ξ_0,η,ξ_1) in the direct sum $F_0\oplus F_{\mathrm{K}}\oplus F_1$ such that[12]

$$(\xi_0\otimes 1_{\mathrm{K}_r},0)=\eta(0)\quad\text{and}\quad\eta(1)=(0,\xi_1\otimes 1_{\mathrm{K}_r}).$$

Let $D(M_r)$ be as in line (11), and define a $D(M_r)$-valued inner product on F by

$$\langle(\xi_0^{(0)},\eta^{(0)},\xi_1^{(0)})\ ,\ (\xi_0^{(1)},\eta^{(1)},\xi_1^{(1)})\rangle:=\Big(\langle\xi_0^{(0)},\xi_0^{(1)}\rangle_{F_0}\ ,\ \langle\eta^{(0)},\eta^{(1)}\rangle_{F_{\mathrm{K}}}\ ,\ \langle\xi_1^{(0)},\xi_1^{(1)}\rangle_{F_1}\Big)$$

Direct checks then show that F is a well-defined Hilbert G-$D(M_r)$-module.

Now, with u as in Lemma 4.15 consider the unitary

$$V:=\begin{pmatrix}0 & u^*\\ u & 0\end{pmatrix}\in\mathcal{L}\big((F_0\otimes_{M_r}\mathrm{K}_r)\oplus(F_1\otimes_{M_r}\mathrm{K}_r)\big),$$

which commutes with the direct sum action of G. This is a self-adjoint unitary, so by the spectral theorem we can write $V=p-q$[13], where p and q are orthogonal projections that commute with the G-action and satisfy $p+q=1$. For $t\in[0,1]$, define $V_t=p+e^{i\pi t}q$, so $\{V_t\}_{t\in[0,1]}$ is a path of G-invariant unitaries connecting $V_0=1$ and $V_1=V$. For $f\in C_0(G^{(2)})$, $\eta\in F_{\mathrm{K}}$, and $t\in[0,1]$, define

$$(M_{\mathrm{K}}(f)\eta)(t):=V_t\big(\pi_0(f)\otimes 1_{\mathrm{K}_r},0\big)V_t^*.$$

This defines a G-equivariant representation $\pi_{\mathrm{K}}:C_0(G^{(2)})\to F_{\mathrm{K}}$. Moreover, it is compatible with the representations π_0 and π_1 of $C_0(G^{(2)})$ on F_0 and F_1 respectively, in the sense that for $f\in C_0(G^{(2)})$ and $(\xi_0,\eta,\xi_1)\in F$, the formula

$$\pi_F(f)(\xi_0,\eta,\xi_1):=(\pi_0(f)\xi_0,\pi_{\mathrm{K}}(f)\eta,\pi_1(f)\xi_1)$$

[12]Note that even though K_r is typically non-unital, $\xi_i\otimes 1_{\mathrm{K}_r}$ still makes sense as an element of $F_i\otimes_{M_r}\mathrm{K}_r$ for $i\in\{0,1\}$ - for example, one can show that if $(u_j)_{j\in J}$ is an approximate unit for K_r, then $(\xi_i\otimes u_j)_{j\in J}$ is Cauchy in $F_i\otimes_{M_r}\mathrm{K}_r$, and then define $\xi_i\otimes 1_{\mathrm{K}_r}$ to be its limit.

[13]There are also concrete formulas: $p=\frac{1}{2}\begin{pmatrix}1 & u^*\\ u & 1\end{pmatrix}$ and $q=\frac{1}{2}\begin{pmatrix}1 & -u^*\\ -u & 1\end{pmatrix}$.

defines a Hilbert G-$D(M)$-module representation $\pi_F : C_0(G^{(2)}) \to \mathcal{L}(F)$. This representation takes values in the compact operators on F, and thus we get a well-defined Kasparov element

$$\widetilde{\psi} := [\pi_F, F, 0] \in KK^G(C_0(G^{(2)}), D(M_r)).$$

The short exact sequence of line (12) and split exactness of KK^G-theory then gives a canonical isomorphism

(35)
$$KK^G(C_0(G^{(2)}), D(M_r)) \cong KK^G(C_0(G^{(2)}), C_0(G)) \oplus KK^G(C_0(G^{(2)}), C(M_r)).$$

Write ψ for the image of $\widetilde{\psi}$ in $KK^G(C_0(G^{(2)}), C(M_r))$ under the canonical quotient map arising from the direct sum decomposition above.

Lemma 4.16. *The element ψ fits into the canonical phantom tower in KK^G as the map labeled ψ_1*

(36)
$$\begin{array}{ccccc} \mathrm{K}_r & & \xrightarrow{\quad\iota\quad} & & C(M_r) \\ & {\scriptstyle M_r}\nwarrow & & \swarrow{\scriptstyle e^0} & \nwarrow{\scriptstyle \psi_1} \\ & C_0(G) & \xleftarrow[\delta_1]{} & & C_0(G^{(2)}). \end{array}$$

Proof. As we already noted, the proof of [37, Lemma 3.2] shows that ψ_1 always exists, and is uniquely determined by the property $\psi_1 \otimes [e^0] = \delta_1$. We thus need to show that $\psi \otimes [e^0] = \delta_1$.

We have canonical quotient maps $f^j : D(M_r) \to C_0(G)$ defined by $f^j : (a_0, b, a_1) \mapsto a_j$ for $j \in \{0, 1\}$. Let $s : C_0(G) \to D(M_r)$, $a \mapsto (a, \mathsf{c}(M_r(a)), a)$ be the canonical splitting of the short exact sequence from line (12). Clearly $f^0 \circ s = f^1 \circ s$, whence the map

$$\cdot \otimes ([f^0] - [f^1]) : KK^G(C_0(G^{(2)}), D(M)) \to KK^G(C_0(G^{(2)}), C_0(G))$$

vanishes on $s_*(KK^G(C_0(G^{(2)}), C_0(G)))$. Let $i : C(M_r) \to D(M_r)$, $(a_0, b) \mapsto (a_0, b, 0)$ be the canonical inclusion. Then according to the isomorphism in line (35) we have $\widetilde{\psi} = i_*(\psi) \oplus s_*(\alpha)$ for some $\alpha \in KK^G(C_0(G^{(2)}), C_0(G))$, so the above-discussed vanishing of $\cdot \otimes ([f^0] - [f^1])$ on the image of s_* gives

$$\widetilde{\psi} \otimes ([f^0] - [f^1]) = i_*(\psi) \otimes ([f^0] - [f^1]) = \psi \otimes ([f^0 \circ i] - [f^1 \circ i]).$$

On the other hand, we clearly have $f^0 \circ i = e^0$ and $f^1 \circ i = 0$, so the above implies that

$$\widetilde{\psi} \otimes ([f^0] - [f^1]) = \psi \otimes [e^0]$$

It thus suffices to show that $\widetilde{\psi} \otimes ([f^0] - [f^1]) = \delta_1$. For this, we note that $\widetilde{\psi}[f^j] = [F_j, \pi_j, 0]$ for $j \in \{0, 1\}$ so we need to show that

$$\delta_1 = [\pi_0, F_0, 0] - [\pi_1, F_1, 0];$$

this is exactly the formula in line (33), so we are done. □

Our next goal is to compute the image of the diagram in line (36) under descent. Unfortunately, this necessitates more notation. Let $\iota : C_0(G^0) \to C_r^*(G)$ denote the canonical inclusion, and let $C(\iota)$ and $D(\iota)$ be the corresponding C^*-algebras from lines (7) and (11). Define X_D to consist of all triples (ξ_0, η, ξ_1) in

$$L^2(G, s) \oplus I\big(L^2(G, s) \otimes_\iota C_r^*(G)\big) \oplus L^2(G, s)$$

such that $\xi_i \otimes 1_{C_r^*(G)} = \eta(i)$ for $i \in \{0, 1\}$. Direct checks based on Lemma 4.13 shows that this is canonically a Morita equivalence $D(M_r) \rtimes_r G$-$D(\iota)$-bimodule. Similarly, if X_C consists of all pairs in $L^2(G, s) \oplus I\big(L^2(G, s) \otimes_\iota C_r^*(G)\big)$ such that

$\xi_0 \otimes 1_{C_r^*(G)} = \eta(0)$ and $\eta(1) = 0$, we see that X_C is a Morita equivalence $C(M_r) \rtimes_r G$-$C(\iota)$ bimodule. Moreover, if X_G is the Morita equivalence $C_0(G) \rtimes_r G$-$C_0(G^0)$ bimodule from Lemma 4.9, then the following diagram (built from the general short exact sequence of line (12)) is easily seen to commute in KK

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(M_r) \rtimes_r G & \longrightarrow & D(M_r) \rtimes_r G & \longrightarrow & C_0(G) \rtimes_r G & \longrightarrow & 0 \\ \| & & \cong \downarrow X_C & & \cong \downarrow X_D & & \cong \downarrow X_G & & \| \\ 0 & \longrightarrow & C(\iota) & \longrightarrow & D(\iota) & \longrightarrow & C_0(G^0) & \longrightarrow & 0. \end{array} \tag{37}$$

For ease of notation let E_s and E_r denote the Hilbert $C_r^*(G)$-modules $L^2(G,s) \otimes_\iota C_r^*(G)$ and $L^2(G,r) \otimes_\iota C_r^*(G)$, respectively. Let E be the collection of triples (ξ_0, η, ξ_1) in

$$L^2(G,s) \oplus I\big(E_s \oplus E_r\big) \oplus L^2(G,r) \tag{38}$$

such that $(\xi_0 \otimes 1_{C_r^*(G)}, 0) = \eta(0)$ and $(0, \xi_1 \otimes 1_{C_r^*(G)}) = \eta(1)$; this is a Hilbert $D(\iota)$-module in the natural way. It is moreover equipped with a left $C_0(G)$-action defined as follows. Let $w : E_s \to E_r$ be the unitary isomorphism from Lemma 4.15, and define

$$v := \begin{pmatrix} 0 & w^* \\ w & 0 \end{pmatrix} \in \mathcal{L}(E_s \oplus E_r).$$

As in the discussion defining $\widetilde{\psi}$, we may write $v = p - q$ for complementary projections p and q, and define $v_t := p + e^{i\pi t} q$. Then the formula

$$\pi_E := M_s \oplus v_t(M_s \otimes 1_{C_r^*(G)}, 0)v_t^* \oplus M_r \tag{39}$$

defines the desired $C_0(G)$-action on E. Define $\widetilde{\Psi} := [\pi_E, E, 0] \in KK(C_0(G), D(\iota))$, and define Ψ to be the image of $\widetilde{\Psi}$ in $KK(C_0(G), C(\iota))$ under the canonical quotient map arising from the direct sum decomposition

$$KK(C_0(G), D(\iota)) \cong KK(C_0(G), C(\iota)) \oplus KK(C_0(G), C_0(G^0))$$

that in turn arises from the split short exact sequence in line (12).

The next lemma is the last main ingredient needed to compute μ_1. To state it, let $\iota : \Sigma C_r^*(G) \to C(\iota)$ and $e^0 : C(\iota) \to C_0(G^0)$ be the canonical maps associated to the mapping cone. Let also $[s]$ and $[r]$ respectively denote the elements $[M_s, L^2(G,s), 0]$ and $[M_r, L^2(G,r), 0]$ of $KK(C_0(G), C_0(G^0))$.

Lemma 4.17. *After applying the canonical Morita isomorphisms $C(M_r) \rtimes_r G \overset{X_C}{\sim} C(\iota)$ discussed above, the Morita isomorphism $\mathrm{K}_r \rtimes_r G \overset{X_{G,\mathrm{K}}}{\sim} C_r^*(G)$ of Lemma 4.13, and $C_0(G) \rtimes_r G \overset{X_G}{\sim} C_0(G^0)$ and $C_0(G^{(2)}) \rtimes_r G \overset{X_{G^{(2)}}}{\sim} C_0(G)$ of Lemma 4.9, the image of the commutative diagram (36) above under descent $j_G : KK^G \to KK$ identifies with*

$$\begin{array}{ccccc} C_r^*(G) & & \xrightarrow{\quad\iota\quad} & & C(\iota) \\ & {\scriptstyle\iota}\nwarrow & & \swarrow{\scriptstyle e^0} & \nwarrow{\scriptstyle\Psi} \\ & & C_0(G^0) & \xleftarrow[[s]-[r]]{} & C_0(G) \end{array} \tag{40}$$

Proof. That the left triangle in line (36) has image equal to the left triangle in line (40) under descent (and modulo the given Morita equivalences) follows from the

commutative diagram of short exact sequences in KK

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma\mathrm{K}_r \rtimes_r G & \longrightarrow & C(M_r) \rtimes_r G & \longrightarrow & C_0(G) \rtimes_r G & \longrightarrow & 0 \\ \| & & \cong \downarrow X_{G,\mathrm{K}} & & \cong \downarrow X_C & & \cong \downarrow X_G & & \| \\ 0 & \longrightarrow & \Sigma C_r^*(G) & \longrightarrow & C(\iota) & \longrightarrow & C_0(G^0) & \longrightarrow & 0. \end{array} \tag{41}$$

We next claim that $j_G(\psi) \otimes_{C(M_r)\rtimes_r G} [X_C] = [X_{G^{(2)}}] \otimes_{C_0(G)} \Psi$. Thanks to the commutative diagram in line (37) above, it suffices to show that

$$j_G(\widetilde{\psi}) \otimes_{D(M_r)\rtimes_r G} [X_D] = [X_{G^{(2)}}] \otimes_{C_0(G)} \widetilde{\Psi}. \tag{42}$$

This will moreover show that the bottom horizontal arrow in line (40) is correctly labeled, as it is clear that if $f^0, f^1 : D(\iota) \to C_0(G^0)$ are the canonical $*$-homomorphisms, then $\widetilde{\Psi} \otimes ([f^0] - [f^1]) = [s] - [r]$; thus to complete the proof it suffices to establish the identity in line (42).

The Kasparov product $j_G(\widetilde{\psi}) \otimes_{D(M_r)\rtimes_r G} [X_D]$ is represented by the triple

$$\big(\pi_F \rtimes_r G \otimes 1, F \rtimes_r G \otimes X_D, 0\big).$$

Our first goal is to identify this triple with the triple

$$\big(\kappa_2 \otimes (1 \oplus v_t(1,0)v_t^* \oplus 1), L^2(G^{(2)}, \mathrm{pr}) \otimes_{\tilde{\pi}_E} E, 0\big),$$

where we emphasise that we are using the representation $\tilde{\pi}_E = (M_s, I(M_s \otimes 1, M_r \otimes 1), M_r)$ (as opposed to π_E defined in line (39)).

In fact we will deal with the ambient modules of E and F respectively, which allows us to treat each component separately to improve readability. We first deal with the first and third components. For these we have isomorphisms

$$(F_0 \rtimes_r G) \otimes_{\kappa_1} L^2(G, s) \to L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_s} L^2(G, s) \tag{43}$$

$$(F_1 \rtimes_r G) \otimes_{\kappa_1} L^2(G, s) \to L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_r} L^2(G, r) \tag{44}$$

identifying the first and third components of $F \rtimes_r G \otimes X_D$ and $L^2(G^{(2)}, \mathrm{pr}) \otimes_{\tilde{\pi}_E} E$, respectively. These isomorphisms are produced in a similar fashion so will only explain the procedure for (44): We apply the isomorphism ω from Lemma 4.13 to the transformation groupoid $G_r \times_r G$ of the left action of G on itself (in place of G) to obtain an isomorphism

$$L^2(G_r \times_r G, S) \otimes_{C_0(G)} C_0(G) \rtimes_{\mathrm{lt}} G \xrightarrow{\cong} F_1 \rtimes_r (G_r \times_r G) \cong F_1 \rtimes_r G,$$

where $S : G_r \times_r G \to G$ denotes the source map of $G_r \times_r G$ given by $S(g,h) = g^{-1}h$. Using this we get a chain of identifications

$$\begin{aligned} (F_1 \rtimes_r G) \otimes_{\kappa_1} L^2(G, s) &\cong L^2(G_r \times_r G, S) \otimes_{M_s} C_0(G) \rtimes_r G \otimes_{\kappa_1} L^2(G, s) \\ &\cong L^2(G_r \times_r G, S) \otimes_{M_s} L^2(G, s) \\ &\cong L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_s} L^2(G, s) \end{aligned}$$

A tedious but routine calculation shows that this isomorphism intertwines the actions $\pi_1 \rtimes G \otimes 1$ and $\kappa_2 \otimes 1$.

It remains to identify the middle components of $F \rtimes_r G \otimes X_D$ and $L^2(G^{(2)}, \mathrm{pr}) \otimes_{\tilde{\pi}_E} E$, respectively. For every $t \in [0,1]$ we have the following chain of isomorphisms, where we use Lemma 4.13 in line 2 and the isomorphisms Θ_0 and Θ_1 from Lemma

4.15 in line 4.

$$\begin{aligned}
&(F_0 \otimes_{M_r} \mathrm{K}_r \oplus F_1 \otimes_{M_r} \mathrm{K}_r) \rtimes_r G \otimes_{Ad\,\omega} E_s \\
\cong&(F_0 \otimes_{M_r} \mathrm{K}_r \oplus F_1 \otimes_{M_r} \mathrm{K}_r) \rtimes_r G \otimes_{id} (L^2(G,r) \rtimes_r G) \\
\cong&\big((F_0 \otimes_{M_r} \mathrm{K}_r) \rtimes_r G \otimes_{id} L^2(G,r) \rtimes_r G\big) \oplus \big((F_1 \otimes_{M_r} \mathrm{K}_r) \rtimes_r G \otimes_{id} L^2(G,r) \rtimes_r G\big) \\
\cong&L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_s \otimes 1} E_s \oplus L^2(G^{(2)}, \mathrm{pr}) \otimes_{M_r \otimes 1} E_r \\
\cong&L^2(G^{(2)}, \mathrm{pr}) \otimes_{(M_s \otimes 1, M_r \otimes 1)} (E_s \oplus E_r)
\end{aligned}$$

The commutative diagram in Lemma 4.15 together with the construction of the families of unitaries $(V_t)_t$ and $(v_t)_t$ imply that $(\Theta_0 \oplus \Theta_1)((V_t \rtimes_r G \otimes 1_{L^2(G,r)\rtimes G}) = (1_{L^2(G^{(2)},\mathrm{pr})} \otimes v_t)(\Theta_0 \oplus \Theta_1)$ and hence the chain of isomorphisms above intertwines the representations $(V_t(\pi_0 \otimes 1, 0)V_t^*) \rtimes_r G \otimes 1$ and $\kappa_2 \otimes v_t(1,0)v_t^*$ of $C_0(G^{(2)}) \rtimes_r G$. This completes the identification of the Kasparov triples.

Finally, we apply a standard trick in KK-theory to replace the representing module by a non-degenerate one, i.e. we pass to the module

$$\overline{\big(\kappa_2 \otimes (1, v_t(1,0)v_t^*, 1)\big)\big(L^2(G^{(2)}, \mathrm{pr}) \otimes_{\tilde{\pi}_E} E\big)}.$$

The latter module however is easily seen to be isomorphic to

$$L^2(G^{(2)}, \mathrm{pr}) \otimes_{\pi_E} E,$$

in such a way that the isomorphism intertwines the representations $\kappa_2 \otimes (1, v_t(1,0)v_t^*, 1)$ and $\kappa_2 \otimes 1_E$. □

Finally, we are ready to give our concrete formula for the comparison map μ_1.

Proposition 4.18. *The map μ_1 from Lemma 4.11 agrees with the canonical homomorphism from Theorem 4.7.*

Proof. According to the description in Theorem 4.7, it will suffice to show that for any compact open bisection V in G, if η is as in Theorem 4.3 we have that $\eta([1_V] \otimes \Psi) = [1_V^*]$ in $V_\infty(C_0(G^0), C_r^*(G))/\approx$.

Now, if we write $\lambda_V : \mathbb{C} \to \mathcal{L}(\pi_E(1_V)E)$ for the unital scalar representation, then

$$[1_V] \otimes \widetilde{\Psi} = [\lambda_V, \pi_E(1_V)E, 0].$$

Recalling (see lines (38) and (39)) that $\pi_E = M_s \oplus v_t(M_s \otimes 1_{C_r^*(G)}, 0)v_t^* \oplus M_r$, we compute that $M_s(1_V) \cdot L^2(G,s) \cong 1_{s(V)}C_0(G^0)$ as a right $C_0(G^0)$-module, and that $M_r(1_V) \cdot L^2(G,r) \cong 1_{r(V)}C_0(G^0)$ as a right $C_0(G^0)$-module. On the other hand $v_t(M_s(1_V) \otimes 1_{C_r^*(G)}, 0)v_t^* \cdot (L^2(G,s) \otimes_\iota C_r^*(G) \oplus L^2(G,r) \otimes_\iota C_r^*(G))$ is isomorphic to $p_t(C_r^*(G) \oplus C_r^*(G))$ as a right $C_r^*(G)$-module, where

$$p_t := \frac{1}{2}\begin{pmatrix} (1+\cos(\pi t))1_{s(V)} & -i\sin(\pi t)1_V^* \\ i\sin(\pi t)1_V & (1-\cos(\pi t))1_{r(V)} \end{pmatrix} \in M_2(C_r^*(G)).$$

It follows that $[1_V] \otimes \widetilde{\Psi} \in KK(\mathbb{C}, D(\iota))$ is represented by the class of the projection

$$\begin{pmatrix} 1_{s(V)} & 0 \\ 0 & 0 \end{pmatrix} \oplus (p_t)_{t\in[0,1]} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1_{r(V)} \end{pmatrix} \in M_2(D(\iota))$$

On the other hand, note that $p_t = u_t p_0 u_t^*$, where

$$u_t := \frac{1}{2}\begin{pmatrix} 2 - 1_{s(V)}(1-e^{i\pi t}) & 1_V^*(1-e^{i\pi t}) \\ 1_V(1-e^{i\pi t}) & 2 - 1_{r(V)}(1-e^{i\pi t}) \end{pmatrix},$$

so $(u_t)_{t\in[0,1]}$ defines a continuous path of unitaries in $M_2(C_r^*(G))$ (or in the unitization of this C^*-algebra if it is not unital). It follows from the definition of η given

in line (14) that

$$\eta([1_V]\otimes\widetilde{\Psi}) = \left[\begin{pmatrix} 1_{s(V)} & 0 \\ 0 & 0 \end{pmatrix} u_1\right].$$

Computing,

$$\begin{pmatrix} 1_{s(V)} & 0 \\ 0 & 0 \end{pmatrix} u_1 = \begin{pmatrix} 0 & 1_V^* \\ 0 & 0 \end{pmatrix},$$

so we see that $\eta([1_V]\otimes\widetilde{\Psi}) = [1_V^*]$. □

4.4. **The HK-conjecture in low dimensions.** In this subsection, we apply the previous result and Theorem 3.36 to deduce consequences for the HK conjecture.

As discussed in Subsection 4.2, a recent result of Proietti and Yamashita in [44] established the existence of a convergent spectral sequence

$$E^2_{p,q} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(A \rtimes_r G) \tag{45}$$

for any G-algebra A, provided that G is a second countable ample groupoid with torsion free isotropy, which satisfies the strong Baum-Connes conjecture.

Combining this with our results from previous sections we obtain the following application to the HK-conjecture. The reader should compare this to [44, Remark 4.5], which establishes a similar result on the HK conjecture under a vanishing hypothesis on $H_k(G)$ for $k \geqslant 3$; the main difference between Theorem 4.19 below and [44, Remark 4.5] is that the former gives a concrete criterion when vanishing holds.

Theorem 4.19. *Let G be a second countable principal ample groupoid with dynamic asymptotic dimension at most two. Then there is a short exact sequence*

$$0 \to H_0(G) \xrightarrow{\mu_0} K_0(C_r^*(G)) \to H_2(G) \to 0,$$

and $\mu_1 : H_1(G) \to K_1(C_r^*(G))$ is an isomorphism. If moreover $H_2(G)$ is free (e.g. if it is finitely generated), then the HK-conjecture holds for G, i.e.

$$K_0(C_r^*(G)) \cong H_0(G) \oplus H_2(G) \text{ and } K_1(C_r^*(G)) \cong H_1(G).$$

Proof. First of all we can apply the spectral sequence (45) since G is principal and any groupoid with finite dynamic asymptotic dimension is in particular amenable (this follows from the proof of [23, Theorem A.9]), and hence satisfies the strong Baum-Connes conjecture by the main result of [53]. Since $H_n(G) = 0$ for all $n \geqslant 3$ by Theorem 3.36 the spectral sequence (45) collapses on the second page and we conclude that $K_1(C_r^*(G)) \cong H_1(G)$ and that $K_0(C_r^*(G))$ fits into a short exact sequence

$$0 \to H_0(G) \to K_0(C_r^*(G)) \to H_2(G) \to 0.$$

If moreover $H_2(G)$ is free abelian (which holds if it is finitely generated, as it is torsion-free by Theorem 3.36), then the sequence above splits and we get a direct sum decomposition $K_0(C_r^*(G)) \cong H_0(G) \oplus H_2(G)$. □

With a view towards the classification program for simple nuclear C^*-algebras, we obtain the following consequence.

Corollary 4.20. *Let G be a second countable, principal, ample groupoid with compact base space and dynamic asymptotic dimension at most one. Then*

$$\mathrm{Ell}(C_r^*(G)) = (H_0(G), H_0(G)^+, [1_{G^0}], H_1(G), M(G), p),$$

where $M(G)$ is the set of all G-invariant probability measures on G^0 and $p : M(G) \times H_0(G) \to \mathbb{R}$ is the pairing given by $p(\mu, [f]_0) = \int f d\mu$.

Proof. We have canonical isomorphisms $\mu_i : H_i(G) \cong K_i(C_r^*(G))$. The isomorphism μ_0 clearly extends to an isomorphism of ordered groups respecting the position of the unit. Since G is principal, there is an affine homeomorphism between the set $M(G)$ of G-invariant probability measures on G^0 and the tracial state space $T(C_r^*(G))$ of the reduced groupoid C^*-algebra (see, for instance, [32, Section 4.1]). Finally, from the definition of μ_0 it is clear that the pairings are compatible. □

5. Examples and applications

In this final section we discuss several applications of our results for specific classes of groupoids and exhibit some interesting examples.

5.1. **Free actions on totally disconnected spaces.** In [10, Theorem 1.3], Conley et. al. show that for a large class of countable groups Γ, any free action on a second countable, locally compact, zero-dimensional space has dynamic asymptotic dimension at most the asymptotic dimension of Γ, and that the latter is finite. If X is compact, for example a Cantor set, then the dynamic asymptotic dimension will therefore be exactly equal to the asymptotic dimension of Γ by [24, Theorem 6.5]. The class described by the authors of [10] is technical and we refer there for details; suffice to say that it includes many interesting examples such as all polycylic groups, all virtually nilpotent groups, the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, and the Baumslag-Solitar group $BS(1,2)$.

Hence for such actions, Theorem 3.36 implies that $H_n(\Gamma, \mathbb{Z}[X]) = 0$ for all $n > \operatorname{asdim}(\Gamma)$.

5.2. **Smale spaces with totally disconnected stable sets.** A Smale space consists of a self-homeomorphism $\varphi : X \to X$ of a compact metric space X, such that the space can be locally decomposed into the product of a coordinate whose points get closer together as φ is iteratively applied, and a coordinate whose points get farther apart under the map φ. We refer to [46] for basic definitions and details. Given a Smale space (X, φ) one can define two equivalence relations on X as follows:

$$x \sim_s y \text{ if and only if } \lim_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) = 0, \text{ and}$$

$$x \sim_u y \text{ if and only if } \lim_{n\to\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0.$$

Let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of a point $x \in X$ respectively. Upon choosing a finite set P of φ-periodic points one can construct étale principal groupoids $G^u(X,P)$ and $G^s(X,P)$ with unit space $X^s(P) = \bigcup_{x\in P} X^s(x)$ and $X^u(P) = \bigcup_{x\in P} X^u(P)$, respectively. In particular, if these unit spaces are totally disconnected, the groupoids are ample. Note further, that for an irreducible Smale space (X, φ), the groupoids $G^u(X,P)$ and $G^s(X,P)$ only depend on P up to equivalence. In particular, the choice of P is irrelevant when computing their homology. Deeley and Strung prove in [12] that for an irreducible Smale space one has the estimate

$$\operatorname{dad}(G^u(X,P)) \leqslant \dim X.$$

Combining this with our Theorem 3.36 and Corollary 4.19, and also [43, Theorem 4.1] we can compute the K-theory of the resulting C^*-algebras from Putnam's homology for Smale spaces.

Corollary 5.1. *Let (X, φ) be an irreducible Smale space with totally disconnected stable sets. Then $H_n^s(X,\varphi) = 0$ for all $n > \dim(X)$; and if $\dim(X) \leqslant 2$ and $H_2^s(X,\varphi)$ is free abelian (e.g. when it is finitely generated), then*

$$K_0(C^*(G^u(X,P))) \cong H_0^s(X,\varphi) \oplus H_2^s(X,\varphi) \text{ and } K_1(C^*(G^u(X,P))) \cong H_1^s(X,\varphi).$$

This result includes most of the previously known examples (see e.g. [56]) that were based on separate computations of the K-theory and homology, and hence provides a more conceptual explanation.

5.3. **Bounded geometry metric spaces.** A metric space X has *bounded geometry* if for each $r > 0$ there is a uniform bound on the cardinalities of all r-balls in X; important examples come from groups with word metric, or from suitable discretisations of Riemannian manifolds.

Skandalis, Tu, and Yu [49] construct an ample groupoid $G(X)$ which captures the coarse geometry of X. In particular, the reduced groupoid C^*-algebra $C^*_r(G(X))$ can be canonically identified with the uniform Roe algebra $C^*_u(X)$. Let us briefly recall the construction. Let βX denote the Stone-Čech compactification of X, i.e. the maximal ideal space of $\ell^\infty(X)$. For each $r > 0$, let E_r be the closure of $\{(x, y) \in X \times X \mid d(x, y) \leqslant r\}$ inside $\beta X \times \beta X$,[14] which is a compact open set. Then the *coarse groupoid* $G(X)$ of X has as underlying set $\bigcup_{r=0}^\infty E_r$. The operations are the restriction of the pair groupoid operations from $\beta X \times \beta X$, and the topology is the weak topology coming from the union $\bigcup_{r=0}^\infty E_r$, i.e. a subset U of G is open if and only if $U \cap E_r$ is open for each r. Then $G(X)$ is a principal, ample, σ-compact groupoid with compact base space homeomorphic to βX, see [47, Theorem 10.20].

Our first goal is to identify the groupoid homology $H_*(G(X))$ with the uniformly finite homology of X introduced by Block and Weinberger in [5, Section 2]. We begin by recalling the relevant definitions. Let $C_n(X)$ denote the collection of all bounded functions $c : X^{n+1} \to \mathbb{Z}$ such that there exists $r > 0$ such that if $c(x_0, ..., x_n) \neq 0$, then the diameter of the set $\{x_0, ..., x_n\}$ is at most r. For each $i \in \{0, ..., n\}$, let $\partial^i : X^{n+1} \to X^n$ be defined by $\partial^i(x_0, ..., x_n) := (x_0, ..., \widehat{x_i}, ..., x_n)$. Define $\partial^i_* : C_n(X) \to C_{n-1}(X)$ by

$$(\partial^i_* c) = \sum_{\partial^i y = x} c(x)$$

and define $\partial : C_n(X) \to C_{n-1}(X)$ by $\partial := \sum_{i=0}^n (-1)^i \partial^i_*$. Then we have $\partial \circ \partial = 0$, so we get a chain complex. The *uniformly finite homology* of X, denoted $H^{\mathrm{uf}}_*(X)$, is by definition the associated homology of this complex.

Having introduced all the main actors we can now prove the following theorem.

Theorem 5.2. *Let $G(X)$ be the coarse groupoid associated to a bounded geometry metric space X. Then there is a canonical isomorphism $H_*(G(X)) \cong H^{\mathrm{uf}}_*(X)$.*

Proof. For brevity let us denote the coarse groupoid by G throughout the proof. Now, for $a \in \mathbb{Z}[EG_n]$, define $\overline{a} \in \mathbb{Z}[EG_n]$ by

$$\overline{a}(g_0, ..., g_n) := \begin{cases} \sum_{h \in G_x} a(h, hg_1, ..., hg_n) & g_0 = x \in G^0 \\ 0 & g_0 \notin G^0 \end{cases}.$$

One can check (we leave this to the reader) that the equivalence classes $[\overline{a}]$ and $[a]$ in $\mathbb{Z}[EG_n]_G$ of $\overline{a}$ and a respectively are the same, and moreover that $\overline{a}$ is the unique element of $[a]$ that is supported on $\{(g_0, ..., g_n) \in EG_n \mid g_0 \in G^0\}$.

We define maps $\alpha : \mathbb{Z}[EG_n]_G \to C_n(X)$ and $\beta : C_n(X) \to \mathbb{Z}[EG_n]_G$ as follows. First,

$$(\alpha[a])(x_0, ..., x_n) := \overline{a}((x_0, x_0), ..., (x_0, x_n)).$$

This makes sense using that G contains the pair groupoid $X \times X$. We note that $\alpha[a]$ is bounded as $\overline{a}$ is. Moreover, the fact that $\overline{a}$ has compact support implies that it is supported in a set of the form $E_0 \times E_{r_1} \times \cdots \times E_{r_n} \cap EG_n$ for compact

[14] The authors of [49] take the closure in $\beta(X \times X)$ instead of in $\beta X \times \beta X$, but by [47, Proposition 10.15] these closures are canonically homeomorphic, so it does not matter which of them one uses.

open sets E_{r_i} as in the definition of G. It follows that $\alpha[a]$ is supported on the set of tuples with diameter at most $2\max\{r_1, ..., r_n\}$ and is thus a well-defined element of $C_n(X)$.

To define β, let first $((x, x_0), (x, x_1), ..., (x, x_n)) \in EG_n$ where each pair (x, x_i) is in the pair groupoid. For $c \in C_n(X)$, define

$$(\beta c)((x, x_0), (x, x_1), ..., (x, x_n)) := \begin{cases} c(x_0, ..., x_n) & x = x_0 \\ 0 & x \neq x_0 \end{cases}.$$

Due to the support condition on elements of $C_n(X)$, there exists $r > 0$ such that c is supported in the set $\{(x_0, ..., x_n) \in X^{n+1} \mid d(x_i, x_j) \leqslant r \text{ for all } i, j\}$. One can check using the bounded geometry condition that this implies that the closure of the support S of βc in $EG_n \cap (X \times X)^{n+1}$ canonically identifies with the Stone-Čech compactification of S, and thus that βc extends uniquely to a function on the compact open set $\overline{S}$ as it is bounded, and so a function on EG_n by setting it to be zero outside $\overline{S}$. We also denote βc the corresponding class in $\mathbb{Z}[EG_n]_G$.

Now, having built the maps α and β, note that both define maps of complexes as the face maps in both cases are given by omitting the i^{th} element in a tuple. To see that they are mutually inverse isomorphisms, one computes directly that $\alpha(\beta(c)) = c$, and that $\beta(\alpha[a]) = [\overline{a}]$; we leave this to the reader. The result follows. □

Having identified the homology groups of the coarse groupoid with a more classical object, we would now like to apply our main results and draw some consequences for the computation of the K-theory groups of uniform Roe algebras $C^*_u(X)$ which can be canonically identified with $C^*_r(G(X))$. Since the spectral sequence (45) is only available in the case of second countable groupoids we need to do some additional work. To this end it is useful to consider a slightly different construction of the coarse groupoid.

Following [49, Section 2.2], let Γ_X denote the collection of all subsets $A \subseteq X \times X$ such that the first coordinate map $r : X \times X \to X$ and second coordinate maps $s : X \times X \to X$ are both injective when restricted to A, and such that $\sup_{(x,y)\in A} d(x, y) < \infty$. As in [49, Section 3.1], every $A \in \Gamma_X$ defines a bijection $t_A : s(A) \to r(A)$ with the property that $\sup_{x\in s(A)} d(x, t_A(x)) < \infty$. Every such bijection extends to a homeomorphism $\phi_A : \overline{s(A)} \to \overline{r(A)}$ between the respective closures in βX. As in [49, Definition 3.1], we write $\mathscr{G}(X)$ for the collection $\{\phi_A \mid A \in \Gamma_X\}$, which is a pseudogroup, i.e. closed under compositions and inverses. As in [49, Section 3.2], the coarse groupoid $G(X)$ of X can be realized as the groupoid of germs associated to this pseudogroup (see [49, Section 2.6] for the construction of the groupoid of germs associated to a pseudogroup and [49, Proposition 3.2] for the identification of the two constructions).

Now, as in [49, Section 3.3], let us say that a sub-pseudogroup $\mathscr{A}$ of $\mathscr{G}(X)$ is *admissible* if $\bigcup_{\phi_A\in\mathscr{A}} \overline{A} = G(X)$. Define $G_{\mathscr{A}}$ to be the spectrum of the C^*-subalgebra of $C_0(G(X))$ generated by $\{\chi_{\overline{A}} \mid \phi_A \in \mathcal{A}\}$, and let $X_{\mathscr{A}}$ be the spectrum of the C^*-subalgebra of $C(\beta X) = \ell^\infty(X)$ generated $\{\chi_{s(\overline{A})} \mid \phi_A \in \mathcal{A}\}$. The following comes from [49, Lemma 3.3] and its proof.

Lemma 5.3. *Let $\mathscr{A}$ be an admissible sub-pseudogroup of $\mathscr{G}(X)$. Then the groupoid operations naturally factor through the canonical quotient maps $G(X) \to G_{\mathscr{A}}$ and $G(X)^{(0)} \to X_{\mathscr{A}}$, making $G_{\mathscr{A}}$ an étale, locally compact, Hausdorff groupoid with base space $X_{\mathscr{A}}$, which is moreover second countable if $\mathscr{A}$ is countable.*

Moreover, the quotient map $p : \beta X = G(X)^{(0)} \to X_{\mathscr{A}}$ gives rise to an action of $G_{\mathscr{A}}$ on βX, and there is a canonical isomorphism of topological groupoids $G(X) \cong \beta X \rtimes G_{\mathscr{A}}$.

Proof. The only part not explicitly in [49, Lemma 3.3] or its proof is the second countability statement. This follows as if $\mathscr{A}$ is countable, then the C^*-subalgebra of $C_0(G(X))$ generated by $\{\chi_{\overline{A}} \mid \phi_A \in \mathcal{A}\}$ is separable. □

Lemma 5.4. *Let X be a bounded geometry metric space. Then there exists a countable admissible sub-pseudogroup $\mathscr{A}$ of $\mathscr{G}(X)$ such that $G_{\mathscr{A}}$ is principal.*

Proof. First, choose a countable admissible sub-pseudogroup $\mathscr{A}'$ of $\mathscr{G}(X)$ as follows. For each $n \in \mathbb{N}$, a greedy algorithm based on bounded geometry (compare the discussion in [49, Section 2.2, part (a)]) gives a finite decomposition of $\{(x,y) \in X \times X \mid d(x,y) \leqslant n\}$ such that

$$\{(x,y) \in X \times X \mid d(x,y) \leqslant n\} = \bigsqcup_{i=1}^{m_n} A_i^{(n)}$$

and so that each $A_i^{(n)}$ is in Γ_X. Let $\mathscr{A}'$ be the sub-pseudogroup of $\mathscr{G}(X)$ generated by all the $A_i^{(n)}$. It is countable (as generated by a countable set) and it is admissible by construction. Given $\phi \in \mathscr{A}'$ we can apply [42, Proposition 2.7] to decompose its domain $dom(\phi) = A_{0,\phi} \sqcup A_{1,\phi} \sqcup A_{2,\phi} \sqcup A_{3,\phi}$ into disjoint clopen sets, where $A_{0,\phi}$ is the set of fixed points of ϕ and $\phi(A_{i,\phi}) \cap A_{i,\phi} = \varnothing$ for $i = 1,2,3$. Let $\mathscr{A}$ be the sub-pseudogroup generated by $\mathscr{A}'$ and $\{\mathrm{id}_{A_{i,\phi}} \mid \phi \in \mathscr{A}', 0 \leqslant i \leqslant 4\}$. Then $\mathscr{A}$ is still countable and admissible. We claim that $G_{\mathscr{A}}$ is principal. So let $[\phi,\omega] \in G_{\mathscr{A}}$ such that its source and range are are equal, i.e. $\phi(\omega) = \omega$. We may assume that $\phi \in \mathscr{A}'$ as there is nothing to show if ϕ was already the identity function on some clopen set. We then have $\phi|_{A_{0,\phi}} = \mathrm{id}_{A_{0,\phi}}$ and hence $[\phi,\omega] = [\mathrm{id},\omega]$ as desired. □

Let us now assume that X has asymptotic dimension at most d, or equivalently (see [24, Theorem 6.4]) that $G(X)$ has dynamic asymptotic dimension at most d. For each $n \in \mathbb{N}$, let $E_n := \overline{\{(x,y) \in X \times X \mid d(x,y) \leqslant n\}}$, where the closure is taken in $\beta X \times \beta X$. Then E_n is a compact open subset of $G(X)$. Hence using the assumption that the dynamic asymptotic dimension of $G(X)$ is at most d, there exists a decomposition $\beta X = U_0^{(n)} \sqcup \cdots \sqcup U_d^{(n)}$ of βX into compact open subsets such that for for each $i \in \{0,\ldots,d\}$ and each $n \in \mathbb{N}$, the subgroupoid of $G(X)$ generated by $\{g \in E_n \mid r(g), s(g) \in U_i^{(n)}\}$ is compact and open. Note that as each $U_i^{(n)}$ is clopen in βX, each $U_i^{(n)}$ is the closure of $V_i^{(n)} := U_i^{(n)} \cap X$ (this follows as clopen sets in βX are in one-one correspondence with arbitrary subsets of X in this way). Let $\mathscr{B}$ denote the sub-pseudogroup of $\mathscr{G}(X)$ generated by $\mathscr{A}$ as in Lemma 5.4, and by $\{\mathrm{id}_{V_i^{(n)}} \mid n \in \mathbb{N}, i \in \{0,\ldots,d\}\}$.

Then we have the following result.

Lemma 5.5. *Let X be a bounded geometry metric space with asymptotic dimension at most d. Then there is a second countable, étale, locally compact, Hausdorff principal groupoid G with dynamic asymptotic dimension at most d, and such that G acts on βX giving rise to a canonical isomorphism $\beta X \rtimes G \cong G(X)$.*

Proof. We claim that $G = G_{\mathscr{B}}$ works. Note that $\mathscr{B}$ is countable (as generated by a countable set) and admissible (as it contains $\mathscr{A}$, which is admissible). Hence most of the statement follows from Lemma 5.3. As we are only adding further identity functions in the passage from $\mathscr{A}$ to $\mathscr{B}$, we also retain principality by the same proof as in Lemma 5.4. We only need to show that the dynamic asymptotic dimension of $G_{\mathscr{B}}$ is at most d.

For each $n \in \mathbb{N}$, let us write $[E_n]$ for the image of $E_n \subseteq G(X)$ under the quotient map $G(X) \to G_{\mathscr{B}}$. Then $[E_n]$ is compact and open: indeed, with notation as in

the construction of $\mathscr{A}$ its characteristic function is equal to

$$\sum_{i=1}^{m_n} \chi_{\overline{A_i^{(n)}}},$$

and so in $C_0(G_{\mathscr{B}}) \subseteq C_0(G(X))$. Moreover, $G_{\mathscr{B}}$ is the union of the $[E_n]$ (as $G(X)$ is the union of the E_n). Hence to show that $G_{\mathscr{B}}$ has dynamic asymptotic dimension at most d, it suffices to show that for each n we can find an open cover $W_0, ..., W_d$ of $X_{\mathscr{B}}$ such that for each i the subgroupoid G_i of $G_{\mathscr{B}}$ generated by

$$\{g \in [E_n] \mid s(g), r(g) \in W_i\}$$

has compact closure. For this, let us take $W_i = U_i^{(n)}$, noting that each $U_i^{(n)}$ makes sense as a clopen subset of $X_{\mathscr{B}}$ by construction of $\mathscr{B}$. Then we have a decomposition

$$X_{\mathscr{B}} = \bigsqcup_{i=0}^{d} U_i^{(n)}$$

coming from the corresponding decomposition of βX. Finally, note that each G_i is contained in the image of the subgroupoid $\widetilde{G_i}$ of $G(X)$ generated by

$$\{g \in E_n \mid s(g), r(g) \in U_i^{(n)}\}$$

under the canonical quotient map $G(X) \to G_{\mathscr{B}}$. As $\widetilde{G_i}$ is compact (by choice of the $U_i^{(n)}$), and as this quotient map is continuous, we are done. □

We can now use this observation to deduce the existence of a convergent spectral sequence also for the (non-second countable) coarse groupoid:

Proposition 5.6. Let X be a discrete metric space with bounded geometry and finite asymptotic dimension. Then there exists a convergent spectral sequence

$$E^2_{p,q} = H^{\mathrm{uf}}_p(X) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*_u(X)).$$

Proof. By Lemma 5.5 we can write $G(X) = G \ltimes \beta X$ for a principal second countable ample groupoid G with finite dynamic asymptotic dimension. Further, we can write βX as an inverse limit $\beta X = \varprojlim Y_i$ of G-invariant second countable spaces Y_i. Since G is in particular amenable, it satisfies the strong Baum-Connes conjecture. Hence (45), for each $i \in I$, provides a convergent spectral sequence

$$E^2_{p,q}(i) = H_p(G, K_q(C(Y_i))) \Rightarrow K_{p+q}(C(Y_i) \rtimes_r G).$$

The spectral sequence (45) is a special case of the ABC spectral sequence constructed by Meyer in [37, Theorem 4.3], and hence it is functorial in the coefficient variable. Consequently, the abelian groups $E^d_{p,q}(i)$ together with the differential maps form a directed system of spectral sequences. Hence we obtain a spectral sequence with $E^d_{p,q} := \lim_i E^d_{p,q}(i)$ in the limit. Each of the spectral sequences $(E^d_{p,q}(i))$ converges by [37, Theorem 5.1] and as explained on page 172 of [37], the associated filtrations are functorial in the appropriate sense. Hence taking limits again, we obtain an induced filtration of $\lim K_{p+q}(C(Y_i) \rtimes_r G)$. Now since $E^2_{p,q}(i) = 0$ for all $p > \mathrm{asdim}(X)$ and all $i \in I$ by Theorem 3.36, the induced filtration of $\lim K_{p+q}(C(Y_i) \rtimes_r G)$ is finite and hence we obtain a convergent spectral sequence in the limit:

$$E^2_{p,q} = \lim_i H_p(G, K_q(C(Y_i))) \Rightarrow \lim_i K_{p+q}(C(Y_i) \rtimes G). \tag{46}$$

There are canonical identifications $\lim_i H_n(G, C(Y_i, \mathbb{Z})) \cong \lim_i H_n(G \ltimes Y_i) = H_n(G(X))$ and by Theorem 5.2 we can identify the latter group with $H^{\mathrm{uf}}_n(X)$. On the right hand-side we have $\lim_i K_*(C(Y_i) \rtimes_r G) = K_*(C^*_r(G(X))) = K_*(C^*_u(X))$ and hence we are done. □

Corollary 5.7. *Let X be a bounded geometry metric space with asymptotic dimension d. Then $H_n^{\mathrm{uf}}(X) = 0$ for all $n > d$ and $H_d^{\mathrm{uf}}(X)$ is torsion-free. Moreover,*

(1) if $\mathrm{asdim}(X) \leqslant 2$ and $H_2^{\mathrm{uf}}(X)$ is free, or finitely generated, then

$$K_0(C_u^*(X)) \cong H_0^{\mathrm{uf}}(X) \oplus H_2^{\mathrm{uf}}(X), \text{ and } K_1(C_u^*(X)) \cong H_1^{\mathrm{uf}}(X),$$

(2) if $\mathrm{asdim}(X) \leqslant 3$, X is non-amenable, and $H_3^{\mathrm{uf}}(X)$ is free, or finitely generated, then

$$K_0(C_u^*(X)) \cong H_2^{\mathrm{uf}}(X), \text{ and } K_1(C_u^*(X)) \cong H_1^{\mathrm{uf}}(X) \oplus H_3^{\mathrm{uf}}(X).$$

Proof. The first case follows from Theorem 3.36 and Corollary 4.19 as in our earlier examples. For (2) note that if $\mathrm{asdim}(X) \leqslant 3$ then the only possibly non-zero differentials on the E^3-page are the maps $d^3_{3,2l} : H_3^{\mathrm{uf}}(X) \to H_0^{\mathrm{uf}}(X)$ for $l \geqslant 0$. The sequence converges on the E^4-page and hence there are short exact sequences

$$0 \to \mathrm{coker}(d^3_{3,0}) \to K_0(C_u^*(X)) \to H_2^{\mathrm{uf}}(X) \to 0, \text{and}$$

$$0 \to H_1^{\mathrm{uf}}(X) \to K_1(C_u^*(X)) \to \ker(d^3_{3,0}) \to 0$$

For a non-amenable space X the group $H_0^{\mathrm{uf}}(X)$ vanishes by [5, Theorem 3.1]. Since $H_3^{\mathrm{uf}}(X)$ is free the result follows. □

Examples 5.8. Let Γ be a countable group equipped with a left invariant bounded geometry metric[15]. Let Γ act on the group $\ell^\infty(\Gamma, \mathbb{Z})$ of bounded $\mathbb{Z}$-valued functions on Γ via the action induced by the left translation action of Γ on itself. Then it is well-known that $H_*^{\mathrm{uf}}(\Gamma)$ identifies with the group homology $H_*(\Gamma, \ell^\infty(\Gamma, \mathbb{Z}))$ of Γ with coefficients in $\ell^\infty(\Gamma, \mathbb{Z})$: see for example [8, last paragraph on page 1515] (this discusses the case of H_*^{uf} with real coefficients, but the same argument works for integer coefficients).

Now, assume that Γ is a d-dimensional Poincaré duality group, for example if Γ is the fundamental group of a closed d-manifold with contractible universal cover. Then

$$H_d^{\mathrm{uf}}(\Gamma) \cong H_d(\Gamma, \ell^\infty(\Gamma, \mathbb{Z})) \cong H^0(\Gamma, \ell^\infty(\Gamma, \mathbb{Z})) \cong \ell^\infty(\Gamma, \mathbb{Z})^\Gamma \cong \mathbb{Z},$$

where the first isomorphism is the general fact noted above, the second is Poincaré duality, the third is the definition of the zeroth cohomology group, and the fourth is straightforward. In particular, $H_d^{\mathrm{uf}}(\Gamma)$ is free.

This discussion applies in particular if Γ is the fundamental group of a closed orientable surface. In this case Γ is quasi-isometric to either the hyperbolic plane or to the Euclidean plane, whence the asymptotic dimension of Γ is two, and we may apply the first part of Corollary 5.7 to conclude that

$$K_0(C_u^*(\Gamma)) \cong H_0^{\mathrm{uf}}(\Gamma) \oplus \mathbb{Z}, \text{ and } K_1(C_u^*(\Gamma)) \cong H_1^{\mathrm{uf}}(\Gamma).$$

If moreover the underlying surface has genus at least two, then Γ is non-amenable, so $H_0^{\mathrm{uf}}(\Gamma)$ vanishes, and $K_0(C_u^*(\Gamma)) \cong \mathbb{Z}$.

The discussion also applies if Γ is the fundamental group of a closed, orientable, hyperbolic 3-manifold. In this case Γ is non-amenable, and Γ is quasi-isometric to hyperbolic 3-space, so of asymptotic dimension three. We may thus apply the second part of Corollary 5.7 to conclude that

$$K_0(C_u^*(\Gamma)) \cong H_2^{\mathrm{uf}}(\Gamma), \text{ and } K_1(C_u^*(\Gamma)) \cong H_1^{\mathrm{uf}}(\Gamma) \oplus \mathbb{Z}.$$

[15]Such a metric exists and is essentially unique by [55, Proposition 2.3.3], for example.

5.4. **Examples with topological property (T).** The HK-conjecture asserts that for a principal ample groupoid we have abstract isomorphisms $\bigoplus_{j\in\mathbb{N}} H_{2j+i}(G) \cong K_i(C_r^*(G))$. If G has homological dimension 1 one might be tempted to strengthen this conjecture and ask for the canonical maps μ_0 and μ_1 to be isomorphisms. Here, we show that this strong version of the conjecture fails.

In order to exhibit the examples we need some preliminary facts about topological property (T). Topological property (T) for groupoids was introduced in [14, Definition 3.6], and we refer the reader there for the definition. Let $\mathcal{R}_G = \{\pi_x \mid x \in G^0\}$ denote the family of regular representations of G, i.e.

$$\pi_x : C_c(G) \to B(\ell^2(G_x)), \quad \pi_x(f)\delta_g = \sum_{h\in G_{r(g)}} f(h)\delta_{hg}.$$

Then we have the following generalization of [14, Proposition 4.19].

Lemma 5.9. *Let G be an ample groupoid with compact unit space acting on a compact space X. If G has property (T) with respect to $\mathcal{R}_G$ then $G \ltimes X$ has property (T) with respect to $\mathcal{R}_{G\ltimes X}$.*

Proof. Let $p : X \to G^0$ denote the anchor map of the action. Let (K, c) be a Kazhdan pair for $\mathcal{R}_G$. Now let

$$L := \{(g, x) \in G \ltimes X \mid g \in K\} = (G \ltimes X) \cap (K \times X),$$

which is compact. We claim that L is a Kazhdan set for $\mathcal{R}_{G\ltimes X}$. Indeed consider the regular representation $\pi_x^{G\ltimes X} : C_c(G \ltimes X) \to B(\ell^2(G_{p(x)}))$ associated with an arbitrary point $x \in X$ and let $\xi \in \ell^2(G_{p(x)})$ be a unit vector. Since (K, c) is Kazhdan for $\mathcal{R}_G$, there exists a function $f \in C_c(G)$ with support in K such that $\|f\|_I \leqslant 1$ and

$$\|\pi^G_{p(x)}(f)\xi - \pi^G_{p(x)}(\Psi(f))\xi\| \geqslant c,$$

where $\Psi : C_c(G) \to C(G^0)$ is given by $\Psi(f)(x) = \sum_{g\in G^x} f(g)$. Since K is compact we can cover it with finitely many compact open bisections $V_1, \dots, V_n$ and using a partition of unity argument, we can write $f = \sum f_i$ where $supp(f_i) \subseteq V_i$. Then there must be some $1 \leqslant i \leqslant n$ such that

$$\|\pi^G_{p(x)}(f_i)\xi - \pi^G_{p(x)}(\Psi(f_i))\xi\| \geqslant \frac{c}{n}.$$

Using that f_i is supported in a bisection one directly verifies that

$$\pi^G_{p(x)}(f_i) = \pi^G_{p(x)}(\Psi(f_i))\pi^G_{p(x)}(1_{V_i}) \text{ and } \Psi(f_i) = \Psi(f_i)1_{r(V_i)}$$

and combining this with the previous observation, we conclude that there exists an i such that

$$\|\pi^G_{p(x)}(1_{V_i})\xi - \pi^G_{p(x)}(1_{r(V_i)})\xi\| \geqslant \frac{c}{n}.$$

Now let $V_i \ltimes X$ denote the compact open set $(V_i \times X) \cap G \ltimes X$ and let $f' := 1_{V_i\ltimes X}$. Then f' is clearly supported in L with $\|f'\|_I \leqslant 1$ and since $\pi_x^{G\ltimes X}(f') = \pi^G_{p(x)}(1_{V_i})$ and $\pi_x^{G\ltimes X}(\Psi(f')) = \pi^G_{p(x)}(1_{r(V_i)})$ we conclude that

$$\|\pi_x^{G\ltimes X}(f')\xi - \pi_x^{G\ltimes X}(\Psi(f'))\xi\| \geqslant \frac{c}{n}.$$

Hence $(L, \frac{c}{n})$ is a Kazhdan pair for $\mathcal{R}_{G\ltimes X}$. □

Now let Γ be a residually finite group and $\mathcal{L} = (N_i)_i$ a sequence of finite index normal subgroups. Let N_∞ denote the trivial subgroup of Γ and let $\pi_i : \Gamma \to \Gamma/N_i$ be the quotient map. We denote by $G_\mathcal{L}$ the associated HLS groupoid, i.e. the group bundle $\bigsqcup_{i\in\mathbb{N}\cup\{\infty\}} \Gamma/N_i$ equipped with the topology generated by the singleton sets $\{(i, \gamma)\}$ for $i \in \mathbb{N}$ and $\gamma \in \Gamma$, and the tails $\{(i, \pi_i(\gamma)) \mid i > N\}$ for each fixed $\gamma \in \Gamma$ and $N \in \mathbb{N}$. It is well-known that this groupoid is Hausdorff if and only if for each

$\gamma \in \Gamma\backslash\{e\}$ the set $\{i \in \mathbb{N} \mid \gamma \in N_i\}$ is finite. This is in particular the case if the sequence is nested and has trivial intersection.

Following a construction of Alekseev and Finn-Sell in [1] we associate a principal groupoid to this data as follows. Let $X := \bigsqcup_{i\in\mathbb{N}\cup\{\infty\}} \Gamma/N_i$. Then X carries a canonical action of the HLS groupoid $G_{\mathcal{L}}$ given by left multiplication. For $\gamma N_i \in \Gamma/N_i$ let

$$Sh(\gamma N_i) = \bigcup_{j\geqslant i} \pi_{i,j}^{-1}(\gamma N_i)$$

be the *shadow* of γN_i in X. Now let $\widehat{X}$ be the spectrum of the smallest $G_{\mathcal{L}}$-invariant C^*-subalgebra $B \subseteq \ell^\infty(X)$ containing

$$\{\delta_x \mid x \in X\} \cup \{1_{Sh(\gamma N_i)} \mid \gamma \in \Gamma, i \in \mathbb{N}\}.$$

Since B is $G_{\mathcal{L}}$-invariant, $\widehat{X}$ also carries an action of $G_{\mathcal{L}}$ and we can form the transformation groupoid $\mathcal{G} := G_{\mathcal{L}} \ltimes \widehat{X}$. As explained just after Remark 2.2 in [1], this groupoid is principal. Moreover, $X \subseteq \widehat{X}$ is a dense open $G_{\mathcal{L}}$-invariant subset with complement $\widehat{X}\backslash X \cong \widehat{\Gamma}_{\mathcal{L}} = \varprojlim \Gamma/N_i$. Hence we obtain isomorphisms

$$\mathcal{G}|_X \cong \bigsqcup_{i\in\mathbb{N}\cup\{\infty\}} \Gamma/N_i \ltimes \Gamma/N_i \quad \text{and} \quad \mathcal{G}|_{\widehat{X}\backslash X} \cong \Gamma \ltimes \widehat{\Gamma}_{\mathcal{L}}.$$

The following result relies on property (τ) as defined in [33, Definition 4.3.1].

Proposition 5.10. Suppose Γ is a finitely generated, residually finite group and $\mathcal{L} = (N_i)_i$ is a sequence of finite index normal subgroups with property (τ). Then the following hold:

(1) $\mathcal{G}$ has topological property (T) with respect to the family of regular representations in the sense of [14].
(2) The sequence

$$K_0(C_r^*(\mathcal{G}|_X)) \to K_0(C_r^*(\mathcal{G})) \to K_0(C_r^*(\mathcal{G}|_{\widehat{X}\backslash X}))$$

is not exact in the middle.

Proof. Since the regular representations of $\mathcal{G}$ extend to $C_r^*(\mathcal{G})$ by definition of the reduced groupoid C^*-algebra, the result follows from [14, Proposition 4.15] and Lemma 5.9. Part (2) follows from (1) and [14, Proposition 7.14]. □

We can now provide some concrete principal ample groupoids where μ_0 is not surjective.

Let $\Gamma = \mathbb{F}_2$ and choose a nested sequence $(N_i)_i$ of finite index normal subgroups in $\mathbb{F}_2$ with property (τ) such that the associated HLS groupoid is Hausdorff.

Example 5.11. To have a concrete example of such a sequence in mind consider the nested family $(L_i)_i$ of finite index normal subgroups $L_i := \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/5^i))$ of $SL_2(\mathbb{Z})$. Embed $\mathbb{F}_2$ in $SL_2(\mathbb{Z})$ as a finite index normal subgroup and let $N_i := L_i \cap \mathbb{F}_2$. Then $(N_i)_i$ is a nested family of finite index normal subgroups of $\mathbb{F}_2$ with trivial intersection. Since $\mathbb{F}_2$ has finite index in $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{Z})$ has property (τ) with respect to the family $(L_i)_i$ we conclude that $\mathbb{F}_2$ has (τ) with respect to the family $(N_i)_i$.

Let us first compute the homology of the associated groupoid $\mathcal{G}$. Consider the long exact sequence in homology

$$\cdots \to H_n(\mathcal{G}|_X) \to H_n(\mathcal{G}) \to H_n(\mathcal{G}|_{\widehat{X}\backslash X}) \to H_{n-1}(\mathcal{G}|_X) \to \cdots \to H_0(\mathcal{G}|_{\widehat{X}\backslash X})$$

corresponding to the decomposition $\widehat{X} = X \sqcup \widehat{X}\backslash X$. Since $\mathcal{G}|_X$ is a disjoint union of principal and proper groupoids, we have

$$H_n(\mathcal{G}|_X) = \bigoplus_{i\in\mathbb{N}\cup\{\infty\}} H_n(\Gamma/N_i \ltimes \Gamma/N_i) = 0 \text{ for all } n \geqslant 1, \text{ and}$$

$$H_0(\mathcal{G}|_X) = \bigoplus_{i\in\mathbb{N}\cup\{\infty\}} H_0(\Gamma/N_i \ltimes \Gamma/N_i) = \bigoplus_{i\in\mathbb{N}\cup\{\infty\}} \mathbb{Z}$$

From the long exact sequence we conclude that for all $n \geqslant 2$ the restriction to the boundary induces isomorphisms

$$H_n(\mathcal{G}) \cong H_n(\mathbb{F}_2 \ltimes \widehat{\mathbb{F}_{2\mathcal{L}}}).$$

It is a well-known fact that $H_n(\mathbb{F}_2 \ltimes \widehat{\mathbb{F}_{2\mathcal{L}}}) \cong H_n(\mathbb{F}_2, C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z}))$ and the homology of the free group $\mathbb{F}_2$ is well-known to be trivial for all $n \geqslant 2$. Hence $H_n(\mathcal{G}) = 0$ for all $n \geqslant 2$.

Now by construction $H_0(\mathbb{F}_2 \ltimes \widehat{\mathbb{F}_{2\mathcal{L}}}) = C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z})/\langle f - \gamma.f \mid \gamma \in \mathbb{F}_2\rangle$ and from the Pimsner-Voiculescu exact sequence for actions of free groups from [41, Theorem 3.5] we obtain $K_0(C(\widehat{\mathbb{F}_{2\mathcal{L}}}) \rtimes_r \mathbb{F}_2) \cong C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z})/im(\beta)$ where

$$\beta : C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z})^2 \to C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z}),\ (f_1, f_2) \mapsto f_1 - a^{-1}.f_1 + f_2 - b^{-1}.f_2$$

We clearly have $im(\beta) = \langle f - \gamma.f \mid \gamma \in \mathbb{F}_2\rangle$ so that after the identification above, μ_0 is the identity.

Now consider the commutative diagram

$$\begin{array}{ccccc} H_0(\mathcal{G}|_X) & \xrightarrow{i} & H_0(\mathcal{G}) & \xrightarrow{p} & H_0(\mathcal{G}|_{\widehat{X}\backslash X}) \\ \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_0^{\mathcal{G}}} & & \downarrow{\scriptstyle \mu_0} \\ K_0(C_r^*(\mathcal{G}|_X)) & \xrightarrow{i_0} & K_0(C_r^*(\mathcal{G})) & \longrightarrow & K_0(C_r^*(\mathcal{G}|_{\widehat{X}\backslash X})) \end{array}$$

The top row is exact in the middle, as it is part of the long exact sequence in homology corresponding to the open invariant subset $X \subseteq \widehat{X}$. The bottom row however is not exact in the middle by Proposition 5.10. The map on the right hand side is an isomorphism by our reasoning above.

We claim that the map $\mu_0^{\mathcal{G}}$ is not surjective. Suppose for contradiction that it was. Let $x \in K_0(C_r^*(\mathcal{G}))$ be an element which maps to zero in $K_0(C_r^*(\mathcal{G}|_{\widehat{X}\backslash X})$ but is not in the image of i_0. Since $\mu_0^{\mathcal{G}}$ is surjective, we can find an element $y \in H_0(\mathcal{G})$ such that $\mu_0^{\mathcal{G}}(y) = x$. But then by commutativity of the right hand square we have $\mu_0(p(y)) = 0$ and since μ_0 is injective we conclude that $p(y)$ is zero and hence $y = i(z)$ for some $z \in H_0(\mathcal{G}|_X)$. Moreover, by commutativity of the left square we have $x = \mu_0^{\mathcal{G}}(y) = i_0(\mu_0(z))$, which contradicts our assumption that $x \notin \mathrm{im}(i_0)$.

Remark 5.12. At this point, it seems there are three known reasons for failure of the HK conjecture. The first, due to Scarparo [48] is the presence of torsion in isotropy groups. The second, due to Deeley [13] is due to torsion phenomena in K-theory; however, Deeley's results do not contradict the "rational" HK conjecture one gets after tensoring with $\mathbb{Q}$, analogously to the classical fact that the Chern character is a rational isomorphism between K-theory and cohomology. The third is exotic analytic phenomena connected to the failure of the Baum-Connes conjecture as discussed above (this is admittedly not exactly a failure of the HK conjecture, but it seems to us as evidence that the HK conjecture should sometimes fail when the Baum-Connes conjecture fails). Based on these counterexamples, the following "folk conjecture" (arrived at independently by several people) seems reasonable: if

G is an ample, second countable groupoid with torsion free isotropy and satisfying the strong Baum-Connes conjecture, then there are isomorphisms

$$K_0(C_r^*(G)) \otimes \mathbb{Q} \cong \bigoplus_{k \text{ even}} H_k(G;\mathbb{Q}) \quad \text{and} \quad K_1(C_r^*(G)) \otimes \mathbb{Q} \cong \bigoplus_{k \text{ odd}} H_k(G;\mathbb{Q}).$$

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