THE UCT PROBLEM FOR NUCLEAR C^*-ALGEBRAS

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ABSTRACT. In recent years, a large class of nuclear C^*-algebras have been classified, modulo an assumption on the Universal Coefficient Theorem (UCT). We think this assumption is redundant and propose a strategy for proving it. Indeed, following the original proof of the classification theorem, we propose bridging the gap between reduction theorems and examples. While many such bridges are possible, various approximate ideal structures appear quite promising.

CONTENTS

1. INTRODUCTION

After decades of work by many hands, a remarkable classification theorem for simple C^*-algebras has emerged.[1] Specifically, assuming the Universal Coefficient Theorem (UCT), when two simple C^*-algebras have finite nuclear dimension, they are isomorphic if and only if they have isomorphic K-theoretic invariants ([17], [12], [32], [5]); without finite nuclear dimension, classification via these invariants is impossible ([26], [33]). Thus the classification of simple nuclear C^*-algebras is complete – modulo the UCT. This stunning fact has renewed interest in the old problem of whether or not every nuclear C^*-algebra satisfies the UCT. The purpose of this note is to review what is known about the UCT and propose a strategy for proving it for all nuclear C^*-algebras.

The UCT is topological in nature, having its roots in Kasparov's KK-theory. Kasparov introduced the KK-group $KK(A, B)$ around 1980 [18] for

[1]All C^*-algebras in this note are assumed separable, with exceptions like multiplier algebras.

the purpose of building and analyzing maps between the K-theory groups of C^*-algebras A and B. In the Cuntz picture [9], an element of $KK(A, B)$ is represented by a quasi-homomorphism $A \to B$ and hence gives rise to a morphism at the level of K-theory $K_*(A) \to K_*(B)$. This induces a group homomorphism

$$\gamma \colon KK_*(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)).$$

Unfortunately γ cannot be an isomorphism, in general, since the left and right hand sides treat short exact sequences differently. Determining if γ is surjective or describing its kernel is the role of the UCT.

In the stable case and Ext picture ([18]), an element of $KK_1(A, B)$ is a C^*-algebraic short exact sequence $0 \to B \to E \to A \to 0$. The boundary maps in the six-term exact sequence for K-theory then provide the homomorphism $K_*(A) \to K_{*+1}(B)$. When these maps vanish, i.e., belong to the kernel of γ, the K-theory of E provides an element of $\mathrm{Ext}(K_*(A), K_{*+1}(B))$. Following the seminal paper of Rosenberg and Schochet [28], we define the *UCT class* to be those C^*-algebras A for which

$$0 \to \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to KK_*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

is exact for any C^*-algebra B.[2] C^*-algebras in the UCT class are said to *satisfy the UCT*.

The strategy we propose for proving nuclear C^*-algebras satisfy the UCT is rooted in the original proof of the stably finite case of the classification theorem for simple C^*-algebras of finite nuclear dimension. For a couple decades, classification was only achieved for C^*-algebras constructed as limits of well-understood building blocks. It started with pioneering work of Elliott [13] on the classification of AF-algebras, i.e., limits of direct sums of matrix algebras. With increasingly sophisticated techniques, classification was achieved for simple $A\mathbb{T}$-algebras of real rank zero, and then, unital simple AH-algebras with no dimension growth ([14, 16, 15]). In 2000, Huaxin Lin made a conceptual leap with the introduction and classification of TAF-algebras ([21]), which are defined by an abstract approximation property as opposed to concrete inductive limit structure. This breakthrough was eventually generalized, leading to the classification of algebras with generalized tracial rank (g-TR) at most one ([17]). Thus, the classes of examples which could be classified via K-theoretic invariants grew over time, getting larger with each decade.

At the same time, very general approximation properties were being introduced and studied ([38], [19], [41]), which led to reduction theorems in the classification program. That is, it was shown that in order to classify algebras in a large class, it suffices to classify a smaller subclass. Perhaps the most influential reduction theorem was due to Winter, who proved that to classify all simple C^*-algebras A with finite nuclear dimension, it suffices to classify $A \otimes \mathcal{U}$ where $\mathcal{U}$ is the universal UHF algebra ([40]). This reduction is quite surprising as algebras of the form $A \otimes \mathcal{U}$ have several special properties not enjoyed in the general finite-nuclear-dimension case such as an abundance of projections and divisible K-theory. In the presence of other

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[2]The reader is warned that our discussion only conveys broad ideas, and sweeps substantial and subtle details under the rug. Please see [28] for a precise treatment.

conditions like real rank zero and quasidiagonality, related reduction theorems inched closer and closer to the abstract approximation properties being classified ([39]). In 2015, a remarkable bridge was constructed by Elliott, Gong, Lin and Niu [12]: if A is simple, unital, satisfies the UCT, has finite nuclear dimension and every tracial state on A is quasidiagonal, then $A \otimes \mathcal{U}$ has generalized tracial rank at most one.
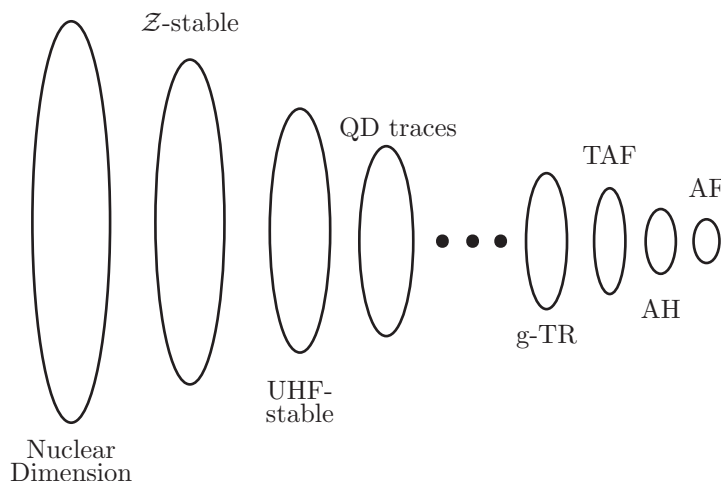

FIGURE 1. Reductions and examples in Classification

We think following this roadmap could lead to a proof of the UCT for all nuclear C*-algebras.[3] In the next section we review existing reduction theorems. In section 3 we review the classes of examples known to satisfy the UCT. In section 4 we discuss possible bridges between reduction theorems and examples. Special attention is given to algebras which admit *approximate ideal structures* as they seem particularly promising. Finally, in section 5 we frame the Künneth formula for tensor products as proof of concept and present evidence that approximate ideal structures could be the key to the UCT.

2. REDUCTION THEOREMS

The first important subclass to which the UCT can be reduced is the so-called *Kirchberg algebras*, i.e., simple nuclear and *purely infinite* C*-algebras. Recall that a simple C*-algebra is purely infinite if every nonzero hereditary sub-C*-algebra contains an infinite projection, that is, a projection which is Murray-von Neumann equivalent to a proper subprojection of itself. Kirchberg algebras enjoy many useful properties, including:

(1) any two nonzero positive elements are Cuntz equivalent ([27, Proposition 4.1.1]);
(2) real rank zero, meaning any self-adjoint element can be approximated in norm by self-adjoint elements with finite spectrum ([27, Proposition 4.1.1]);

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[3]During the preparation of this article, Huaxin Lin proposed a different strategy in conference lectures. His interesting ideas will not be covered here.

(3) tensorial absorption of the Cuntz algebra $\mathcal{O}_\infty$ and thus the Jiang-Su algebra $\mathcal{Z}$ ([27, Theorem 7.2.6]),

Using the fact that A is KK-equivalent to $A \otimes \mathcal{O}_\infty$ and Kirchberg's celebrated $\mathcal{O}_2$-embedding theorem ([27, Theorem 6.3.11]), an inductive limit construction shows that every nuclear C*-algebra is KK-equivalent to a Kirchberg algebra ([27, Proposition 8.4.5]). Hence we have:

Theorem 2.1 (Kirchberg). *The UCT holds for all nuclear C*-algebras if and only if it holds for all unital Kirchberg C*-algebras.*

In fact, Kirchberg reduced even further, to the simplest possible K-groups.

Theorem 2.2. *([27, Corollary 8.4.6]) The UCT holds for all nuclear C*-algebras if and only if it holds for all unital Kirchberg C*-algebras with trivial K-theory (i.e., they are all isomorphic to $\mathcal{O}_2$).*

Our second reduction theorem deals with algebras at the opposite end of the spectrum from those which are simple and purely infinite.

Definition 2.3. A C*-algebra is called *RFD* or *residually finite-dimensional* if it embeds into $\prod M_{k_i}$ for a sequence $(k_i)_{i \in \mathbb{N}}$ of integers, where M_{k_i} denotes the C*-algebra of $k_i \times k_i$-matrices.

Equivalently, a C*-algebra is RFD if it has a separating family of finite-dimensional representations. Using Voiculescu's stunning result that cones are always quasidiagonal ([35]), Dadarlat established the following reduction theorem.

Theorem 2.4. *([10, proof of Lemma 2.4]) The UCT holds for all nuclear C*-algebras if and only if it holds for all nuclear RFD C*-algebras.*

For our third reduction theorem we need Lin's groundbreaking tracial-approximation idea.

Definition 2.5. ([21, page 694]) A C*-algebra is *TAF* or *tracially approximately finite-dimensional* if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ containing a non-zero element, and any full $a \in A_+$, there exists a finite dimensional C*-subalgebra $B \subset A$ with $1_B = p$ such that, for all $x \in \mathcal{F}$, we have
   (1) $\|px - xp\| < \varepsilon$,
   (2) the distance from pxp to B is no more than ε, and
   (3) $n[1-p] \leq [p]$ in the Murray-von Neumann semigroup of A and $1-p$ is equivalent to a projection in the hereditary subalgebra generated by a.

The following theorem of Dadarlat, which relies on Theorem 2.4 and utilizes another inductive limit construction, is an analogue of Theorem 2.1 in the stably finite case.

Theorem 2.6. *([10, Theorem 1.2]) The UCT holds for all nuclear C*-algebras if and only if it holds for all nuclear TAF C*-algebras.*

In fact ([10, Theorem 1.2]), analogously to Theorem 2.2, it suffices to prove the UCT for a simple, unital, nuclear TAF C*-algebra with the same K-theory as the universal UHF algebra $\mathcal{Q}$ (and so that it is isomorphic to $\mathcal{Q}$).

Our last reduction theorem requires another groundbreaking idea: non-commutative topological covering dimension.

Definition 2.7. ([41, Definition 2.1]) The *nuclear dimension* of a separable C^*-algebra A is the infimum of all natural numbers d such that there is a sequence $(F_i)_{i \in \mathbb{N}}$ of finite-dimensional C^*-algebras, a sequence of completely positive contractions $(\psi_i \colon A \to F_i)_{i \in \mathbb{N}}$, and $(d+1)$ sequences $(\phi_i^{(l)} \colon F_i \to A)_{i \in \mathbb{N}}$ of order-zero completely positive contractions for $l = 0, 1, \ldots, d$ such that

$$\left\| \left(\phi_i^{(0)} + \ldots + \phi_i^{(d)} \right) \circ \psi_i(a) - a \right\| \to 0 \quad \text{as } i \to \infty$$

for any $a \in A$.

Note that having finite nuclear dimension implies nuclearity. In [4, Theorem G] it was shown that every Kirchberg algebra has nuclear dimension one, hence Theorem 2.1 implies the following.

Theorem 2.8. *The UCT holds for all nuclear C^*-algebras if and only if it holds for all simple unital C^*-algebras with nuclear dimension 1.*

There are several other open problems which are equivalent to the UCT for nuclear C*-algebras. Since they are not reduction theorems in the sense we are considering the reader is referred to ([3, Introduction]) for a nice summary.

3. EXAMPLES

Rosenberg and Schochet observed that abelian C*-algebras satisfy the UCT in ([28, page 439]). Essentially every other known example is derived from this case using a variety of permanence properties. In this section we recall the main examples, then review the long list of permanence properties enjoyed by the UCT class.

A C*-algebra is *type I* if its double dual is a type I von Neumann algebra. Basic examples include abelian C*-algebras and the compact operators on a Hilbert space. Another important class of examples, particularly for classification, are subhomogeneous C*-algebras, i.e., subalgebras of $C(X) \otimes M_n(\mathbb{C})$ for some space X and $n \in \mathbb{N}$.

Theorem 3.1. *([28, page 439]) Type I C*-algebras satisfy the UCT.*

Groupoid[4] C^*-algebras ([24], [37]) provide our next class of examples. For simplicity, we will stick to the amenable case (as it corresponds to nuclearity, at least in the unital case).

Theorem 3.2. [34] *Let G be an amenable groupoid. Then its C^*-algebra satisfies the UCT.*

Importantly, the previous result was partially generalized by Barlak and Li to *twisted* étale groupoid C*-algebras in [2]. This allowed a connection with the notion of Cartan subalgebras.

Definition 3.3. A maximal abelian subalgebra $B \subset A$ is called a *Cartan* subalgebras if its normalizer generates A, it is the image of a conditional expectation, and it contains an approximate unit for A.

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[4]All groupoids are locally compact, Hausdorff, and second countable.

By work of Renault ([25]), Cartan subalgebras induce twisted groupoid structures (which are amenable in the nuclear case) and hence we have the following:

Theorem 3.4. *If A is nuclear and has a Cartan subalgebra, then A satisfies the UCT.*

The converse is also true when A is simple and has finite nuclear dimension [31], [20].

Though narrow in scope when compared to the previous examples, Eckhardt and Gillaspy used special properties of nilpotent groups to prove the following interesting theorem [11].

Theorem 3.5. *Let G be a finitely generated nilpotent group and π an irreducible representation of G. Then $C_\pi^*(G)$ satisfies the UCT.*

3.1. **Permanence Properties.** Here are the known permanence properties of the nuclear UCT class. Definitions and descriptions of their utility follow.

- KK-equivalence
- Tensor products
- Inductive limits
- Two out of three in a short exact sequence
- Crossed products by $\mathbb{Z}$ or $\mathbb{R}$
- Internal approximation by subalgebras

C*-algebras A and B are said to be KK-equivalent if there exists an invertible element in $KK(A, B)$. These equivalence classes are large, giving one lots of room to explore when searching for new examples within a class. For instance, the Kirchberg algebra $\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$! More generally, Rosenberg and Schochet proved that every C*-algebra in the UCT class is KK-equivalent to an abelian C*-algebra ([27, pages 455-456], and also [30, Proposition 5.3]).

When A and B satisfy the UCT, so does their minimal (and maximal) tensor product $A \otimes B$, since A and B are KK-equivalent to abelian C*-algebras. In particular, the stabilization of something in the UCT class remains in the UCT class.

If $A_1 \to A_2 \to A_3 \to \cdots$ is an inductive system and each A_i satisfies the UCT, then so does their inductive limit ([28, Proposition 2.3]). It follows that AF algebras, and their generalizations using subhomogeneous building blocks, satisfy the UCT.

If $0 \to A \to D \to B \to 0$ is short exact and two of the algebras A, D or B satisfy the UCT, so does the third. In particular, the UCT class is closed under extensions and taking quotients by ideals in the UCT class ([28, Proposition 2.3]).

If A satisfies the UCT and α is an action of either $\mathbb{Z}$ or $\mathbb{R}$, then the crossed products $A \rtimes_\alpha \mathbb{Z}$ or $A \rtimes_\alpha \mathbb{R}$ satisfy the UCT ([28, Propositions 2.6 and 2.7]). One can show Cuntz algebras satisfy the UCT this way, since their stabilizations are isomorphic to crossed products of AF algebras ([27, page 87]).

The internal-approximation permanence property (which generalizes the inductive limit result) is a theorem of Dadarlat [10, Theorem 1.1].

Theorem 3.6. *Let A be a nuclear C^*-algebra. Assume for any finite set $\mathcal{F} \subset A$ and any $\epsilon > 0$ there is a C^*-subalgebra B of A satisfying the UCT and such that $\text{dist}(a, B) < \epsilon$ for all $a \in \mathcal{F}$. Then A satisfies the UCT.*

Taken together, these permanence properties are wide ranging and exceedingly useful. For instance, Tu's proof of the UCT for C*-algebras associated to amenable groupoids first uses Kasparov's so-called Dirac-dual Dirac method to construct a C^*-algebra $A(G)$ which is KK-equivalent to $C^*(G)$. He then observes that $A(G)$ is an inductive limit of type I C*-algebras, completing the proof.

4. Possible Bridges

Summarizing section 2, we know the UCT holds for all nuclear C*-algebras if and only if it holds for any of the following subclasses:

- Kirchberg algebras (with trivial K-theory);
- nuclear RFD algebras;
- simple, nuclear, unital TAF algebras;
- simple, unital C*-algebras with nuclear dimension one.

In section 3 we saw that the following examples, and anything built out of them via appropriate permanence properties, satisfy the UCT:

- Type I C*-algebras;
- $C^*(G)$, where G is an amenable groupoid;
- any C*-algebra with a Cartan subalgebra.

Any KK-equivalence from the first group to the second would prove the UCT for all nuclear C*-algebras. For instance, one could try to prove that every Kirchberg algebra is KK-equivalent to something with a Cartan subalgebra. Or perhaps there is a notion of "tracial Cartan subalgebra" which still allows one to prove the UCT, thereby adding another bullet point to the second group, and for which every TAF algebra is KK-equivalent to an algebra with this property. There are lots of possibilities.

4.1. **Approximate ideal structures.** In the classification program, bridging the gap between reduction theorems and examples took decades of hard work and experimentation. The same could be true for the UCT, but there is a potential bridge that seems particularly promising. It is based on approximate ideal structures, which we now describe.

To motivate the ideas, assume that $A = I + J$, where both I and J are ideals in a nuclear C^*-algebra A. Applying the two out of three principle twice, one gets a Mayer-Vietoris sequence in KK-theory which implies that if I, J and $I \cap J$ satisfy the UCT, then so does A. In other words, we can add another bootstrap operation: if A is built from ideals which satisfy the UCT, then it does too. Obviously when A is simple, this is no help. But remarkably many simple C*-algebras have *approximate ideal structures*.

Definition 4.1. Let $\mathcal{QC}$ be the set of C*-algebras arising from quotients of cones over finite-dimensional algebras. That is, $C \in \mathcal{QC}$ if and only if there is a finite-dimensional C*-algebra F and a surjective $*$-homomorphism $C_0(0, 1] \otimes F \to C$.

Definition 4.2. [5] We say A admits an *approximate ideal structure over* $\mathcal{QC}$ if for every $\epsilon > 0$ and finite-dimensional subspace $X \subset A$ there is a triple (h, C, D) consisting of a positive contraction h in the multiplier algebra of A and C^*-subalgebras C and D of A from the class $\mathcal{QC}$ such that

(1) h multiplies C and D into themselves;
(2) $\|[h, x]\| \leq \epsilon \|x\|$ for all $x \in X$;
(3) $d(hx, C) \leq \epsilon \|x\|$ and $d((1-h)x, D) \leq \epsilon \|x\|$ for all $x \in X$;
(4) $d((1-h)hx, C \cap D) \leq \epsilon \|x\|$ for all $x \in X$.

If one can only arrange the first three conditions, we say A admits a *weak approximate ideal structure over* $\mathcal{QC}$.

If $C, D \subset A$ are actually ideals, a simple approximate-unit exercise shows all four conditions are satisfied.

The following result, pointed out by Winter, shows the ubiquity of weak approximate ideal structures ([36, Corollary A.2]).

Theorem 4.3. *If the nuclear dimension of A is 1, then A admits a weak approximate ideal structure over* $\mathcal{QC}$. *Conversely, if A admits a weak approximate ideal structure over* $\mathcal{QC}$, *then it has finite nuclear dimension.*

Since a *simple* C^*-algebra with finite nuclear dimension automatically has nuclear dimension ≤ 1 ([6], [7]), it follows that simple C^*-algebras have a weak ideal structure over $\mathcal{QC}$ if and only if they have nuclear dimension ≤ 1. Hence we have a new reduction theorem.

Theorem 4.4. *If the UCT holds for all simple, unital C^*-algebras with a weak approximate ideal structure over* $\mathcal{QC}$, *then it holds for all nuclear C^*-algebras.*

Note that the subspace X in Definition 4.2 is finite dimensional, hence approximate ideal structures are local in nature. One option for a global notion is as follows.

Definition 4.5. We say A has a *uniform approximate ideal structure over* $\mathcal{QC}$ if it has an approximate ideal structure over $\mathcal{QC}$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any subalgebras $C, D \subset A$, with $C, D \in \mathcal{QC}$, and any auxiliary C^*-algebra B we have that whenever $c \in C \otimes B$, $d \in D \otimes B$ and $\|c - d\| < \delta$, one can find $x \in (C \cap D) \otimes B$ such that $\|x - c\| < \varepsilon$ and $\|x - d\| < \varepsilon$.

In the next section we will explain why we are optimistic that every C^*-algebra with a uniform approximate ideal structure over $\mathcal{QC}$ satisfies the UCT. Thus we propose attacking the UCT problem for nuclear C^*-algebras via the following open problems.

- Is every (simple, unital) C^*-algebra admitting a weak approximate ideal structure over $\mathcal{QC}$ KK-equivalent to one admitting an approximate ideal structure over $\mathcal{QC}$?
- Is every C^*-algebra admitting an approximate ideal structure over $\mathcal{QC}$ KK-equivalent to one admitting a uniform approximate ideal structure over $\mathcal{QC}$?

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[5] This can be done in much greater generality ([36]).

- Does every C*-algebra admitting a uniform approximate ideal structure over $\mathcal{QC}$ satisfy the UCT?

5. The Künneth formula as proof of concept

A C^*-algebra A *satisfies the Künneth formula*[6] ([29]) for tensor products if for all C^*-algebras B with $K_*(B)$ free, the canonical product [7]

$$\pi : K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

is an isomorphism. Equivalently (using Schochet's method of geometric resolutions [29, Section 3]), A satisfies the Künneth formula if for every B there is a short exact sequence

$$0 \to K_*(A) \otimes K_*(B) \xrightarrow{\pi} K_*(A \otimes B) \to \mathrm{Tor}(K_{*+1}(A), K_*(B)) \to 0,$$

where the first map is the product map. Since every abelian C*-algebra satisfies the Künneth formula ([1],[29, Proposition 2.11]) and every C*-algebra satisfying the UCT is KK-equivalent to an abelian C*-algebra, it follows that the UCT implies the Künneth formula. As such, the Künneth formula is an important test case when trying to establish the UCT for a new class of examples such as those defined by approximate ideal structures. However, we view it as more than a necessary condition; it is proof of concept, at least at a metamathematical level. Indeed, due to the dualities between $\otimes$ and Hom (and therefore also between Ext and Tor) in abelian group theory, it is natural to think of the Künneth formula as sort of dual to the UCT. Thus, if one has a proof of the Künneth formula for a new class of examples, "duality" suggests the UCT should also hold.[8]

Theorem 5.1. *([36, Theorem 1.4]) If A admits a uniform ideal structure over $\mathcal{QC}$, then A satisfies the Künneth formula.*

The proof of this result, due to the third author, follows a strategy pioneered by Yu in the context of the coarse Baum-Connes conjecture ([42]) and generalized by Oyono-Oyono and Yu ([22]); it was also these authors who first approached the Künneth formula from this angle ([23]). Namely, one applies an approximate Mayer-Vietoris sequence in K-theory to the approximate ideal structure of A ([36]). To prove the UCT, one should prove an approximate Mayer-Vietoris sequence in K-homology, then mimic the K-theory arguments. As such, we regard Theorem 5.1 as strong evidence that the UCT holds for C*-algebras admitting a uniform ideal structure over $\mathcal{QC}$.

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[6]We warn the reader that this is *not* equivalent to the KK-theoretic Künneth formula from [28, page 439].

[7]The symbol "$\otimes$" should be understood as the tensor product of graded abelian groups on the left, and as the spatial tensor product of C^*-algebras on the right.

[8]The reader is warned not to take this too literally. There are examples of C*-algebras that satisfy the Künneth formula, but not the UCT ([8, page 492] and [30, Théorème 4.1 and page 571]).

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