Complexity rank for C^*-algebras

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June 14, 2022

Abstract

Complexity rank for C^*-algebras was introduced by the second author and Yu for applications towards the UCT: very roughly, this rank is at most n if you can repeatedly cut the C^*-algebra in half at most n times, and end up with something finite dimensional. In this paper, we study this ("strong") complexity rank, and also a "weak complexity rank" that we introduce here.

We first show that for a large class of C^*-algebras, weak complexity rank one is equivalent to the conjunction of nuclear dimension one and real rank zero. In particular, this shows that the UCT for all nuclear C^*-algebras is equivalent to equality of the weak and strong complexity ranks for Kirchberg algebras with zero K-theory groups. However, we also show using a K-theoretic obstruction (torsion in K_1) that weak complexity rank one and strong complexity rank one are not the same in general.

We then use the Kirchberg-Phillips classification theorem to compute the strong complexity rank of all UCT Kirchberg algebras: it is always one or two, with the rank one case occurring if and only if the K_1-group is torsion free.

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1 Introduction

Background

In recent work, the second author and Yu [31] introduced the notion of decomposability of a C^*-algebra over a class of C^*-algebras. This has two sources of inspiration: the first are corresponding notions of decomposability in coarse geometry introduced by Guentner, Tessera, and Yu [16, 17] and in dynamics introduced by Guentner, the second author, and Yu [18]; the second is the class of C^*-algebras of nuclear dimension (at most) one as introduced by Winter and Zacharias [36].

Before going on with the general discussion, let us state the formal definition. For a subset S of a C^*-algebra A and $a \in A$, write "$a \in_\epsilon S$" to mean that there is $s \in S$ with $\|a - s\| < \epsilon$.

Definition 1.1. Let A be a unital C^*-algebra, and let $\mathcal{C}$ be a class of unital C^*-algebras. Then A is *decomposable* over $\mathcal{C}$ if for every finite subset X of A and every $\epsilon > 0$ there exist C^*-subalgebras C, D, and E of A that are in the class $\mathcal{C}$ and contain 1_A, and a positive contraction $h \in E$ such that:

(i) $\|[h, x]\| < \epsilon$ for all $x \in X$;

(ii) $hx \in_\epsilon C$, $(1-h)x \in_\epsilon D$, and $h(1-h)x \in_\epsilon E$ for all $x \in X$;

(iii) for all e in the unit ball of E, $e \in_\epsilon C$ and $e \in_\epsilon D$.

In words, the definition says that one can use an almost central element (h above) to locally cut the C^*-algebra A into two pieces (C and D above) with well-behaved approximate intersection (E above).

The main application of this notion is to the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet [27]. For this paper we do not need any details about the UCT; suffice to say that the UCT is a K-theoretic property that a C^*-algebra may or may not have, and that whether or not the UCT holds for all nuclear C^*-algebras is an important open question. The following theorem is the main result of [31].

Theorem 1.2. *If A is a separable unital C^*-algebra that decomposes over the class of nuclear UCT C^*-algebras, then A itself is nuclear and satisfies the UCT.*

Moreover, all nuclear C^*-algebras satisfy the UCT if and only if any unital Kirchberg algebra[1] *with zero K-theory decomposes over the class of finite-dimensional C^*-algebras.*

Due to the importance of the UCT, it thus becomes interesting to better understand the class of C^*-algebras that decompose over finite-dimensional C^*-algebras. Inspired by this and coarse geometry [17, Definition 2.9], it is natural to introduce a complexity hierarchy on C^*-algebras: we say a C^*-algebra has complexity rank zero if it is locally finite-dimensional[2], and has complexity rank at most $n+1$ if it decomposes over the class of C^*-algebras of complexity rank at most n; having complexity rank at most one is then the same as decomposing over the class of finite-dimensional C^*-algebras. One of our goals in this paper is to better understand the complexity rank for Kirchberg algebras, partly due to the connections to the UCT, and partly for the intrinsic interest of complexity rank as an invariant in its own right.

Results

We first aim to make the connection between decomposability over the class of finite-dimensional C^*-algebras and nuclear dimension one more precise. For this purpose, we introduce the notion of weak decomposability: this is the variant of Definition 1.1 where all conditions

[1] A unital C^*-algebra A is a *Kirchberg algebra* if it is separable, nuclear, and if for any non-zero $a \in A$ there are $b, c \in A$ with $bac = 1_A$.

[2] In the separable case, this is the same as being an AF C^*-algebra.

involving E are dropped (in particular, h is just a positive contraction in A). The corresponding notion of weak complexity rank one turns out to be very closely related to nuclear dimension one: here is the precise result.

Theorem 1.3. *If A is a separable unital C^*-algebra with real rank zero and nuclear dimension at most one, then A has weak complexity rank at most one.*

If A is a separable unital C^*-algebra with weak complexity rank at most one, then A has nuclear dimension at most one. If in addition A is simple with at most finitely many (and possibly zero) extreme tracial states, then A has real rank zero.

Having established this, it becomes very natural to ask if weak complexity rank and complexity rank are actually the same: indeed, if they were, Theorem 1.2 (plus the fact that all Kirchberg algebras have nuclear dimension one [6, Theorem G] and real rank zero [37]) would imply the UCT for all nuclear C^*-algebras. The next theorem shows that equality of the weak complexity rank and complexity rank in general is too much to ask for.

Theorem 1.4. *Let A be a unital C^*-algebra of complexity rank at most one. Then $K_1(A)$ is torsion free.*

As there are Kirchberg algebras with arbitrary countable K-theory groups [24, Section 3], it follows from this and Theorem 1.3 that complexity rank and weak complexity rank are indeed genuinely different. Nonetheless, Theorems 1.2 and 1.3 show that the UCT for all nuclear C^*-algebras is equivalent to equality of the weak and strong complexity ranks for Kirchberg algebras with zero K-theory.

We are unable to shed any more light on the complexity rank of general Kirchberg algebras than given by the above theorems; thus we do not make progress on the UCT problem for general nuclear C^*-algebras. However, if we allow ourselves to *assume* the UCT, and thus give ourselves access to the Kirchberg-Phillips classification theorem [22], then the situation is completely different: we get a complete answer.

Theorem 1.5. *All unital UCT Kirchberg algebras have complexity rank one or two. Moreover, the rank one case occurs if and only if the K_1 group of the C^*-algebra is torsion free.*

This theorem provides a striking contrast to the case of nuclear dimension / weak complexity rank, which are both always one for Kirchberg algebras.

Outline of the paper

In Section 2 we discuss the main definitions, and give some basic reformulations (the version of decomposability used in this introduction is one of the stronger ones). We establish some basic consequences of weak complexity rank for nuclear dimension and existence of projections, and show that the complexity rank is subadditive on tensor products.

In Section 3 we study the class of C^*-algebras with weak complexity rank one in detail, and in particular establish Theorem 1.3. Most of the section does not need anything beyond basic facts about nuclear dimension, as established in the seminal paper [36]. However, the results going from weak complexity rank one to real rank zero are different: they use deep structure results from [34, 26, 29].

In Section 4 we use techniques from controlled K-theory as developed in [30] to establish Theorem 1.4. This and the results of the previous section are interesting partly as they allow us to distinguish weak complexity rank and complexity rank, and partly as they will be used for our results on Kirchberg algebras.

In Section 5, we establish Theorem 1.5. This theorem again uses substantial ingredients, most notably the Kirchberg-Phillips classification theorem [22] (see also [25, Chapter 8] and [15]) is used; we note that we need the existence and uniqueness theorems for morphisms, not 'only' the fact that UCT Kirchberg algebras are classified by K-theory. As well as this, we also need Rørdam's crossed product models for Kirchberg algebras [24], and a technique developed by Enders [14] to estimate the nuclear dimension of Kirchberg algebras.

Finally in the short Section 6, we list some natural questions.

Notation and conventions

The symbol A is reserved throughout for a C^*-algebra.

Let $\epsilon > 0$. For $a, b \in A$, we write "$a \approx_\epsilon b$" if $\|a - b\| < \epsilon$. For a subset S of A and $a \in A$, we write "$a \in_\epsilon S$" if there exists $s \in S$ such that $\|a - s\| < \epsilon$. For subspaces S and T of A, we write "$S \subseteq_\epsilon T$" if for all elements s of the unit ball of S, there exists t in the unit ball of T with $\|s - t\| < \epsilon$.

The unitization of a C^*-algebra A is denoted A^+; we use this convention even if A already has a unit, in which case A^+ is canonically isomorphic to $A \oplus \mathbb{C}$. The multiplier algebra of a C^*-algebra A is written $M(A)$. The symbol $\mathcal{K}$ denotes the compact operators on a separable infinite-dimensional Hilbert space. For C^*-algebras A and

B, $A \otimes B$ is always the spatial (i.e. minimal) tensor product. For a unitary $u \in M(A)$, $\mathrm{Ad}_u : A \to A$ denotes the conjugation automorphism defined by $a \mapsto uau^*$.

For a C^*-algebra A, $K_0(A)$ and $K_1(A)$ are its even and odd (topological) K-theory groups, and $K_*(A) := K_0(A) \oplus K_1(A)$ is the corresponding graded group; here 'graded' means that the direct sum decomposition is remembered as part of the structure. Homomorphisms $\alpha : K_*(A) \to K_*(B)$ will always be assumed to be graded, i.e. satisfying $\alpha(K_i(A)) \subseteq K_i(A)$ for $i \in \{0, 1\}$.

Acknowledgments

The authors gratefully acknowledge support from the US NSF under DMS 1901522. The second author is grateful to Dominik Enders, Wilhelm Winter, and Guoliang Yu for conversations (in some cases, occurring some time ago) that influenced the results in this paper.

2 Definitions and basic properties

In this section, we introduce the main definitions that we will study in this paper.

Definition 2.1. Let $\mathcal{C}$ be a class of C^*-algebras. A C^*-algebra A is *locally* in $\mathcal{C}$ if for any finite subset X of A and any $\epsilon > 0$ there is a C^*-subalgebra C of A that is in $\mathcal{C}$, and such that $x \in_\epsilon C$ for all $x \in X$.

Definition 2.2. Let $\mathcal{C}$ be a class of unital C^*-algebras. A unital C^*-algebra A *decomposes over* $\mathcal{C}$ if for every finite subset X of A and every $\epsilon > 0$ there exist C^*-subalgebras C, D, and E of A that are in the class $\mathcal{C}$, and a positive contraction $h \in A$ such that:

(i) $\|[h, x]\| < \epsilon$ for all $x \in X$;

(ii) $hx \in_\epsilon C$, $(1-h)x \in_\epsilon D$, and $h(1-h)x \in_\epsilon E$ for all $x \in X$;

(iii) $E \subseteq_\epsilon C$ and $E \subseteq_\epsilon D$;

(iv) for all $e \in E_1$, $he \in_\epsilon E$.

We now come to the fundamental definition for this paper.

Definition 2.3. For an ordinal number α:

(i) if $\alpha = 0$, let $\mathcal{D}_0$ be the class of unital C^*-algebras that are locally finite dimensional;

(ii) for any $\alpha > 0$, let $\mathcal{D}_\alpha$ be the class of unital C^*-algebras that decompose over C^*-algebras in $\bigcup_{\beta<\alpha} \mathcal{D}_\beta$.

A unital C^*-algebra has *finite complexity* if it is in $\mathcal{D}_\alpha$ for some α, in which case its *complexity rank* is the smallest possible α.

Remark 2.4. Definition 2.3 is partly motivated by a notion of geometric complexity due to Guentner, Tessera, and Yu [17, Definition 2.9]. In our previous work [31], we showed that if X is a bounded geometry metric space then the geometric complexity of X in the sense of [17, Definition 2.9] is an upper bound for the complexity rank of the uniform Roe algebra $C^*_u(X)$. We will not pursue this further here, however.

We record two straightforward lemmas. These will not be used until much later in the paper.

Lemma 2.5. *Let $A_1, ..., A_n$ be unital C^*-algebras, and $A = A_1 \oplus \cdots \oplus A_n$ be their direct sum. Then for any ordinal α, A is in $\mathcal{D}_\alpha$ if and only if each A_i is in $\mathcal{D}_\alpha$.*

Proof. This follows from a straightforward transfinite induction on α that we leave to the reader. □

Lemma 2.6. *Any unital C^*-algebra that is locally in $\mathcal{D}_\alpha$ is in $\mathcal{D}_\alpha$. In particular, $\mathcal{D}_\alpha$ is closed under inductive limits.*

Proof. As the definitions are all local in nature, this is immediate. □

2.1 Stronger formulations

In this subsection, we show that the definition of decomposability bootstraps up to stronger versions of itself. We then show that the class of C^*-algebras of complexity rank at most one admits a particularly nice characterization.

We need four very well-known lemmas; we record them for the reader's convenience as we will use them over and over again.

Lemma 2.7. *Let a and b be bounded operators on a Hilbert space with b normal. Then any z in the spectrum of a is contained within distance $\|a-b\|$ of the spectrum of b.*

Proof. We need to show that if z is further than $\|a-b\|$ from the spectrum of b, then $a-z$ is invertible. Indeed, in this case the continuous functional calculus implies that $\|(b-z)^{-1}\| < \|a-b\|^{-1}$. Hence

$$\|(a-z)(b-z)^{-1} - 1\| \leq \|(a-z)-(b-z)\|\|(b-z)^{-1}\| < 1,$$

whence $(a-z)(b-z)^{-1}$ is invertible, and so $a-z$ is invertible too. □

Lemma 2.8. *Let $a \in A$ be an element in a C^*-algebra, let $\epsilon > 0$, and let B be a C^*-subalgebra of A such that $a \in_\epsilon B$.*

(i) If a is positive, then there is positive $b \in B$ such that $\|b\| \leq \|a\|$ and $a \approx_{2\epsilon} b$.

(ii) If a is a projection and $\epsilon < 1/2$, there is a projection $p \in B$ such that $a \approx_{2\epsilon} p$.

Proof. For part (i), let $b_0 \in B$ be such that $a \approx_\epsilon b_0$. Let $b_1 = \frac{1}{2}(b_0 + b_0^*)$, which is self-adjoint and still satisfies $b_1 \approx_\epsilon a$. Then b_1 has spectrum contained in $[-\epsilon, \|a\| + \epsilon]$ by Lemma 2.7. Hence if $f : \mathbb{R} \to \mathbb{R}$ is the function that is 0 on $(-\infty, 0]$, linear on $[0, \|a\|]$, and 1 on $[\|a\|, \infty)$, then by the functional calculus $b := f(b_1)$ is a positive contraction such that $b \approx_\epsilon b_1 \approx_\epsilon a$.

Part (ii) is essentially the same: this time b_1 as above has spectrum contained in $(-\epsilon, \epsilon) \cup (1-\epsilon, 1+\epsilon)$, and if $f = \chi_{(1/2,\infty)}$, then $p := f(b_1)$ is a projection in B such that $p \approx_{2\epsilon} a$. ☐

Lemma 2.9. *Let a be a self-adjoint element of a C^*-algebra A such that $\|a^2 - a\| \leq \epsilon < 1/4$. Then there is a projection $p \in A$ such that $p \approx_{\sqrt{\epsilon}} a$.*

Proof. Let t be in the spectrum of a. Then $|t(1-t)| \leq \epsilon$, so either $|t| \leq \sqrt{\epsilon}$, or $|1-t| \leq \sqrt{\epsilon}$, and so the spectrum of a is contained in $[-\sqrt{\epsilon}, \sqrt{\epsilon}] \cup [1-\sqrt{\epsilon}, 1+\sqrt{\epsilon}]$. As $\sqrt{\epsilon} < 1/2$, the characteristic function χ of $(1/2, \infty)$ is continuous on the spectrum of a, and the functional calculus implies that $p := \chi(a)$ satisfies $p \approx_{\sqrt{\epsilon}} a$. ☐

Lemma 2.10. *Let A be a C^*-algebra. Let p, q be projections in A, and assume that $\|p - q\| \leq \epsilon < 1/4$. Then there is a unitary $u \in A^+$ (or in A itself if it is already unital) with $\|u - 1\| \leq 10\epsilon$ such that $p = uqu^*$.*

Proof. Passing to A^+ if necessary, we may assume A is unital. Let $v = (1-p)(1-q) + pq$. Then one computes that $v - 1 = p(q-p) + (p-q)q$, so $\|v - 1\| < 2\epsilon < 1$. Hence v is invertible. Moreover, one checks that $vp = pq = qv$, so $vpv^{-1} = q$ and also $v^*vp = v^*qv = (qv)^*v = (vp)^*v = pv^*v$. Let now $u := v(v^*v)^{-1/2}$. Then u is unitary, and the previous computations show that $up = v(v^*v)^{-1/2}p = vp(v^*v)^{-1/2} = qv(v^*v)^{-1/2} = qu$, so $upu^* = q$. Note moreover that

$$\|v^*v - 1\| \leq \|v^* - 1\|\|v\| + \|v - 1\| < \epsilon(1 + 1 + 2\epsilon) \leq 2\epsilon(1 + \epsilon) < 3\epsilon$$

as $\epsilon < 1/4$. Hence by the functional calculus

$$(1+3\epsilon)^{-1/2} \leq (v^*v)^{-1/2} \leq (1-3\epsilon)^{-1/2}$$

and so by elementary estimates using that $\epsilon < 1/4$, $\|1-(v^*v)^{-1/2}\| < 4\epsilon$. It follows that

$$\|1-u\| \leq \|v\|\|1-(v^*v)^{-1/2}\| + \|1-v\| \leq (1+2\epsilon)4\epsilon + 2\epsilon = 10\epsilon$$

as claimed. □

We hope the following lemma clarifies the definition of decomposability.

Lemma 2.11. *Let $\mathcal{C}$ be a class of unital C^*-algebras. A unital C^*-algebra A* decomposes over *$\mathcal{C}$ if and only if it satisfies the following condition.*

For every finite subset X of A and every $\epsilon > 0$ there exist C^*-subalgebras C, D, and E of A that are in the class $\mathcal{C}$, and a positive contraction $h \in A$ such that:

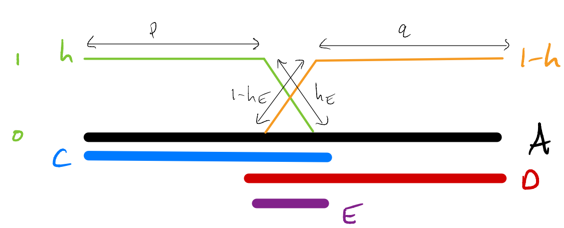
(i) $\|[h,x]\| < \epsilon$ *for all* $x \in X$*;*

(ii) $hx \in_\epsilon C$, $(1-h)x \in_\epsilon D$, *and* $h(1-h)x \in_\epsilon E$ *for all* $x \in X$*;*

(iii) $E \subseteq_\epsilon C$ *and* $E \subseteq_\epsilon D$*;*

(iv) $h = h_E + p$ *and* $1 - h = (1_E - h_E) + q$*, where* h_E *is a positive contraction in* E*, and* $p \in C$ *and* $q \in D$ *are projections that are orthogonal to* 1_E*, and satisfy* $1_A = 1_E + p + q$*.*

Schematically, we thus have a spectral decomposition of h that 'looks like'[3] the following.



[3]The picture is maybe slightly misleading in that E is not obviously contained in (or equal to) the intersection $C \cap D$; however, this can be arranged if $\mathcal{C}$ is the class of finite-dimensional C^*-algebras as in Lemma 2.14 below.

Proof of Lemma 2.11. Let $\epsilon > 0$, and let X be a finite subset of A. Let $\delta > 0$, to be determined in the course of the proof in a way depending only on ϵ. Let C, D, and E be C^*-algebras in $\mathcal{C}$ and h be a positive contraction that have the properties in Definition 2.2 with respect to the finite set $X \cup \{1_A\}$ and δ. Throughout the proof, the notation δ_n refers to a quantity that converges to zero as δ tends to zero, and that depends only on δ.

Now, as $1_E \in_\delta C$, Lemma 2.8 part (ii) gives δ_1 and a projection $p_E \in C$ such that $\|p_E - 1_E\| < \delta_1$. Hence by Lemma 2.10 there is a unitary $u \in A$ and $\delta_2 > 0$ such that $\|u - 1_A\| < \delta_2$, and so that $u1_E u^* = p_E$. Similarly, there is a projection $q_E \in D$ and a unitary $v \in A$ such that $\|v - 1_A\| < \delta_3$ for some δ_3, and such that $v1_E v^* = q_E$. Hence replacing C with u^*Cu and D with v^*Dv, we may assume that C, D, E and h satisfy the conditions in Definition 2.2 for X and some $\delta_4 > 0$, and moreover that $1_E \in C \cap D$.

As $1_E h 1_E \in_\delta E$, Lemma 2.8 gives a positive contraction $h_E \in E$ with $1_E h 1_E \approx_{2\delta} h_E$. Moreover, as $h1_E \in_\delta E$, we have $(1_A - 1_E)h1_E \approx_\delta 0$ and taking adjoints gives $1_E h(1_A - 1_E) \approx_\delta 0$. Hence if we write $h_{E^\perp} := (1_A - 1_E)h(1_A - 1_E)$ then

$$h \approx_{2\delta} 1_E h 1_E + (1_A - 1_E)h(1_A - 1_E) \approx_{2\delta} h_E + h_E^\perp.$$

Replacing h with $h_E + h_{E^\perp}$, we may assume h is a sum of two positive contractions, one of which is in E, and one of which is orthogonal to E; in particular, h multiplies E into itself. Note then that

$$h(1-h) = h_E(1_E - h_E) + h_{E^\perp}^2 - h_{E^\perp}$$

and so $h_E(1_E - h_E) + h_{E^\perp}^2 - h_{E^\perp} \in_{\delta_4} E$. As $h_E(1_E - h_E)$ is in E, we thus see that $h_{E^\perp}^2 - h_{E^\perp} \approx_{\delta_4} 0$. As long as $\delta_4 < 1/4$ Lemma 2.9 implies there is δ_5 and a projection $p \in (1_A - 1_E)A(1_A - 1_E)$ such that $p \approx_{\delta_5} h_{E^\perp}$ Now, as $h = h \cdot 1_A \in_{\delta_5} C$ and as $h_E \in E \subseteq_{\delta_4} C$, we have that there is δ_6 such that $p \in_{\delta_6} C$. As $1_E \in C$ and as p is orthogonal to 1_E, Lemma 2.8 part (ii) a projection $p_C \in (1_C - 1_E)C(1_C - 1_E)$ and $\delta_7 > 0$ such that $p_C \approx_{\delta_7} p$. Hence Lemma 2.10 gives a unitary $u \in (1_A - 1_E)A(1_A - 1_E)$ and δ_8 such that $\|(1_A - 1_E) - u\| < \delta_8$, and such that $up_C u^* = p$. Replacing C by $(1_E + u)C(1_E + u^*)$, we may assume that C contains p.

On the other hand, we have $1_A - h = 1_E - h_E + (1_A - 1_E - p)$. Write $q = (1_A - 1_E - q)$. Arguing analogously to the above, we also see that $q \in_{\delta_9} D$ for some δ_9, and so that there exists a unitary $v \in (1_A - 1_E)A(1_A - 1_E)$ such that $\|(1_A - 1_E) - q\| < \delta_{10}$ for some δ_{10}.

Replacing D by $(1_E + v)D(1_E + v^*)$ and taking the original δ small enough, we are done. $\square$

We are now able to deduce the definition of decomposability that we used in the introduction.

Corollary 2.12. *Let $\mathcal{C}$ be a class of unital C^*-algebras. A unital C^*-algebra A* decomposes over $\mathcal{C}$ *if and only if it satisfies the following condition.*

For every finite subset X of A and every $\epsilon > 0$ there exist C^*-subalgebras C, D, and E of A that are in the class $\mathcal{C}$ and contain 1_A, and a positive contraction $h \in E$ such that:

(i) $\|[h, x]\| < \epsilon$ *for all* $x \in X$*;*

(ii) $hx \in_\epsilon C$, $(1 - h)x \in_\epsilon D$, *and* $h(1 - h)x \in_\epsilon E$ *for all* $x \in X$*;*

(iii) $E \subseteq_\epsilon C$ *and* $E \subseteq_\epsilon D$*.*

Proof. Note that from the proof of the previous lemma, we may assume that $1_E \in C \cap D$ in addition to the conclusions stated there. Then replace E with $\mathbb{C}p \oplus E \oplus \mathbb{C}q$, C with $\operatorname{span}\{C, 1_A\}$ and D with $\operatorname{span}(D, 1_A)$ $\square$

In the remainder of this section, we show that complexity rank at most one bootstraps up to a stronger version of itself. For this, we need to recall a theorem of Christensen [12, Theorem 5.3] about perturbing almost inclusions of finite dimensional C^*-algebras to honest inclusions.

Theorem 2.13 (Christensen)**.** *Let A be a C^*-algebra, and let E and C be C^*-subalgebras of A with E finite-dimensional. If $0 < \epsilon \leq 10^{-4}$ and $E \subseteq_\epsilon C$, then there exists a partial isometry $v \in A$ such that $\|v - 1_E\| < 120\sqrt{\epsilon}$ and $vEv^* \subseteq C$.* $\square$

Proposition 2.14. *A unital C^*-algebra A has complexity rank at most one if and only if it has the following property.*

For any finite subset X of the unit ball of A and any $\epsilon > 0$ there exist finite-dimensional C^*-subalgebras C, D and E of A that contain the unit and a positive contraction $h \in E$ such that:

(i) $\|[h, x]\| < \epsilon$ *for all* $x \in X$*;*

(ii) $hx \in_\epsilon C$, $(1 - h)x \in_\epsilon D$ *and* $(1 - h)hx \in_\epsilon E$ *for all* $x \in X$*;*

(iii) *E is contained in both C and D.*

Proof. The property in the statement clearly implies complexity rank one, so it suffices to show that complexity rank one implies this property. Assume A has complexity rank at most one. Let X be a finite subset of the unit ball of A, and let $\epsilon > 0$. Fix $\delta > 0$, to be chosen by the rest of the proof in a way depending only on ϵ. Throughout the proof, anything called "δ_n" for some n is a positive constant that depends only on the original δ, and tends to zero as δ tends to zero.

Let h_0, C_0, D_0, and E_0 satisfy the conclusion of Lemma 2.11 for X and δ; in particular, then, each of C_0, D_0, and E_0 are unital and locally finite-dimensional C^*-algebras and we can write $h = h_{E_0} + p$, where $h_{E_0} \in E_0$ is a positive contraction, $p \in C_0$ is a projection that is orthogonal to 1_E, and $q := 1_A - 1_E - p$ is a projection in D_0.

Choose a finite-dimensional C^*-subalgebra E_1 of E_0 that contains the unit 1_{E_0} of E_0 (whence 1_{E_0} is also the unit of E_1), and is such that $h(1-h)x \in_{2\delta} E_1$ for all $x \in X$, and such that $h_{E_0} \in_{2\delta} E_1$. Choose a finite-dimensional C^*-subalgebra C_1 of C_0 such that $E_1 \subseteq_{2\delta} C_1$, $hx \in_{2\delta} C_1$ for all $x \in X$, so that $p \in_{2\delta} C_1$ and so that $1_{E_0} \in_{2\delta} C_1$. As $1_{E_0} \in_{2\delta} C_1$, Lemma 2.8 gives a projection $p_{CE} \in C_1$ such that $\|1_{E_0} - p_{CE}\| < 4\delta$. As long as δ is suitably small, Lemma 2.10 gives $\delta_1 > 0$ and a unitary $u \in A$ such that $\|u - 1_A\| < \delta_1$ and so that $up_{CE}u^* = 1_{E_0}$. Define $C_2 := uC_1u^*$. Then $1_{E_0} \in C_2$, and for some $\delta_2 > 0$, we have that $E_1 \subseteq_{\delta_2} C_2$, $hx \in_{\delta_2} C_2$ for all $x \in X$, and that $p \in_{\delta_2} C_2$. As $p \in_{\delta_2} (1_A - 1_{E_0})C_2(1_A - 1_{E_0})$, we similarly find a projection $p_C \in (1_A - 1_{E_0})C_2(1_A - 1_{E_0})$ and a unitary $v \in (1_A - 1_{E_0})A(1_A - 1_{E_0})$ such that for some $\delta_3 > 0$, $\|v - (1_A - 1_{E_0})\| < \delta_3$, and such that $vp_Cv^* = p$. Define $C_3 := (1_{E_0} + v)C_2(1_{E_0} + v^*)$. Then C_3 is a finite-dimensional C^*-subalgebra of A that contains p and 1_{E_0}, and such that there is $\delta_4 > 0$ such that $E_1 \subseteq_{\delta_4} C_3$ and $hx \in_{\delta_4} C_3$ for all $x \in X$. Analogously, find a finite-dimensional C^*-subalgebra D_3 of D_0 that contains q and 1_{E_0}, and such that there is $\delta_4 > 0$ such that $E_1 \subseteq_{\delta_4} D_3$, $(1-h)x \in_{\delta_4} D_3$ for all $x \in X$.

Now, let E_2 be the (finite-dimensional) C^*-subalgebra of A spanned by E_1 and p and q, and let C_4 (respectively, D_4) be the (finite-dimensional) C^*-subalgebra of A spanned by C_3 (respectively D_3) and 1_A. These C^*-algebras E_2, C_4, and D_4 satisfy the following conditions: all contain 1_{E_0}, p and q (and therefore 1_A); $E_2 \subseteq_{\delta_4} D_4$ and $(1-h)x \in_{\delta_4} D_4$ for all $x \in X$; $E_2 \subseteq_{\delta_4} C_4$ and $hx \in_{\delta_4} C_4$ for all $x \in X$; $h_{E_0} \in_{2\delta} E_2$. Choose a positive contraction h_E in $1_{E_0}E_21_{E_0} = E_1$ such that $h_E \approx_{2\delta} h_{E_0}$.

Define $E := E_2$. Using Theorem 2.13 if $\delta_4 \leq 10^{-4}$ there exists a partial isometry $w_C \in A$ such that $w_CE_2w_C^* \subseteq C_4$, $w_C^*w_C = 1_{E_2}$, and so that $\|w_C - 1_{E_2}\| \leq 120\sqrt{\delta_4} =: \delta_5$. As $1_{E_2} = 1_A$, w_C must be unitary as long as δ is small enough that $120\sqrt{\delta_4} < 1$. Assuming this, define $C_5 := w_C^* C_3 w_C$, so C contains E, and satisfies $hx \in_{\delta_6} C$ for some $\delta_6 > 0$. Similarly, there is a unitary $w_D \in A$ such that $\|w_D - 1_{E_2}\| \leq \delta_5$, and so that $w_D E w_D^* \subseteq D_4$. Define $D := w_D^* D_3 w_D$. At this point, the reader can check that the C^*-subalgebras C, D, and E together with $h := h_E + p$ satisfy the conditions in the statement with respect to some $\delta_7 > 0$. Taking the original δ suitably small, we are done. $\square$

2.2 Weak finite complexity

The main motivation for introducing finite complexity is that it gives a sufficient condition for a C^*-algebra to satisfy the UCT. In contrast, the following weaker version of finite complexity does not obviously have any K-theoretic consequences. Instead, we introduce it mainly as it serves as a bridge between complexity rank and some more established "dimension notions" like nuclear dimension and real rank; these relations will be clarified in the rest of this subsection, and in Section 3 below.

Definition 2.15. Let $\mathcal{C}$ be a class of unital C^*-algebras. A unital C^*-algebra A *weakly decomposes over* $\mathcal{C}$ if for every finite subset X of A and every $\epsilon > 0$ there exist C^*-subalgebras C and D of A that are in the class $\mathcal{C}$, and a positive contraction $h \in A$ such that:

(i) $\|[h, x]\| < \epsilon$ for all $x \in X$;

(ii) $hx \in_\epsilon C$ and $(1 - h)x \in_\epsilon D$ for all $x \in X$.

In other words, weak decomposability is like decomposability, but with the conditions on E dropped.

Definition 2.16. For an ordinal number α:

(i) if $\alpha = 0$, let $\mathcal{WD}_0$ be the class of unital C^*-algebras that are locally finite-dimensional;

(ii) for any $\alpha > 0$, let $\mathcal{WD}_\alpha$ be the class of unital C^*-algebras that weakly decompose over C^*-algebras in $\bigcup_{\beta<\alpha} \mathcal{WD}_\beta$.

A C^*-algebra D has *weak finite complexity* if it is in $\mathcal{WD}_\alpha$ for some α, in which case its *weak complexity rank* is the smallest possible α.

Clearly the weak complexity rank of a C^*-algebra is bounded above by its complexity rank. We will see later in the paper (see Corollary 4.2) that the two are genuinely different.

In the remainder of this subsection, we discuss two basic consequences of weak finite complexity: the first gives a weak existence of projections property (see [2] for background), and the second gives bounds on nuclear dimension (see [36] for background).

First, we look at existence of projections properties, establishing a weak version (see Subsection 3.2 below for a stronger conclusion under much stronger hypotheses). For the statement, we say that *projections separate traces* in a C^*-algebra A if whenever τ_1, τ_2 are tracial states on A such that $\tau_1(p) = \tau_2(p)$ for all projections p in A, then $\tau_1 = \tau_2$.

Lemma 2.17. *If A is a unital C^*-algebra with finite weak complexity, then the span of the projections in A is dense. In particular, projections in A separate traces.*

Proof. We proceed by transfinite induction on the weak complexity rank. The base case is clear, as is the case of a limit ordinal. For the case of a successor ordinal, let $a \in A$ be arbitrary, let $\epsilon > 0$, and let h, C and D be as in the definition of weak decomposability with respect to $X = \{a\}$ and $\epsilon/3$. Choose $c \in C$ and $d \in D$ with $\|ha - c\| < \epsilon/3$ and $\|(1-h)a - d\| < \epsilon/3$. The inductive hypothesis implies that the span of the projections in C and D are dense, so each of c and d can be approximated within $\epsilon/6$ by a linear combination of projections. Hence $c + d$ can be approximated within $\epsilon/3$ by a linear combination of projections. Putting this together with the fact that $\|a - (c+d)\| < 2\epsilon/3$, we are done. $\square$

It is shown in [31, Lemma 7.3] that C^*-algebras of finite complexity are always nuclear. Here we give a more precise version of this result. First we need to recall the definition of nuclear dimension from [36, Definition 2.1].

Definition 2.18. A completely positive map $\phi : A \to B$ between C^*-algebras has *order zero* if whenever $a, b \in A$ are positive elements such that $ab = 0$, we have that $\phi(a)\phi(b) = 0$.

A C^*-algebra A has *nuclear dimension at most n* if for any finite subset X of A and any $\epsilon > 0$ there exists a finite dimensional C^*-algebra F and completely positive maps

$$\begin{tikzcd} A \arrow[dr, "\psi"'] & & A \\ & F \arrow[ur, "\phi"'] & \end{tikzcd}$$

such that:

(i) $\phi(\psi(x)) \approx_\epsilon x$ for all $x \in X$;

(ii) ψ is contractive;

(iii) F splits as a direct sum of ideals $F = F_0 \oplus \cdots \oplus F_n$ such that each restriction $\phi|_{F_i}$ is contractive and order zero.

Proposition 2.19. *Let α be an ordinal number.*

(i) If $\alpha = n \in \mathbb{N} \cup \{0\}$, then any C^*-algebra in $\mathcal{WD}_n$ has nuclear dimension at most $2^n - 1$;

(ii) For any ordinal α, any C^*-algebra in $\mathcal{WD}_\alpha$ is locally in the class of C^*-algebras that are both in $\mathcal{WD}_\alpha$ and have finite nuclear dimension.

Proof. We first establish part (i) by induction on n. If A belongs to $\mathcal{D}_0$, then it is locally finite dimensional, and this implies nuclear dimension zero: this is essentially contained in [36, Remark 2.2 (iii)], but we give an argument for the reader's convenience. Let $X \subseteq A$ be a finite subset and let $\epsilon > 0$. Choose a finite-dimensional C^*-subalgebra F of A such that $x \in_\epsilon F$ for all $x \in X$. Let $\psi : A \to F$ be any choice of conditional expectation (such exists by the finite-dimensional case of Arveson's extension theorem - see for example [8, Theorem 1.6.1]), and let $\phi : F \to A$ be the inclusion $*$-homomorphism; it is straightforward to see that these maps have the right properties.

Assume then that $N \geq 1$, and the result has been established for all $n < N$. Let a finite subset X of A and $\epsilon > 0$ be given; we may assume X consists of contractions. Let C and D be C^*-subalgebras of A in some class $\mathcal{WD}_n$ for some $n \leq N$, and let $h \in A$ be a positive contraction as in the definition of weak decomposability with respect to the finite subset X and parameter $\epsilon^2/(25{\cdot}2^{2N})$. The inductive hypothesis implies that C and D have nuclear dimension at most $2^{N-1} - 1$. Choose a set $X_C \subseteq C$ such that for each $x \in X$, there is $x_C \in X_C$ such that $\|hx - x_C\| < \epsilon/(4 \cdot 2^N)$. Using finite nuclear dimension, choose completely positive maps $\psi_C : C \to F_C$ and $\phi_C : F_C \to C$ such that ψ_C is contractive, such that $\phi_C(\phi_C(x)) \approx_{\epsilon/8} x$ for all $x \in X_C$, and such that F_C decomposes into 2^{N-1} ideals such that the restriction of ϕ_C to each ideal is contractive and order zero. Let X_D, ψ_D, ϕ_D, and F_D have analogous properties with respect to D and with h replaced by $1 - h$.

Now, using Arveson's extension theorem, we may extend each of ψ_C and ψ_D to contractive completely positive (ccp) maps defined on all of A (we keep the same notation for the extensions). Define $F := F_C \oplus F_D$, and

$$\psi : A \to F, \quad a \mapsto \psi_C(h^{1/2}ah^{1/2}) + \psi_D((1-h)^{1/2}a(1-h)^{1/2}),$$

which is easily seen to be ccp. Define moreover

$$\phi : F \to A, \quad (f_C, f_D) \mapsto \phi_C(f_C) + \phi_D(f_D).$$

To show that A has nuclear dimension at most $2^N - 1$, it suffices to show that $\phi(\psi(x)) \approx_\epsilon x$ for any $x \in X$; the remaining properties are easily verified. First note that as $\|[h, x]\| < \epsilon^2/(25 \cdot 2^{2N})$, we have that $\|[h^{1/2}, x]\| < \epsilon/(4 \cdot 2^N)$ and $\|[(1-h)^{1/2}, x]\| < \epsilon/(4 \cdot 2^N)$ by the main result of [21]. Hence

$$\begin{aligned}\psi(x) &= \psi_C(h^{1/2}xh^{1/2}) + \psi_D((1-h)^{1/2}x(1-h)^{1/2}) \\ &\approx_{\epsilon/(4\cdot 2^N)} \psi_C(hx) + \psi_D((1-h)x).\end{aligned}$$

Choose $x_C \in X_C$ and $x_D \in X_D$ such that

$$\|hx - x_C\| < \epsilon/(4 \cdot 2^N) \quad \text{and} \quad \|(1-h)x - x_D\| < \epsilon/(4 \cdot 2^N) \tag{1}$$

so we get

$$\psi(x) \approx_{\epsilon/(2\cdot 2^N)} \psi_C(x_C) + \psi_D(x_D).$$

As $\|\phi\| \leq 2^N$, this implies that

$$\phi(\psi(x)) \approx_{\epsilon/2} \phi(\psi_C(x_C) + \psi_D(x_D)) = \phi_C(\psi_C(x_C)) + \phi_D(\psi_C(x_D)).$$

By choice of ϕ_C and ψ_C, we have that $\phi_C(\psi_C(x_C)) \approx_{\epsilon/8} x_C$, and similarly for x_D, whence

$$\phi(\psi(x)) \approx_{3\epsilon/4} x_C + x_D.$$

Finally, using line (1) and that $N \geq 1$, we see that $x_C + x_D \approx_{\epsilon/4} hx + (1-h)x = x$, and so $\phi(\psi(x)) \approx_\epsilon x$ as required.

Part (ii) can be proved using transfinite induction: essentially the same argument as used above for case (i) works. □

2.3 Tensor products

In this subsection we establish a permanence result for the complexity rank of tensor products: see Proposition 2.22 below. For readability, we just state the result for complexity rank, but it holds for weak complexity rank as well, with a (simpler) version of the same proof.

The key ingredient we need is a result of Christensen on inclusions of tensor products of nuclear C^*-algebras: it follows by combining [12, Proposition 2.6 and Theorem 3.1].

Proposition 2.20 (Christensen)**.** *Let E and C be C^*-subalgebras of a C^*-algebra A such that $E \subseteq_\epsilon C$ for some $\epsilon > 0$, and let B be a C^*-algebra. Assume moreover that E and B are nuclear. Then $E \otimes B \subseteq_{6\epsilon} C \otimes B$.* □

Lemma 2.21. *Let B be a nuclear and unital C^*-algebra, and assume that A is a unital C^*-algebra that decomposes over some class $\mathcal{C}$ of nuclear and unital C^*-algebras. Then $A \otimes B$ decomposes over the class of C^*-algebras $C \otimes B$ with C in $\mathcal{C}$.*

Proof. Let X be a finite subset of $A \otimes B$, and let $\epsilon > 0$. Up to an approximation, we may assume every element of X is a finite sum of elementary tensors. Fix such a finite sum $x = \sum_{i=1}^n a_i \otimes b_i$ for each $x \in X$, and let X_A be the finite subset of A consisting of all the elements a_i appearing in such a sum for some $x \in X$. Let M be the maximum of the sums $\sum_{i=1}^n \|b_i\|$ as x ranges over X. We claim that if $\delta = \min\{\epsilon/M, \epsilon/6\}$ and if E, C, and D are C^*-subalgebras of A in the class $\mathcal{C}$ that satisfy the conditions in the definition of decomposability with respect to X_A and δ, then $E \otimes B$, $C \otimes B$, $D \otimes B$, and $h \otimes 1_B$ satisfy the conditions in the definition of decomposability with respect to X and ϵ; this will suffice to complete the proof.

Let us check the conditions from Definition 2.2. For condition (i), if $x = \sum_{i=1}^n a_i \otimes b_i$ is one of our fixed representations of an element of X, then

$$\|[h \otimes 1_B, x]\| \leq \sum_{i=1}^n \|[a_i, h]\|\|b_i\| < \delta \sum_{i=1}^n \|b_i\| < \epsilon$$

by assumption on δ. For condition (ii), note that for $x = \sum_{i=1}^n a_i \otimes b_i \in X$ and each i, there is $c_i \in C$ with $ha_i \approx_\delta c_i$. Hence

$$\left\|(h \otimes 1_B)x - \sum_{i=1}^n c_i \otimes b_i\right\| = \sum_{i=1}^n \|ha_i - c\|\|b_i\| < \epsilon$$

by choice of δ, and so $(h \otimes 1_B)x \in_\epsilon C \otimes B$. Similarly, $(1_{A\otimes B} - h \otimes 1_B)x \in_\epsilon D \otimes B$ and $h \otimes 1_B(1_{A\otimes B} - h \otimes 1_B)x \in_\epsilon E \otimes B$ for all $x \in X$. For condition (iii), we have that $E \otimes B \subseteq_\epsilon C \otimes B$ and $E \otimes B \subseteq_\epsilon D \otimes B$ by choice of δ, Proposition 2.20, and the assumption that B and everything in $\mathcal{C}$ is nuclear. Condition (iv) is straightforward so we are done. □

Proposition 2.22. *If A is in $\mathcal{D}_\alpha$ and B is in $\mathcal{D}_\beta$, then $A \otimes B$ is in $\mathcal{D}_{\alpha+\beta}$.*

Proof. We first assume $\alpha = 0$ and proceed by transfinite induction on β. The base case $\beta = 0$ says that a tensor product of unital locally finite-dimensional C^*-algebras is unital and locally finite-dimensional, which is straightforward. Assume $\beta > 0$, and let B be a C^*-algebra in $\mathcal{D}_\beta$. Using Lemma 2.19, B is nuclear. Hence by Lemma 2.21 $A \otimes B$ decomposes over the class of C^*-algebras of the form $A \otimes C$, with $C \in \bigcup_{\gamma<\beta} \mathcal{D}_\gamma$. The inductive hypothesis therefore implies that $A \otimes B$ decomposes over the class $\bigcup_{\gamma<\beta} \mathcal{D}_\gamma$, so $A \otimes B$ is in $\mathcal{D}_\beta$ by definition.

Now fix β, and proceed by transfinite induction on α. The base case $\alpha = 0$ follows from the discussion above. For $\alpha > 0$, the inductive step follows directly from Lemma 2.21 just as in the case above, so we are done. □

3 Weak complexity rank one

In this section, we study the special case that a C^*-algebra has weak complexity rank one. Let us first recall a definition from [7].

Definition 3.1. A C^*-algebra A has *real rank zero* if any self-adjoint element of A can be approximated arbitrarily well by a self-adjoint element with finite spectrum

The following theorem is our main goal in this section.

Theorem 3.2. *Let A be a separable, unital C^*-algebra with real rank zero and nuclear dimension at most one. Then A has weak complexity rank at most one.*

Conversely, let A be a separable, unital C^*-algebra with weak complexity rank at most one. Then A has nuclear dimension at most one. If in addition A is simple and either (i) has at most finitely many (possibly zero) extremal tracial states, or (ii) is AH with slow dimension growth, then it has real rank zero.

It is conceivable that weak complexity rank at most one implies real rank zero in general: see Remark 3.15 below for some further comments.

Remark 3.3. Weak complexity rank zero is the same as being locally finite dimensional by definition, and this is in turn equivalent to having nuclear dimension zero by a slight elaboration on [36, Remark 2.2 (iii)]. Hence if one replaces "at most one" by "one" everywhere it appears in Theorem 3.2, the theorem is still correct.

3.1 Sufficient conditions

In this subsection, we focus on establish the sufficient condition for a C^*-algebra to have weak complexity rank one at most one from Theorem 3.2.

We need a lemma about order zero maps in the presence of real rank zero.

Lemma 3.4. *Let A be a C^*-algebra of real rank zero, and let $\phi : F \to A$ be an order zero ccp map from a finite-dimensional C^*-algebra F into A. Then for any $\epsilon > 0$ there exists a positive contraction $h \in A$ with finite spectrum and a $*$-homomorphism $\pi : F \to M(C^*(\phi(F))) \cap \{h\}'$ such that*

$$\|\phi(f) - h\pi(f)\| \leq \epsilon\|f\|$$

for all $f \in F$.

Without h having finite spectrum, the previous result is well-known (even with $\epsilon = 0$), and due to Winter [32, 4.1.1] (see also Winter and Zacharias [35], which removes the assumption that F is finite dimensional). The reader might also usefully compare the lemma above to [33, Lemma 2.4], which gives a very similar result for real rank zero codomains.

Proof of Lemma 3.4. Let $C^*(\phi(F))$ be the C^*-subalgebra of A generated by the image of ϕ, and let $M(C^*(\phi(F)))$ be its multiplier algebra. By the basic structure theorem for order zero ccp maps [35, Theorem 2.3], there exists a positive contraction $h_0 \in A$ and a $*$-homomorphism $\pi : F \to M(C^*(\phi(F))) \cap \{h_0\}'$ such that

$$\phi(f) = h_0\pi(f)$$

for all $f \in F$. Write $F = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, and let

$$\{e_{ij}^{(l)} \mid l \in \{1, ..., k\}, i, j \in \{1, ..., n_k\}\}$$

be a set of matrix units for F. Define $m_{ij}^{(l)} := \pi(e_{ij}^{(l)}) \in M(C^*(\phi(F)))$. For each l, let $(b_\lambda^{(l)})$ be a net of positive contractions in $C^*(\phi(F))$ that converges to $m_{11}^{(l)}$ in the strict topology; for simplicity, we assume that the index set for all these nets is the same as l varies. Replacing each $b_\lambda^{(l)}$ with $m_{11}^{(l)} b_\lambda^{(l)} m_{11}^{(l)}$, we may assume that $b_\lambda^{(l)} \leq m_{11}^{(l)}$ for all λ and all l. Let λ be large enough that $\|b_\lambda^{(l)} h_0 b_\lambda^{(l)} - m_{11}^{(l)} h_0 m_{11}^{(l)}\| < \epsilon/2$, which exists by strict convergence. Note that $b_\lambda^{(l)} h_0 b_\lambda^{(l)}$ is an element of the hereditary C^*-subalgebra $\overline{b_\lambda^{(l)} A b_\lambda^{(l)}}$ of A. Hence using that real rank zero passes to hereditary subalgebras (see for example [7, Theorem 2.6, (iii)]), we may find a positive contraction $h_{11}^{(l)} \in \overline{b_\lambda^{(l)} A b_\lambda^{(l)}}$ with finite spectrum such that $\|h_{11}^{(l)} - m_{11}^{(l)} h_0 m_{11}^{(l)}\| < \epsilon$. Define now

$$h := \sum_{l=1}^{k} \sum_{j=1}^{n} m_{j1}^{(l)} h_{11}^{(l)} m_{1j}^{(l)}. \tag{2}$$

We claim that this h (and the original π) have the right properties.

We have to show that:

(i) the image of π commutes with h;

(ii) h has finite spectrum;

(iii) $\|\phi(f) - h\pi(f)\| \leq \epsilon \|f\|$ for all $f \in F$.

Indeed, for (i), note that for any $m_{ij}^{(l)}$,

$$\begin{aligned} m_{ij}^{(l)} h &= \sum_{k=1}^{n} m_{ij}^{(l)} m_{k1}^{(l)} h_{11}^{(l)} m_{1k}^{(l)} = m_{i1}^{(l)} h_{11}^{(l)} m_{1j}^{(l)} = \sum_{k=1}^{n} m_{k1}^{(l)} h_{11}^{(l)} m_{1k}^{(l)} m_{ij}^{(l)} \\ &= h m_{ij}^{(l)}. \end{aligned}$$

As the $m_{ij}^{(l)}$ span $\pi(F)$, this implies that h commutes with $\pi(F)$, and thus that π takes image in $\{h\}'$ as we needed.

For (ii), note that as $h_{11}^{(l)} \in b_\lambda^{(l)} A b_\lambda^{(l)}$, and as $b_\lambda^{(l)} \leq m_{11}^{(l)}$, we have that $h_{11}^{(l)} \leq m_{11}^{(l)}$. Hence the elements $m_{j1}^{(l)} h_{11}^{(l)} m_{1j}^{(l)}$ are mutually orthogonal and unitarily equivalent (say when considered in the bounded operators under an appropriate Hilbert space representation) to $h_{11}^{(l)}$ for all j and l. It follows that the spectrum of h is the union of the spectra of the $h_{11}^{(l)}$ as l varies, so finite.

For (iii), note that

$$\|\phi(f) - h\pi(f)\| = \|h_0\pi(f) - h\pi(f)\| \leq \|h_0 - h\| \|\pi(f)\| \leq \|h - h_0\| \|f\|.$$

Hence it suffices to prove that $\|h - h_0\| < \epsilon$. For this, note that as h commutes with $\pi(F)$ and as $h \leq \sum_{l=1}^{k} \sum_{j=1}^{n} m_{jj}^{(l)}$, we have that

$$\begin{aligned} h &= \Big(\sum_{l=1}^{k} \sum_{j=1}^{n} m_{jj}^{(l)} \Big) h = \sum_{l=1}^{k} \sum_{j=1}^{n} m_{j1}^{(l)} m_{1j}^{(l)} h = \sum_{l=1}^{k} \sum_{j=1}^{n} m_{j1}^{(l)} h m_{1j}^{(l)} \\ &= \sum_{l=1}^{k} \sum_{j=1}^{n} m_{j1}^{(l)} m_{11}^{(l)} h m_{11}^{(l)} m_{1j}^{(l)} \end{aligned}$$

Hence

$$h - h_0 = \sum_{l=1}^{k}\sum_{j=1}^{n} m_{j1}^{(l)} h_{11}^{(l)} m_{1j}^{(l)} - \sum_{l=1}^{k}\sum_{j=1}^{n} m_{j1}^{(l)} m_{11}^{(l)} h_0 m_{11}^{(l)} m_{1j}^{(l)}$$

and so

$$\|h - h_0\| = \Big\| \sum_{l=1}^{k}\sum_{j=1}^{n} m_{j1}^{(l)} (h_{11}^{(l)} - m_{11}^{(l)} h_0 m_{11}^{(l)}) m_{1j}^{(l)} \Big\|$$

As the terms $m_{j1}^{(l)}(h_{11}^{(l)} - m_{11}^{(l)} h m_{11}^{(l)}) m_{1j}^{(l)}$ are mutually orthogonal as j and l vary, this equals

$$\sup_{l,j} \|m_{j1}^{(l)}(h_{11}^{(l)} - m_{11}^{(l)} h_0 m_{11}^{(l)}) m_{1j}^{(l)}\| \leq \|h_{11}^{(l)} - m_{11}^{(l)} h_0 m_{11}^{(l)}\| < \epsilon,$$

and we are done. □

For the next result, let $A_\infty := \prod_{\mathbb{N}} A / \oplus_{\mathbb{N}} A$ denote the quotient of the product of countably many copies of a C^*-algebra A by the direct sum. We identify A with its image in A_∞ under the natural diagonal embedding, and write $A_\infty \cap A'$ for the relative commutant. More generally, if (B_n) is a sequence of C^*-algebras, we also write $B_\infty := \prod_{\mathbb{N}} B_n / \oplus_{\mathbb{N}} B_n$ for the associated quotient.

Proposition 3.5. *Let A be a separable, unital C^*-algebra with real rank zero and nuclear dimension one. Then there exists a positive contraction $h \in A_\infty \cap A'$ and sequences (C_n) and (D_n) of finite-dimensional C^*-subalgebras of A such that $ha \in C_\infty$ and $(1-h)a \in D_\infty$ for all $a \in A$.*

Proof. Since A is separable and of nuclear dimension one, by [36, Theorem 3.2] there exists a sequence (ψ_n, ϕ_n, F_n) where:

(i) each F_n is a finite-dimensional C^*-algebra that decomposes as a direct sum of ideals $F_n = F_n^{(0)} \oplus F_n^{(1)}$;

(ii) each ψ_n is a ccp map $A \to F_n$ such that the induced diagonal map $\overline{\psi} : A \to F_\infty$ is order zero;

(iii) each ϕ_n is a map $F_n \to A$, such that the restriction $\phi_n^{(i)}$ of ϕ_n to $F_n^{(i)}$ is ccp and order zero for $i \in \{0, 1\}$;

(iv) for all $a \in A$, $\phi_n \psi_n(a) \to a$ as $n \to \infty$.

For $i \in \{0,1\}$, we will also need to consider the (order zero, ccp) maps $\overline{\phi^{(i)}} : (F^{(i)})_\infty \to A_\infty$ induced from $\phi_n^{(i)} : F_n \to A$, and the canonical projection $*$-homomorphism $\kappa^{(i)} : F_\infty \to F_\infty^{(i)}$.

As for each n the map $\phi_n^{(0)} : F_n^{(0)} \to A$ is ccp and order zero, by [35, Theorem 2.3] there exists a positive contraction $h_n^{(0)} \in A$ and a $*$-homomorphism $\pi_n^{(0)} : F_n^{(0)} \to M\Big(C^*\big(\phi_n^{(0)}(F_n^{(0)})\big)\Big) \cap \{h_n^{(0)}\}'$ such that

$$\phi_n^{(0)}(b) = h_n^{(0)}\pi_n^{(0)}(b)$$

for all $b \in F_n^{(0)}$. As in [35, Corollary 3.1], the formula

$$\rho_n^{(0)}(f \otimes b) := f(h_n^{(0)})\pi_n^{(0)}(b)$$

determines a $*$-homomorphism

$$\rho_n^{(0)} : C_0(0,1] \otimes F_n^{(0)} \to A.$$

Define $S_n := \rho_n^{(0)}(C_0(0,1] \otimes F_n^{(0)})$ and $R_n := \rho_n^{(1)}(C_0(0,1] \otimes F_n^{(1)})$.

As in (the proof of) [30, Proposition A.1], the element

$$h := \overline{\phi^{(0)}} \circ \kappa^{(0)} \circ \overline{\psi}(1)$$

is a positive contraction in $A_\infty \cap A'$, and has the property that for all $a \in A \subseteq A_\infty$,

$$ha = \overline{\phi^{(0)}} \circ \kappa^{(0)} \circ \overline{\psi}(a) \quad \text{and} \quad (1-h)a = \overline{\phi^{(1)}} \circ \kappa^{(1)} \circ \overline{\psi}(a).$$

In particular, we see that $ha \in S_\infty$ and $(1-h)a \in R_\infty$ for all $a \in A$.

From Lemma 3.4, since A has real-rank zero, for each n there exists a positive contraction $\eta_n^{(0)} \in A$ with finite spectrum that commutes with the image of $\pi_n^{(0)}$ and such that

$$\|\phi_n^{(0)}(b) - \eta_n^{(0)}\pi_n^{(0)}(b)\| \leq \frac{\|b\|}{n} \tag{3}$$

for all $b \in F_n^{(0)}$. Define a new map $\sigma_n^{(0)} : C_0(0,1] \otimes F_n^{(0)} \to A$ by sending $f \otimes b \mapsto f(\eta_n^{(0)})\pi_n^{(0)}(b)$. This factors through a finite-dimensional C^*-algebra as in the diagram below

$$\begin{tikzcd}
C_0(0,1] \otimes F_n^{(0)} \arrow[r, "\sigma_n^{(0)}"] \arrow[d] & A \\
C\big(\mathrm{spec}(\eta_n^{(0)})\big) \otimes F_n^{(0)} \arrow[ur] &
\end{tikzcd}$$

and so the image of $\sigma_n^{(0)}$ is a finite-dimensional C^*-subalgebra of A. Define C_n to be the image of $\sigma_n^{(0)}$, and let $C_\infty := \prod_{\mathbb{N}} C_n / \oplus_{\mathbb{N}} C_n$ denote the corresponding C^*-subalgebra of A_∞. Working instead with

$i = 1$, we choose $\eta_n^{(1)}$ and use it to define $\sigma_n^{(1)}$, D_n, and D_n^∞ precisely analogously.

Let $a \in A \subseteq A_\infty$, so that $ha \in S_\infty$ and denote by $b := [\kappa^{(0)} \circ \overline{\psi}(a)] \in (F^{(0)})_\infty$. Choose a sequence (b_n) in $\prod_{\mathbb{N}} F_n^{(0)}$ that lifts b and that satisfies $\|b_n\| \leq \|a\|$ for all n. For a sequence (a_n) in $\prod_{\mathbb{N}} A_n$, let us write $[(a_n)]$ for the corresponding element of A_∞. Then we compute that in A_∞

$$\begin{aligned} ha - [(\eta_n^{(0)}\pi_n^{(0)}(b_n))] &= [\phi_n^{(0)}(b_n)] - [(\eta_n^{(0)}\pi_n^{(0)}(b_n))] \\ &= [(\phi_n^{(0)} - \eta_n^{(0)}\pi_n^{(0)})(b_n)]. \end{aligned}$$

Line (3) implies that

$$\|(\phi_n^{(0)} - \eta_n^{(0)}\pi_n^{(0)})(b_n)\| \leq \frac{\|b_n\|}{n} \leq \frac{\|a\|}{n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence $ha = [\phi_n^{(0)}(b_n)] = [(\eta_n^{(0)}\pi_n^{(0)}(b_n))] \in C_\infty$. A similar argument shows that $(1-h)a \in D_\infty$ and we are done. □

From Proposition 3.5 we have the following.

Theorem 3.6. *If A is a unital separable C^*-algebra with real rank zero and nuclear dimension at most one, then A has weak complexity rank at most one.*

Proof. Let (C_n) and (D_n) and h be as in the conclusion of Proposition 3.5. Lift h to a positive contraction (h_n) in $\prod_{\mathbb{N}} A_n$. Then one checks that directly that for any finite subset X and $\epsilon > 0$ there is N so that for all $n \geq N$, C_n, D_n, and h_n satisfy the conditions needed for weak complexity rank at most one. □

The following corollary gives an interesting class of C^*-algebras with weak complexity rank one that we will use later. For the statement, recall that a unital C^*-algebra A is a *Kirchberg algebra* if it is separable, nuclear, and if for any non-zero $a \in A$, there exist $b, c \in A$ such that $bac = 1_A$ (note that this last condition implies simplicity). See for example [25, Chapter 4] for background on this class of C^*-algebras.

Corollary 3.7. *Any unital Kirchberg algebra has weak complexity rank one.*

Proof. Kirchberg algebras have real rank zero by the main result of [37] and nuclear dimension one by [6, Theorem G], whence weak complexity rank at most one by Theorem 3.2. Kirchberg algebras do not have complexity rank zero as they are not locally finite dimensional. □

3.2 Necessary conditions

We now establish a partial converse to Theorem 3.6. First, recall that Proposition 2.19 shows that if A has weak complexity rank at most one, then it has nuclear dimension one. To establish a converse to Theorem 3.6, we therefore need to show that weak complexity rank one implies real rank zero. We can do this for some special classes of C^*-algebras, but not in general; moreover, the proofs of our main results (see Corollary 3.8 and Proposition 3.9 below) are not self-contained, but rely on deep structural results for simple nuclear C^*-algebras. We have generally tried to explain the properties we use as we need them: the most glaring omission is probably any discussion of $\mathcal{Z}$-stability, which we just use as a black box.

First, let us look at the class of simple AH C^*-algebras [4]: see for example [25, Section 3.1] for background and definitions, including the notion of slow dimension growth (see [25, Definition 3.1.1]) that we use below. Suffice to say here that the class of AH C^*-algebras with slow dimension growth is large, well-studied, and contains many interesting examples such as irrational rotation algebras.

Corollary 3.8. *Let A be a simple, separable, unital AH algebra. Then A has weak complexity rank at most one if and only if has real rank zero and slow dimension growth.*

Proof. Let A be as in the assumptions, and assume in addition that A has slow dimension growth and real rank zero. Then A is $\mathcal{Z}$-stable by [34, Corollary 6.5] or [29, Corollary 1.3], whence has nuclear dimension at most one by [10, Theorem B]. Hence if A has real rank zero, it has weak complexity rank at most one by Theorem 3.6. Conversely, say A has weak complexity rank at most one. Then it has nuclear dimension at most one by Proposition 2.19, whence is $\mathcal{Z}$-stable by [34, Corollary 6.3], whence has slow dimension growth by [34, Corollary 6.5] or [29, Corollary 1.3] again. As we now know that A has slow dimension growth, and as projections separate traces by Lemma 2.17, it has real rank zero by [4, Theorem 2]. □

The next result takes more effort to establish (and similarly to the above, relies on deep work of others). For a unital C^*-algebra A, we let $T(A)$ denote its tracial state space, which is a (possibly empty) convex, weak-$*$ compact subset of the dual A^*.

Proposition 3.9. *Let A be a simple, separable, unital C^*-algebra with weak complexity rank at most one, and such that $T(A)$ is either empty,* *or has finitely many extreme points. Then A has real rank zero.*

To establish this, we will need some facts about Cuntz (sub)equivalence and dimension functions. We will recall the facts we need: the reader can see [23] or [1] for further background on these topics.

Let us write $M_\infty(A)$ for the purely $*$-algebraic direct limit of the sequence $(M_n(A))$ of C^*-algebras, with connecting maps given by

$$M_n(A) \to M_{n+1}(A), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

As any finite collection of elements of the $*$-algebra $M_\infty(A)$ is contained in a C^*-subalgebra, it makes sense to speak of norms, the subset $M_\infty(A)_+$ of positive elements, functional calculus, and so on.

For $a, b \in M_\infty(A)_+$, we say a is *Cuntz subequivalent* to b, and write $a \precsim b$, if there is a sequence (r_n) in $M_\infty(A)_+$ such that $r_n b r_n^*$ converges in norm to a. We say a and b are *Cuntz equivalent*, and write $a \sim b$, if $a \precsim b$ and $b \precsim a$. Note that $\precsim$ is a transitive and reflexive relation, and $\sim$ is an equivalence relation. For $a, b \in M_n(A)$, we define

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{2n}(A).$$

If $a, b \in M_\infty(A)$ we abuse notation by writing "$a \oplus b$" for 'the' corresponding element of $M_\infty(A)$ defined by identifying a and b with elements of some subalgebra $M_n(A)$; although this is technically not well-defined, any two choices will be Cuntz equivalent.

We record some basic properties of Cuntz subequivalence in the following lemma. For a self-adjoint element $a \in M_\infty(A)$, let us write a_+ for its positive part. Note that if $a \in M_\infty(A)_+$, then for any $\epsilon > 0$, the element $(a - \epsilon)_+$ is in $M_\infty(A)$, not just in $M_\infty(A^+)$ (with A^+ as usual the unitization of A).

Lemma 3.10. *Let A be a C^*-algebra, and let $a, b \in M_\infty(A)_+$. The following hold:*

(i) If $a \leq b$, then $a \precsim b$.

(ii) $a + b \precsim a \oplus b$.

(iii) If $\|a - b\| \leq \epsilon$, then $(a - \epsilon)_+ \precsim b$.

Proof. Part (i) is [23, Lemma 2.3] or [1, Lemma 2.8], part (ii) is [1, Lemma 2.10], and part (iii) follows directly from [23, Proposition 2.2] or [1, Theorem 2.13]. ∎

We recall a useful estimate of Pedersen, which is (a special case of) the main result of [21].

Lemma 3.11. *Let a and b be bounded operators on a Hilbert space with $b \geq 0$. Then*

$$\|[a, b^{1/2}]\| \leq \frac{5}{4}\|a\|^{1/2}\|[a, b]\|^{1/2}. \qquad \square$$

Lemma 3.12. *Let A be a unital C^*-algebra with weak complexity rank at most one. Then for any $a \in M_\infty(A)_+$ and $\epsilon > 0$ there is a projection $p \in M_\infty(A)$ such that*

$$(a-\epsilon)_+ \precsim p \precsim a \oplus a.$$

Proof. Fix n so that a is in $M_n(A)$. Note that $M_n(A)$ also has weak complexity rank one by (an easy variant of) Proposition 2.22. Let $\delta = \epsilon/10$. The definition of weak complexity rank at most one and Lemma 3.11 give a positive contraction $h \in M_n(A)$ and finite-dimensional C^*-subalgebras C, D of $M_n(A)$ such that $h^{1/2}ah^{1/2} \in_\delta C$ and $(1-h)^{1/2}a(1-h)^{1/2} \in_\delta D$, and such that $\|[a, h^{1/2}]\| < \delta$ and $\|[a,(1-h)^{1/2}]\| < \delta$. Lemma 2.8 then gives positive contractions $c \in C$ and $d \in D$ such that $\|h^{1/2}ah^{1/2} - c\| < 2\delta$ and $\|(1-h)^{1/2}a(1-h)^{1/2} - d\| < 2\delta$. Note that

$$a \approx_{2\delta} h^{1/2}ah^{1/2} + (1-h)^{1/2}a(1-h)^{1/2} \approx_{4\delta} c+d \approx_{4\delta} (c-2\delta)_+ + (d-2\delta)_+$$

whence by Lemma 3.10 part (iii), $(a-10\delta)_+ \precsim (c-2\delta)_+ + (d-2\delta)_+$, and so by part (ii) of Lemma 3.10

$$(a-10\delta)_+ \precsim (c-2\delta)_+ \oplus (d-2\delta)_+. \tag{5}$$

On the other hand, $\|c - h^{1/2}ah^{1/2}\| < 2\delta$, whence part (iii) of Lemma 3.10 again gives $(c-2\delta)_+ \precsim h^{1/2}ah^{1/2} \precsim a$ (the second Cuntz subequivalence is clear from the definition), and similarly $(d-2\delta)_+ \precsim a$; combining this with line (5) gives

$$(a-10\delta)_+ \precsim (c-2\delta)_+ \oplus (d-2\delta)_+ \precsim a \oplus a.$$

Finally, note that as $(c-2\delta)_+$ is contained in a finite-dimensional C^*-algebra, it has finite spectrum, whence it is Cuntz equivalent to its support projection $p_C \in C$; similarly $(d-2\delta)_+$ is Cuntz equivalent to its support projection p_D. Setting $p := p_C \oplus p_D$, we are done. $\square$

We need some more terminology. Let A be a unital C^*-algebra, and extend any $\tau \in T(A)$ to $M_n(A)$ by the usual formula $\tau : (a_{ij})_{i,j=1}^n \mapsto$

$\sum_{i=1}^{n}\tau(a_{ii})$. These extensions are compatible with the inclusion maps in line (4), so piece together to define a positive trace $\tau : M_\infty(A) \to \mathbb{C}$. For $\epsilon > 0$, let $f : [0,\infty) \to [0,1]$ be the continuous function which is zero on $[0,\epsilon/2]$, 1 on $[\epsilon,\infty)$ and linear on $[\epsilon/2,\epsilon]$. For $a \in M_\infty(A)_+$, we define an affine function

$$\widehat{a} : T(A) \to [0,\infty), \quad a \mapsto \lim_{\epsilon\to 0}\tau(f_\epsilon(a))$$

(the limit exists as the collection $(\tau(f_\epsilon(a))_{\epsilon>0}$ is bounded, and increasing as ϵ tends to zero). Note that as $\widehat{a}$ is the pointwise limit of the increasing sequence of continuous, uniformly bounded functions $(\tau \mapsto \tau(f_{1/n}(a)))_{n=1}^{\infty}$, it is lower semi-continuous and bounded. Write now $\mathrm{LAFF}_b(T(A))^+$ for the collection of all lower semi-continuous, positive, bounded, affine functions from $T(A)$ to $\mathbb{R}$, so the above process defines a map

$$\iota : M_\infty(A)_+ \to \mathrm{LAFF}_b(T(A))^+, \quad a \mapsto \widehat{a}. \tag{6}$$

Lemma 3.14 below records the properties of this map that we will need. Before stating it, we need a well-known fact about spectral projections: it is essentially the same as [5, Lemma II.2.1][4].

Lemma 3.13. *Let $0 \leq a \leq b$ be bounded operators on a Hilbert space, and let $0 \leq \lambda < \mu$ be real numbers. Let $p := \chi_{(\mu,\infty)}(a)$ and $q := \chi_{(\lambda,\infty)}(b)$ be the spectral projections corresponding to the intervals (μ,∞) and (λ,∞) respectively. Then:*

(i) $\|p(1-q)\| \leq \sqrt{\lambda/\mu}$;

(ii) there is a partial isometry v in the C^*-algebra A generated by p and q such that $v^*v = p$ and $vv^* \leq q$.

Proof. We note that $\mu p \leq a \leq b$ and $\lambda(1-q) \geq b(1-q) = (1-q)b(1-q)$. Hence

$$\|p(1-q)p\| = \|(1-q)p(1-q)\| \leq \frac{1}{\mu}\|(1-q)b(1-q)\| \leq \frac{\lambda}{\mu}.$$

Part (i) now follows from the C^*-identity. On the other hand, the above inequalities imply that $\|p - pqp\| \leq \lambda/\mu < 1$, whence pqp is invertible in pAp. Let $x \in pAp$ be such that $xpqp = p = pqpx$. Then one can check directly that $v := qx^{1/2}$ has the properties required by part (ii). ☐

[4]The authors of that paper attribute the argument to Zsido.

Lemma 3.14. *For a unital C^*-algebra A, the map ι in line (6) above has the following properties.*

(i) If $a, b \in M_\infty(A)_+$ satisfy $a \precsim b$, then $\widehat{a} \leq \widehat{b}$.

(ii) For $a, b \in M_\infty(A)_+$, $\widehat{a \oplus b} = \widehat{a} + \widehat{b}$.

(iii) For $a \in M_\infty(A)_+$, $\widehat{(a - \epsilon)_+}$ converges pointwise to $\widehat{a}$ as $\epsilon \to 0$.

(iv) If A is separable, $\mathcal{Z}$-stable, simple, unital, finite and exact, and if $LAFF_b(T(A))^{++}$ consists of the strictly *positive elements of $LAFF_b(T(A))^+$, then the co-restricted map*

$$M_\infty(A)_+ \to LAFF_b(T(A))^{++}, \quad a \mapsto \widehat{a}$$

is surjective.

Proof. We start with part (i). Although this is well-known we could not find exactly what we wanted in the literature[5], so give an argument here for the reader's convenience. Say then that $a \precsim b$, so there is a sequence (r_n) such that $r_n b r_n^*$ converges to a. We may assume that a, b, and all the r_n are in $M_m(A)$ for some fixed m.

Fix $\epsilon > 0$ and $r = r_n$ for some n. Then for $\tau \in T(A)$, using that τ is a trace we see that

$$\tau(f_\epsilon(rbr^*)) = \tau(f_\epsilon(b^{1/2}r^*rb^{1/2})) \tag{7}$$

(indeed, it suffices to check this when f_ϵ is replaced a polynomial, which is straightforward). Let $p := \chi_{(\epsilon/2,\infty)}(b^{1/2}r^*rb^{1/2})$ and $q := \chi_{(\epsilon/3,\infty)}(b\|r\|^2)$, considered as elements in A^{**}. Then, using Lemma 3.13 for the second inequality, we see that

$$\tau(f_\epsilon(b^{1/2}r^*rb^{1/2})) \leq \tau(p) \leq \tau(q) \leq \tau(f_{\epsilon/6}(b\|r\|^2)) = \tau(f_{\epsilon/(6\|r\|^2)}(b)).$$

Combining this with line (7) implies that

$$\tau(f_\epsilon(rbr^*)) \leq \tau(f_{\epsilon/(6\|r\|^2)}(b)) \leq \widehat{b}(\tau). \tag{8}$$

for arbitrary $r = r_n$ and ϵ.

Let now δ be arbitrary, and choose ϵ such that

$$\widehat{a}(\tau) \leq \tau(f_\epsilon(a)) + \delta/2.$$

As the map $c \mapsto f_\epsilon(c)$ depends continuously on the input, and as the restriction of τ to $M_m(A)$ is continuous there is n such that for $r = r_n$

$$\widehat{a}(\tau) \leq \tau(f_{\epsilon_0}(a)) + \delta/2 \leq \tau(f_\epsilon(rbr^*)) + \delta.$$

[5]Compare [5, Theorem II.2.2] or [13, Proposition 2.1] for closely related results.

Combining this with line (8), we see that $\widehat{a}(\tau) \leq \widehat{b}(\tau) + \delta$, and as δ was arbitrary, we are done with part (i).

Part (ii) is straightforward from the fact that $f_\epsilon(a \oplus b) = f_\epsilon(a) \oplus f_\epsilon(b)$ for any a, b and ϵ. Part (iii) follows directly from the dominated convergence theorem, once we have used the Gelfand-Naimark theorem to convert it to a problem about integration. Finally, part (iv) is [9, Theorem 5.5]. □

We are now ready for the proof of Proposition 3.9.

Proof of Proposition 3.9. Assume first that $T(A)$ is empty. Then as A has finite nuclear dimension by Proposition 2.19, A is purely infinite by [36, Theorem 5.4], so has real rank zero by the main result of [37].

Assume then that $\tau_1, ..., \tau_m$ are the extremal traces of A for some positive integer m. As A has finite nuclear dimension it is in particular exact. Moreover, if $\mathcal{Z}$ is the Jiang-Su algebra, then as A is simple, unital and has finite nuclear dimension, it is $\mathcal{Z}$-stable by [34, Corollary 6.3]. Write $\mathrm{Aff}(T(A))$ for the collection of all affine continuous functions from $T(A)$ to $\mathbb{R}$. Note that if $p \in M_\infty(A)_+$ is a projection then the map $\widehat{p}$ is in $\mathrm{Aff}(T(A))$, and it is not difficult to check that we get a well-defined homomorphism of abelian groups

$$\iota_K : K_0(A) \to \mathrm{Aff}(T(A)), \quad [p] \mapsto \widehat{p}.$$

Using [26, Theorem 7.2], it suffices to prove that this map has uniformly dense image.

Let then $\epsilon > 0$. For each i, let $\delta_i \in \mathrm{Aff}(T(A))$ be the function determined by $\delta_i(\tau_j) = \delta_{ij}$, where the δ_{ij} on the right is the Kronecker δ-function. To show that ι_K has uniformly dense image, it suffices to prove that for any $N \in \mathbb{N}$, and any $\epsilon > 0$, we can find a projection $p \in M_\infty(A)$ such that $\|\widehat{p} - \delta_i/N\| < \epsilon$. Let then ϵ and N be given. Let $M \in \mathbb{N}$ be a multiple of N that is so large that $1/M < \epsilon$. Using part (iv) of Lemma 3.14, there is $x \in M_\infty(A)_+$ such that with ι as in line (6), $\iota(x)(\tau_i) = 1/2M$, and $\widehat{x}(\tau_j) < \epsilon/8M$ for $i \neq j$. Using part (iii) of Lemma 3.14 there is $\delta > 0$ such that for each j, $\tau_j(x) \geq \tau_j((x-\delta)^+) \geq \tau_j(x)/2$, which implies that

$$\iota(x)/2 \leq \iota((x - \delta)_+) \leq \iota(x). \tag{9}$$

Lemma 3.12 gives a projection $p_0 \in M_\infty(A)_+$ such that $(x - \delta)_+ \precsim p_0 \precsim x \oplus x$. Hence by parts (i) and (ii) of Lemma 3.14 and line (9) we see that $\iota(x)/2 \leq \iota(p_0) \leq 2\iota(x)$. As $1/M < \epsilon$ and as $\iota(x)(\tau_i) = 1/2M$, we have that $1/4M \leq \iota(p_0)(\tau_i) \leq 1/M < \epsilon$. Hence there is $k \in \mathbb{N}$ with $k \leq 4M/N$ and $|k\iota(p_0)(\tau_i) - 1/N| < \epsilon$. Define

$$p := \underbrace{p_0 \oplus \cdots \oplus p_0}_{k \text{ times}},$$

so the previous sentence says that

$$|\iota(p)(\tau_i) - 1/N| < \epsilon. \tag{10}$$

Moreover, for $j \neq i$, as $\iota(x)(\tau_j) < \epsilon/8M$, we have

$$\iota(p)(\tau_j) \leq 2k\iota(x)(\tau_j) < \epsilon. \tag{11}$$

Lines (10) and (11) together imply that $\iota(p)$ is within ϵ of δ_i/N in the uniform norm, which completes the proof. $\square$

Remark 3.15. We do not know if (weak) complexity rank at most one implies real rank zero in general. This seems an interesting question: for example, the uniform Roe algebra $C_u^*(|\mathbb{Z}|)$ of the integers has complexity rank at most one by [31, Example A.9] (and therefore complexity rank one as it is easy to see that it is not locally finite-dimensional - compare also [20, Theorem 2.2]); whether or not $C_u^*(|\mathbb{Z}|)$ has real rank zero is quite an interesting problem for the reasons discussed below [20, Question 3.10]. On the other hand, the uniform Roe algebra of $\mathbb{Z}^2$ has complexity rank at most two by [31, Example A.9] again, and does not have real rank zero by [20, Theorem 3.1], so it is certainly not true that (weak) finite complexity implies real rank zero in general.

4 Torsion in odd K-theory

In this section, we show that the K_1-group of a C^*-algebra with complexity rank at most one is torsion free. This seems to be of interest in its own right, and is also a key ingredient in our computation of the complexity rank of UCT Kirchberg algebras.

Here is the main theorem of this section. We thank Ian Putnam for suggesting to one of the authors that something like this should be true.

Theorem 4.1. *Let A be a unital C^*-algebra with complexity rank at most one. Then $K_1(A)$ is torsion-free.*

Before we get into the proof of us, let us first show that weak complexity rank and complexity rank are genuinely different.

Corollary 4.2. *There are C^*-algebras with weak complexity rank one that do not have complexity rank one.*

Proof. Any unital Kirchberg algebra has weak complexity rank one by Corollary 3.7. A Kirchberg algebra can have any countable abelian group as its K_1-group (see [24, Theorem 3.6], or the exposition in [25, Proposition 4.3.3]), so by Theorem 4.1 there are Kirchberg algebras that do not have complexity rank one. $\square$

Throughout this section, if $a \in M_n(A)$, then $a^{\oplus k}$ is the diagonal matrix with all entries a in $M_k(M_n(A)) = M_{kn}(A)$. We write the unit in $M_n(A^+)$ as 1_n. We will rely heavily on ideas from [30]: we will give precise statements for what we need, but some proofs just refer to that paper. The methods of proof we use rely on K-theory groups based on idempotents and invertibles, not just projections and unitaries as is common in C^*-algebra K-theory. We recommend [3, Chapters 5 and 8] as a background reference.

The following two lemmas are contained in the proof of [30, Lemma 2.4] (see also [3, Proposition 4.3.2] for the second).

Lemma 4.3. *For any $c \geq 1$ and $\epsilon > 0$ there exists $\delta > 0$ with the following property. Let A be a C^*-algebra, B be a C^*-subalgebra, and let $e \in M_n(A)$ be an idempotent with $\|e\| \leq c$ and $e \in_\delta M_n(B)$. Then there is an idempotent $f \in M_n(B)$ with $\|e - f\| < \epsilon$.* $\square$

Lemma 4.4. *Let $d \geq 1$, and let A be a C^*-algebra. If $e, f \in M_n(A)$ are idempotents that satisfy $\|e\| \leq d$, $\|f\| \leq d$, and $\|e-f\| \leq (2d+1)^{-1}$ then the classes $[e]$ and $[f]$ in $K_0(A)$ are the same.* $\square$

Now, assume $c \geq 1$, $\epsilon \in (0, (4c+6)^{-1})$, and let δ have the property in Lemma 4.3 for this c and ϵ. Assume B is a C^*-subalgebra of A, and that $e \in M_n(A)$ is an idempotent with $\|e\| \leq c$ and $e \in_\delta M_n(B)$. Then Lemma 4.3 gives an idempotent $f \in M_n(B)$ with $\|f - e\| < \epsilon$, and so in particular $\|f\| \leq d := c + 1$. Moreover, if $f' \in M_n(B)$ is another idempotent satisfying $\|f' - e\| < \epsilon$ then $\|f - f'\| < 2\epsilon < (2c+3)^{-1} = (2d+1)^{-1}$, so Lemma 4.4 implies that $[f] = [f']$ in $K_0(B)$. In conclusion, we get a well-defined class in $K_0(B)$ associated to e.

The following is [30, Definition 2.5].

Definition 4.5. Assume $c \geq 1$, $\epsilon \in (0, (4c+6)^{-1})$, and let δ have the property in Lemma 4.3 for this c and ϵ. Let B be a C^*-subalgebra of A, and let $e \in M_n(A)$ be an idempotent such that $\|e\| \leq c$, and $e \in_\delta M_n(B)$. We write $\{e\}_B$ for the class in $K_0(B)$ of any idempotent f in $M_n(B)$ that satisfies $\|e - f\| < \epsilon$ as in the above discussion.

The following is [30, Definition 2.6]

Definition 4.6. Let $c \geq 1$, let $\epsilon \in (0, (4c+6)^{-1})$, and let $\delta > 0$ satisfy the condition in Lemma 4.3. Let A be a C^*-algebra, and let C and D be C^*-subalgebras of A. Let $u \in M_n(A^+)$ be an invertible element for some n. Then an element $v \in M_{2n}(A^+)$ is a (δ, c, C, D)-*lift* of u if

(i) $\|v\| \leq c^{1/2}$ and $\|v^{-1}\| \leq c^{1/2}$;

(ii) $v \in_\delta M_{2n}(D^+)$;

(iii) $v \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \in_\delta M_{2n}(C^+)$;

(iv) $v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \in_\delta M_{2n}((C \cap D)^+)$;

(v) the K-theory class

$$\partial_v(u) := \left\{ v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right\}_{(C \cap D)^+} - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0((C \cap D)^+)$$

(see Definition 4.5 for the left hand term) is actually in the subgroup $K_0(C \cap D)$.

We need another definition.

Definition 4.7. Let C and D be C^*-subalgebras of a C^*-algebra A, with corresponding inclusion maps $\iota^C : C \to A$ and $\iota^D : D \to A$. Let $\sigma : K_1(C) \oplus K_1(D) \to K_1(A)$ be the map defined by $\sigma := \iota^C_* + \iota^D_*$.

The following result is contained in the proof of [30, Proposition 2.7].

Lemma 4.8. *Let $c \geq 1$, and let $\epsilon \in (0, (4c+6)^{-1})$. Then there is a $\delta > 0$ depending only on ϵ and c, and with the following property. Let A be a C^*-algebra and let $u \in M_n(A)$ be an invertible element of norm at most c. Let C and D be C^*-subalgebras of A, and let $v \in M_{2n}(A^+)$ be a (δ, c, C, D)-lift of u. If the K-theory class*

$$\partial_v(u) := \left\{ v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right\}_{(C \cap D)^+} - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \right]$$

is zero, then the class $[u] \in K_1(A)$ is in the image of the map σ from Definition 4.7. □

We need a little more notation before we recall another result from [30].

Definition 4.9. Let A be a C^*-algebra, let h be a positive contraction in A, and let u be an invertible element of $M_n(A^+)$. Let[6] $a = h + (1-h)u \in M_n(A)$, $b = h + (1-h)u^{-1} \in M_n(A)$, and define

$$v(u,h) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_{2n}(A^+).$$

The following result is contained in the proof of [30, Proposition 3.6].

Lemma 4.10. *For any $\delta > 0$ and $n \in \mathbb{N}$ there exists $\gamma > 0$ with the following property. Let A be a C^*-algebra and $u \in M_n(A^+)$ be a unitary. Write $u = \lambda + y$ where $\lambda \in M_n(\mathbb{C})$ is a scalar matrix, and $y \in M_n(A)$. Let $X \subseteq A$ be the finite subset of A consisting of the matrix entries of y.*

Then if (h, C, D) is a triple satisfying the conditions in Lemma 2.14 with respect to X and $\epsilon = \gamma$, we have that $v(u,h)$ is a $(\delta, 64, C, D)$ lift of u. □

For $k \in \mathbb{N}$, let $s_k \in M_{2k}(\mathbb{C})$ be the (unitary) permutation matrix determined by

$$(z_1, z_2, ..., z_k, z_{k+1}, ..., z_{2k}) \mapsto (z_1, z_3, ..., z_{2k-1}, z_2, z_4, ..., z_{2k}).$$

For any C^*-algebra A and any $n \in \mathbb{N}$, we abuse notation by identifying s_k with the element $s_k \otimes 1_{M_n(A)}$ of $M_{kn}(A) = M_k(\mathbb{C}) \otimes M_n(A)$.

The following fact is closely related to [30, Lemma 4.2][7]. The proof consists in direct checks that we leave to the reader.

Lemma 4.11. *Let $c \geq 1$, and $\epsilon \in (0, (4c+6)^{-1})$. Let A be a C^*-algebra, let $u \in M_n(A^+)$ be unitary, and let $v \in M_{2n}(A^+)$ be a (δ, c, C, D)-lift of u, where δ, C, and D satisfy the conditions in Definition 4.6. Then the following hold:*

(i) For any $k \in \mathbb{N}$, $s_k(v^{\oplus k})s_k^*$ is a (δ, c, C, D)-lift of $u^{\oplus k}$.

(ii) If in addition (h, C, D) satisfies the conditions in Lemma 4.10 with respect to appropriate X and γ, and (with notation as in Definition 4.9) $v = v(u,h)$, then $s_k(v^{\oplus k})s_k^* = v(u^{\oplus k}, h)$.

[6] Here we conflate a contraction $h \in A$ with the corresponding diagonal matrix $h \otimes 1_n \in M_n(A)$

[7] The statement of that lemma claims that the element we have called s_k is self-inverse, which is clearly wrong. However, this does not significantly affect that lemma, having replaced $s(v^{\oplus k})s$ with $s(v^{\oplus k})s^*$ as appropriate.

(iii) The K-theory classes $\partial_{s(v^{\oplus k})s^*}(u^{\oplus k})$ *and* $k\cdot\partial_v(u)$ *are equal in* $K_0(C\cap D)$. $\square$

Proof of Theorem 4.1. Let $\kappa\in K_1(A)$ be such that $n\cdot\kappa=0$ for some $n\in\mathbb{N}$; our goal is to show that $n=0$ or $\kappa=0$. Let $w\in M_m(A^+)$ be a unitary element such that $[w]=\kappa$. As $U_m(\mathbb{C})$ is connected, we may assume that w is of the form 1_m+y for some $y\in M_m(A)$ (whence also $w^{-1}=w^*=1_m+y^*$). As $[w^{\oplus n}]=n\cdot\kappa=0$ we have that $w^{\oplus n}\oplus 1_{r'}$ is homotopic through unitaries to $1_{s'}$ for some integers $r',s'\geq 1$. Letting $r=r'$ and $s=s'+(n-1)r'$, we have that $(w\oplus 1_r)^{\oplus n}$ is homotopic through unitaries to $w^{\oplus n}\oplus 1_r'\oplus 1_{(n-1)r'}$, and thus to 1_s. Define $u:=w\oplus 1_r$, whence $[u^{\oplus n}]=n\cdot\kappa=0$ and $u^{\oplus n}$ is homotopic to 1_s. Let $(u_t')_{t\in[0,1]}$ denote a path of unitary elements in $M_s(A^+)$ with $u_0'=u^{\oplus n}$ and $u_1'=1_s$. Let $\pi:M_s(A^+)\to\mathbb{C}$ be the canonical quotient map. As $\pi(u_0')=\pi(u_1')=1_s$, defining $u_t:=u_t'\pi(u_t')^{-1}$ for $t\in[0,1]$ gives a path of unitaries in $M_s(A^+)$ that connects $u^{\oplus n}$ and 1_s, and moreover so that u_t is of the form $u_t=1_s+y_t$ for some continuous path $(y_t)_{t\in[0,1]}$ in $M_s(A)$.

Let $c=8$, and let $\epsilon=(12c+18)^{-1}$. Let $\delta>0$ have the property in Lemma 4.8 with respect to this ϵ. Let $\gamma>0$ be as in Lemma 4.10 with respect to c and δ. Choose a finite partition $0=t_0<...<t_k=1$ of $[0,1]$ such that for any $t\in[t_i,t_{i+1}]$ we have $\|u_t-u_{t_i}\|<\gamma/2$. With (y_t) as above, for each $i\in\{0,...,k\}$, let $X_t\subseteq A$ be the finite subset consisting of all matrix entries of y_t. Let $X:=\bigcup_{i=1}^k X_{t_i}$. Let (h,C,D) be a triple satisfying the conditions in Proposition 2.14 with respect to the finite set X and error parameter $\gamma/2$. Note that for any t and any $x\in X_t$ there is i and x_{t_i} such that $\|x-x_{t_i}\|<\gamma/2$. It follows that (h,C,D) satisfies the conditions in Lemma 2.14 with respect to the (possibly infinite) set $\bigcup_t X_t$ and the error parameter γ.

At this point Lemma 4.10 gives us that (with notation as in Definition 4.9) $v_t:=v(u_t,h)$ is a $(\delta,8,C,D)$-lift for u_t for all t. The choice of δ then gives us elements

$$\partial_{v_t}(u_t):=\left\{v_t\begin{pmatrix}1_n&0\\0&0\end{pmatrix}v_t^{-1}\right\}_{(C\cap D)^+}-\left[\begin{pmatrix}1_n&0\\0&0\end{pmatrix}\right]\in K_0(C\cap D)$$

for all t. We claim that

$$\partial_{v_1}(u_1)=\partial_{v_0}(u_0). \tag{12}$$

The path

$$[0,1]\to M_{2s}(A^+),\quad t\mapsto v_t\begin{pmatrix}1_n&0\\0&0\end{pmatrix}v_t^{-1}$$

is continuous, whence there exists a finite partition $0 = t_0 < \cdots < t_l = 1$ such that

$$\left\| v_{t_{j+1}} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_{t_{j+1}}^{-1} - v_{t_j} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_{t_j}^{-1} \right\| < \epsilon$$

for all $j \in \{0, ..., l-1\}$. Hence if f_j and f_{j+1} are idempotents in $M_{2n}(B^+)$ satisfying

$$\left\| f_j - v_{t_j} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_{t_j}^{-1} \right\| < \epsilon$$

then $\|f_j - f_{j+1}\| < 3\epsilon = (4c+6)^{-1}$. Hence $[f_j] = [f_{j+1}]$ in $K_0((C \cap D)^+)$ for all j by Lemma 4.4, whence the claim.

Now, that $\partial_{v_1}(u_1) = 0$ by definition of v_1 and the fact that $u_1 = 1_s$. Hence by the claim from line (12) that we just established,

$$\partial_{v_0}(u_0) = 0. \tag{13}$$

Let $v = v(u, h)$. Then Lemma 4.11 part (i) implies that $\partial_{s_n(v^{\oplus n})s_n^*}(u^{\oplus n})$ makes sense, and part (ii) of Lemma 4.11 and the fact that $u_0 = u^{\oplus n}$ implies that the classes $\partial_{s_n(v^{\oplus n})s_n^*}(u^{\oplus n})$ and $\partial_{v_0}(u_0)$ are equal. Hence $\partial_{s_n(v^{\oplus n})s_n^*}(u^{\oplus n}) = 0$ by line (13). On the other hand, part (iii) of Lemma 4.11 implies that $n \cdot \partial_v(u) = \partial_{s_n(v^{\oplus n})s_n^*}(u^{\oplus n})$ so we get

$$n \cdot \partial_v(u) = 0. \tag{14}$$

Now, as $C \cap D$ is finite-dimensional, $K_0(C \cap D)$ is torsion-free, so line (14) forces $n = 0$ or $\partial_v(u) = 0$. If $n = 0$ we are done, so assume $\partial_v(u) = 0$. From Lemma 4.8, we thus have that $[u]$ is in the range of σ. However, the domain of σ is $K_1(C) \oplus K_1(D)$, which is zero as C and D are finite-dimensional. Hence $[u] = 0$; as $u = w \oplus 1_r$, this implies that $[w] = 0$ too, and we are done. □

5 Kirchberg algebras

Our goal in this section is to study the complexity rank of Kirchberg algebras. Recall that a C^*-algebra is a *Kirchberg algebra* if it is simple, separable, nuclear and purely infinite. Our theorems will only apply to unital Kirchberg algebras, but the proof uses the non-unital case, so we do not include unitality in the definition of Kirchberg algebras.

The following theorem is our goal in this section.

Theorem 5.1. *Let A be a unital UCT Kirchberg algebra. Then A has complexity rank one or two. Moreover, it has complexity rank two if and only if $K_1(A)$ contains non-trivial torsion elements.*

There is quite a striking contrast here with the theory of nuclear dimension (and with weak complexity rank). Indeed, all Kirchberg algebras (regardless of the UCT) have nuclear dimension one by [6, Theorem G]; as a consequence of this and real rank zero, all Kirchberg algebras have *weak* complexity rank one as recorded in Corollary 3.7 above. As already noted in the introduction, proving Theorem 5.1 (or even an a priori much weaker statement, such as that a Kirchberg algebra with zero K-theory has finite complexity) *without the UCT assumption* would imply the UCT for all nuclear C^*-algebras.

The proof of Theorem 5.1 will make repeated use of (part of) the Kirchberg-Phillips classification theorem [22]; see also the exposition in [25, Chapter 8] and the newer approach in [15].

5.1 The rank one case (after Enders)

In this subsection, we adapt ideas of Enders [14] to establish the following theorem.

Theorem 5.2. *Let A be a unital UCT Kirchberg algebra with torsion free K_1 group. Then A has complexity rank one.*

Throughout this subsection we will be dealing with (sometimes large) matrices, so adopt some notation for convenience. Let e_{ij} denote the matrix units in $M_n(\mathbb{C})$, and for $j \in \{-(n-1), ..., 0, 1, ..., n-1\}$, write d_j for the matrix which has ones on the j^{th} subdiagonal and is zero elsewhere, i.e. $d_j := \sum_{i=1}^{n-j} e_{(i+j)i}$. Note for example that d_0 is the identity, and that $d_{-1} = e_{12} + e_{23} + \cdots + e_{(n-1)n}$ is the matrix with ones on the first *super*diagonal and zeros elsewhere. We will also identify $M_n(M(A \rtimes \mathbb{Z}))$ with $M_n(\mathbb{C}) \otimes M(A \rtimes \mathbb{Z})$ and write things like

$$d_1 \otimes 1 + d_{-(n-1)} \otimes u^n \in M_n(\mathbb{C}) \otimes M(A \rtimes \mathbb{Z}).$$

for the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & u^n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in M_n(M(A \rtimes \mathbb{Z})). \tag{15}$$

We need a definition from [14, Definition 1.1]. For a C^*-algebra B and $b_1, ..., b_n \in B$, we will also write $\mathrm{diag}(b_1, ..., b_n)$ for the diagonal matrix in $M_n(B)$ with entries $b_1, ..., b_n$, i.e. for

$$\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \in M_n(B)$$

Definition 5.3. Let A be a C^*-algebra equipped with an action α of $\mathbb{Z}$, and let $n \in \mathbb{N}$. Let ι_n be the $*$-homomorphism

$$\iota_n : A \rtimes \mathbb{Z} \to M_n(A \rtimes \mathbb{Z})$$

determined by the formulas

$$\iota_n(a) := \mathrm{diag}(\alpha^{-1}(a), \alpha^{-2}(a), ..., \alpha^{-n}(a))$$

for $a \in A$ and

$$\iota_n(u) := d_1 \otimes 1 + d_{-(n-1)} \otimes u^n$$

for $u \in M(A \rtimes \mathbb{Z})$ the canonical unitary implementing the $\mathbb{Z}$ action[8].

The key technical result is as follows: although somewhat different from the conclusions of Ender's arguments, it follows the same basic strategy.

Lemma 5.4. *Let A be an AF C^*-algebra equipped with a $\mathbb{Z}$-action. Let X be a finite subset of $A \rtimes \mathbb{Z}$, and assume there exists a projection $p \in A$ such that $px = xp = x$ for all $x \in X$. Let $\epsilon > 0$.*

Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, if $\iota_n : A \rtimes \mathbb{Z} \to M_n(A \rtimes \mathbb{Z})$ is as in Definition 5.3, and $q := \iota_n(p)$, then there exists a positive contraction $h \in q(M_n(A \rtimes \mathbb{Z}))q$ and AF C^*-subalgebras C and D of $q(M_n(A \rtimes \mathbb{Z}))q$ with the following properties:

(i) $\|[h, \iota_n(x)]\| < \epsilon$ *for all* $x \in X$*;*

(ii) $h\iota_n(x) \in_\epsilon C$, $(1-h)\iota_n(x) \in_\epsilon D$, *and* $(1-h)h\iota_n(x) \in_\epsilon C \cap D$ *for all* $x \in X$*;*

(iii) $E := C \cap D$ *is an AF C^*-subalgebra of* $qM_n(A \rtimes \mathbb{Z})q$*;*

(iv) h *multiplies* E *into itself.*

[8]See line (15) for $\iota_n(u)$ written out as a matrix.

Proof. Define a unitary $v \in M_n(\mathbb{C}) \otimes M(A \rtimes \mathbb{Z}) = M(M_n(A \rtimes \mathbb{Z}))$ by

$$v := d_1 \otimes u^{-1} + d_{-(n-1)} \otimes u^{n-1} = \begin{pmatrix} 0 & \dots & \dots & 0 & u^{n-1} \\ u^{-1} & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^{-1} & 0 \end{pmatrix}.$$

Write $n = 2m$ if n is even and $n = 2m + 1$ if n is odd, and note that

$$v^m = d_m \otimes u^{-m} + d_{-(n-m)} \otimes u^{n-m}. \tag{16}$$

Let $j_n : M_n(A) \hookrightarrow M_n(A \rtimes_\alpha \mathbb{Z})$ denote the canonical inclusion and define two $*$-homomorphisms

$$\Lambda_n^0, \Lambda_n^1 : M_n(A) \to M_n(A \rtimes_\alpha \mathbb{Z}), \quad \Lambda_n^0 := j_n, \quad \Lambda_n^1 := Ad_{v^m} \circ j_n.$$

Let us compute the image of Λ_n^1 more concretely. Write elements in $M_n(A)$ in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{(n-m)+m}(A) \tag{17}$$

where writing n as the sum of $n-m$ and m in the subscript on the right records the sizes of the blocks. Using line (16), one then computes that Λ_n^1 acts via sending the matrix in line (17) to the element

$$\begin{pmatrix} \alpha^{n-m}(d) & 0 \\ 0 & \alpha^{-m}(a) \end{pmatrix} + \begin{pmatrix} 0 & \alpha^{n-m}(c) \\ 0 & 0 \end{pmatrix} \cdot u^n + \begin{pmatrix} 0 & 0 \\ \alpha^{-m}(b) & 0 \end{pmatrix} \cdot u^{-n} \tag{18}$$

in $M_{m+(n-m)}(A \rtimes \mathbb{Z})$ (note the switch from "$(n-m)+m$" to "$m+(n-m)$"). Define also

$$q := \iota_n(p) = \mathrm{diag}(\alpha^{-1}(p), ..., \alpha^{-n}(p)).$$

One checks directly that q multiplies $\Lambda_n^0(M_n(A))$ into itself, while the fact that q multiplies $\Lambda_n^1(M_n(A))$ into itself follows from the formula in line (18) above. Hence $C := q(\Lambda_n^0(M_n(A)))q$ and $D := q(\Lambda_n^1(M_n(A)))q$ are well-defined AF subalgebras of $M_n(A \rtimes \mathbb{Z})$. Note moreover that with respect to the decomposition in line (17), the intersection of C and D can be concretely described as the set

$$E := \left\{ q \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} q \;\middle|\; a \in M_m(A),\ d \in M_{n-m}(A) \right\}$$

and is in particular also an AF algebra.

For $1 \leq i \leq n$, define scalars $h_i \in [0,1]$ by

$$h_i := \begin{cases} 0 & 1 \leq i \leq \left\lfloor \frac{n}{6} \right\rfloor \\ \dfrac{i - \left\lfloor \frac{n}{6} \right\rfloor}{\left\lfloor \frac{2n}{6} \right\rfloor - \left\lfloor \frac{n}{6} \right\rfloor} & \left\lfloor \frac{n}{6} \right\rfloor \leq i \leq \left\lfloor \frac{2n}{6} \right\rfloor \\ 1 & \left\lfloor \frac{2n}{6} \right\rfloor \leq i \leq \left\lfloor \frac{4n}{6} \right\rfloor \\ \dfrac{\left\lfloor \frac{5n}{6} \right\rfloor - i}{\left\lfloor \frac{5n}{6} \right\rfloor - \left\lfloor \frac{4n}{6} \right\rfloor} & \left\lfloor \frac{4n}{6} \right\rfloor \leq i \leq \left\lfloor \frac{5n}{6} \right\rfloor \\ 0 & \left\lfloor \frac{5n}{6} \right\rfloor \leq i \leq n \end{cases} \tag{19}$$

and let $h \in M_n(A)$ be defined by

$$h := \mathrm{diag}(h_1, ..., h_n)q = \mathrm{diag}(h_1\alpha^{-1}(p), ..., h_n\alpha^{-n}(p)).$$

Note that h multiplies C and D into themselves, whence it also multiplies E into itself.

We claim now that for n suitably large, C, D, E, and h have the properties claimed in the statement of the lemma. We have already observed properties (iii) and (iv), so it remains to check properties (i) and (ii).

Let us look first at property (i). As $q\iota_n(x) = \iota_n(px) = \iota_n(x) = \iota_n(xp) = \iota_n(x)q$ for all $x \in X$, and as q commutes with

$$h^{(0)} := \mathrm{diag}(h_1, ..., h_n) \tag{20}$$

it suffices to show that

$$[h^{(0)}, \iota_n(x)] \to 0 \quad \text{as} \quad n \to \infty. \tag{21}$$

For this, we may assume that every element of X is a contraction of the form au^k for some $a \in A$ and $k \in \mathbb{N} \cup \{0\}$. Let K be the maximal such k appearing in an exponent for some $x \in X$.

Fix then $x = a \cdot u^k \in X$ and compute $h^{(0)} \cdot \iota_n(x)$:

$$\begin{aligned} h^{(0)} \cdot \iota_n(x) &= h \cdot \iota_n(a \cdot u^k) \\ &= h^{(0)} \cdot \iota_n(a)\iota_n(u^k) \\ &= h^{(0)}\mathrm{diag}(\alpha^{-1}(a), ..., \alpha^{-n}(a))\big(d_k \otimes 1_n + d_{-n+k} \otimes u^n\big) \\ &= \begin{pmatrix} 0 & 0 \\ g_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & g_2 \\ 0 & 0 \end{pmatrix} \cdot u^n \end{aligned} \tag{22}$$

where g_1 is the $(n-k) \times (n-k)$ matrix

$$g_1 := \mathrm{diag}(h_{k+1}\alpha^{-(k+1)}(a), h_{k+2}\alpha^{-(k+2)}(a), ..., h_n\alpha^{-n}(a)) \tag{23}$$

and g_2 is the $k \times k$ matrix

$$g_2 = \mathrm{diag}(h_1\alpha^{-1}(a), ..., h_k\alpha^{-k}(a)). \tag{24}$$

If we choose n large enough so that $k \leq K << \lfloor \frac{n}{6} \rfloor$, then $\left(\begin{smallmatrix} 0 & g_2 \\ 0 & 0 \end{smallmatrix}\right) \cdot u^n = 0$, since $h_i = 0$ for $1 \leq i \leq \lfloor \frac{n}{6} \rfloor$. Hence for large enough n

$$h^{(0)} \cdot \iota_n(x) = \begin{pmatrix} 0 & 0 \\ g_1 & 0 \end{pmatrix}. \tag{25}$$

Computing $\iota_n(x) \cdot h^{(0)}$ is similar: if again, n is large enough so that $k \leq K << \lfloor \frac{n}{6} \rfloor$, we have

$$\iota_n(x) \cdot h^{(0)} = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix},$$

where e is the $(n-k) \times (n-k)$ matrix defined by

$$e := \mathrm{diag}(h_1\alpha^{-(k+1)}(a), h_2\alpha^{-(k+2)}(a), ..., h_{n-k}\alpha^{-n}(a)).$$

At this point, if we let $M_n := \max\{\lfloor \frac{2n}{6} \rfloor - \lfloor \frac{n}{6} \rfloor, \lfloor \frac{5n}{6} \rfloor - \lfloor \frac{4n}{6} \rfloor\}$, then we compute that for all large enough n,

$$\begin{aligned} \|h^{(0)} \cdot \iota_n(x) - \iota_n(x) \cdot h^{(0)}\| &= \left\| \begin{pmatrix} 0 & 0 \\ g_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \right\| \\ &= \|g_1 - e\| \\ &= \max_{k+1 \leq i \leq n} \left\| h_{i-k}\alpha^{-i}(a) - h_i\alpha^{-i}(a) \right\| \\ &\leq \max_{k+1 \leq i \leq n} \left\| h_{i-k} - h_i \right\| \left\| \alpha^{-i}(a) \right\| \\ &\leq \frac{K}{M_n} \to 0 \text{ as } n \to \infty, \end{aligned}$$

which completes the proof of condition (i).

We now look at condition (ii). Define $C^{(0)} := \Lambda_n^0(M_n(A))$, $D^{(0)} := \Lambda_n^1(M_n(A))$ and $E^{(0)} := C^{(0)} \cap D^{(0)}$. Then with $h^{(0)}$ as in (20), it suffices to show that for n suitably large and any $x \in X$, $h^{(0)}x \in_\epsilon C^{(0)}$, $(1 - h^{(0)})x \in_\epsilon D^{(0)}$, and that $h^{(0)}(1 - h^{(0)})x \in_\epsilon E^{(0)}$. We may again assume that every element of X is a contraction of the form au^k for some $a \in A$ and $k \in \mathbb{N}$, and let K be the maximal such k appearing.

First, note that it follows from the computation of $h^{(0)} \cdot \iota_n(x)$ in line (25) that for any x, $h^{(0)} \cdot \iota_n(x) \in C^{(0)}$. To see that $(1 - h^{(0)}) \cdot \iota_n(x) \in D_n$, analogously to lines (22) and (23) and (24) above, we compute that

$$(1 - h^{(0)}) \cdot \iota_n(x) = \begin{pmatrix} 0 & 0 \\ f_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & f_2 \\ 0 & 0 \end{pmatrix} \cdot u^n \tag{26}$$

where f_1 is the $(n-k) \times (n-k)$ matrix given by

$$f_1 = \operatorname{diag}((1-h_{k+1})\alpha^{-(k+1)}(a), (1-h_{k+2})\alpha^{-(k+2)}(a), ..., (1-h_n)\alpha^{-n}(a))$$

and as long as n is chosen large enough so that $k \leq K << \lfloor \frac{n}{6} \rfloor$, f_2 is the $k \times k$ matrix given by

$$f_2 := \operatorname{diag}(\alpha^{-(k+1)}(a), ..., \alpha^{-n}(a)).$$

Note that $(1-h_i) = 0$ for $\lfloor \frac{2n}{6} \rfloor + 1 \leq i \leq \lfloor \frac{4n}{6} \rfloor$. Thus the matrix $\left(\begin{smallmatrix} 0 & 0 \\ f_1 & 0 \end{smallmatrix}\right)$ can be written as the following sum

$$\begin{aligned}
\begin{pmatrix} 0 & 0 \\ f_1 & 0 \end{pmatrix} &= \Big(d_k \cdot \operatorname{diag}\big((1-h_{k+1})\alpha^{-(k+1)}(a), ..., (1-h_{\lfloor \frac{n}{3} \rfloor})\alpha^{-\lfloor \frac{n}{3} \rfloor}(a), 0, ..., 0\big)\Big) \\
&+ \Big(d_k \cdot \operatorname{diag}\big(0, ..., 0, (1-h_{\lfloor \frac{2n}{3} \rfloor})\alpha^{-\lfloor \frac{2n}{3} \rfloor}(a), ..., (1-h_n)\alpha^{-n}(a)\big)\Big) \\
&= \begin{pmatrix} f_3 & 0 \\ 0 & f_4 \end{pmatrix},
\end{aligned}$$

where f_3 is an $m \times m$ matrix built from the entries of the first summand in the middle line above, and f_4 is an $(m+1) \times (m+1)$ matrix built from the entries in the second summand in the middle above. Comparing this with line (26) above, we see that $(1-h^{(0)}) \cdot \iota_n(x) \in D$.

Finally, we consider E. For any $x \in X$ (assumed as usual to be of the special form au^k), we already have that

$$(1-h^{(0)}) \cdot \iota_n(x) = \begin{pmatrix} f_3 & 0 \\ 0 & f_4 \end{pmatrix} + \begin{pmatrix} 0 & f_2 \\ 0 & 0 \end{pmatrix} \cdot u^n.$$

Multiplying by $h^{(0)}$ on the left will make the second term zero as $h_i = 0$ for $1 \leq i \leq \lfloor \frac{n}{6} \rfloor$ and n has been chosen so that $k \leq K << \lfloor \frac{n}{6} \rfloor$. Thus $h^{(0)}(1-h^{(0)}) \cdot \iota_n(x) \in E$ and we are done. ☐

Corollary 5.5. *Let A be an AF C^*-algebra equipped with a $\mathbb{Z}$-action. Let X be a finite subset of $A \rtimes \mathbb{Z}$, and assume there exists a projection $p \in A$ such that $px = xp = x$ for all $x \in X$. Let $\epsilon > 0$. For each $n \in \mathbb{N}$, define*

$$\phi_n : M_n(A \rtimes \mathbb{Z}) \oplus M_{n+1}(A \rtimes \mathbb{Z}) \to M_{2n+1}(A \rtimes \mathbb{Z}), \quad (a,b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

let $\omega : M_n(A \rtimes \mathbb{Z}) \to M_n(A \rtimes \mathbb{Z})$ be any $*$-isomorphism, let ι_n be as in Definition 5.3, and define

$$\kappa_n := \phi_n \circ ((\omega \circ \iota_n) \oplus \iota_{n+1}) : A \rtimes \mathbb{Z} \to M_{2n+1}(A \rtimes \mathbb{Z}).$$

Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, if $q := \kappa_n(p)$ there is a positive contraction $h \in q(M_{2n+1}(A \rtimes \mathbb{Z}))q$ and AF C^*-subalgebras C and D of $q(M_{2n+1}(A \rtimes \mathbb{Z}))q$ with the following properties:

(i) $\|[h, \kappa_n(x)]\| < \epsilon$ *for all* $x \in X$*;*

(ii) $h\kappa_n(x) \in_\epsilon C$, $(1-h)\kappa_n(x) \in_\epsilon D$, *and* $(1-h)h\kappa_n(x) \in_\epsilon C \cap D$ *for all* $x \in X$*;*

(iii) $E := C \cap D$ *is an AF algebra;*

(iv) h *multiplies* E *into itself.*

Proof. Let N be large enough so that the conclusion of Lemma 5.4 holds for all $n \geq N$ with respect to the given X, ϵ and p. Fix $n \geq N$. Let C_n, D_n be subalgebras of $\iota_n(p)(M_n(A \rtimes \mathbb{Z}))\iota_n(p)$ and h_n a positive contraction in $\iota_n(p)(M_n(A \rtimes \mathbb{Z}))\iota_n(p)$ with the properties in Lemma 5.4 and similarly for C_{n+1}, D_{n+1} and h_{n+1} with respect to $\iota_{n+1}(p)(M_{n+1}(A \rtimes \mathbb{Z}))\iota_{n+1}(p)$.

Define

$$C := \phi_n(\omega(C_n) \oplus C_{n+1}), \quad D := \phi_n(\omega(D_n) \oplus D_{n+1})$$

and

$$h := \phi_n(\omega(h_n) \oplus h_{n+1}).$$

Direct checks show these elements have the right properties. □

Enders computes the effect of ι_n on K-theory: the following result is a special case of [14, Proposition 3.2].

Lemma 5.6. *Let A be a C^*-algebra with $K_1(A) = 0$, and equipped with an action of $\mathbb{Z}$. Let $\iota_n : A \rtimes \mathbb{Z} \to M_n(A \rtimes \mathbb{Z})$ be as in Definition 5.3 and let $i_n : A \rtimes \mathbb{Z} \to M_n(A \rtimes \mathbb{Z})$ be the standard top-left-corner inclusion. Then as maps on K-theory, $(\iota_n)_* = n \cdot (i_n)_*$.* □

For the next step, we need to use part of the Kirchberg-Phillips classification theorem. For the reader's convenience, we state the versions of the Kirchberg-Phillips theorem we will use, and how to deduce them from the literature.

Theorem 5.7 (Kirchberg-Phillips)**.** *(i) Let A and B be stable Kirchberg algebras. Then for any invertible element of x of $KK(A, B)$, there exists a $*$-isomorphism $\phi : A \to B$ such that $[\phi] = x$.*

(ii) Let A and B be stable UCT Kirchberg algebras. Then for any graded isomorphism $\alpha : K_*(A) \to K_*(B)$, there exists a $*$-isomorphism $\phi : A \to B$ that induces α.

(iii) Let A and B be stable UCT Kirchberg algebras, and let $\phi, \psi : A \to B$ be $*$-isomorphisms that induce the same class in $KK(A, B)$. Then there is a sequence of unitaries (u_n) in the multiplier algebra of B such that $u_n \phi(a) u_n^* \to \psi(a)$ as $n \to \infty$ for all $a \in A$.

Proof. Parts (i) and (ii) are exactly [25, Theorem 8.4.1, (i) and (ii)]. Part (iii) can be deduced from [25, Theorem 8.2.1 (ii)][9]. □

The next result again follows Enders' work: the proof proceeds along similar lines to [14, Proof of Theorem 4.1]

Corollary 5.8. *Let A be an AF algebra equipped with an action of $\mathbb{Z}$ so that the associated crossed product $A \rtimes \mathbb{Z}$ is a Kirchberg algebra*[10]. *Let X be a finite subset of $A \rtimes \mathbb{Z}$, and assume there exists a projection $p \in A$ such that $px = xp = x$ for all $x \in X$. Then for any $\epsilon > 0$ there exist AF C^*-subalgebras C and D of $p(A \rtimes \mathbb{Z})p$ and a positive contraction $h \in p(A \rtimes \mathbb{Z})p$ such that the following hold:*

(i) for all $x \in X$, $\|[h, x]\| < \epsilon$;

(ii) for all $x \in X$, $hx \in_\epsilon C$, $(1-h)x \in_\epsilon D$, $h(1-h)x \in_\epsilon C \cap D$;

(iii) $E := C \cap D$ is an AF algebra;

(iv) h multiplies E into itself.

Proof. We first follow the argument of [14, Theorem 4.1]. Let N be large enough so that the conclusion of Corollary 5.5 holds for the given X and p, and parameter $\epsilon/2$, and fix any $n \geq N$.

Note first that as $M_n(A \rtimes \mathbb{Z})$ is a stable UCT Kirchberg algebra, Theorem 5.7, part (ii) implies there is a $*$-isomorphism $\omega : M_n(A \rtimes \mathbb{Z}) \to M_n(A \rtimes \mathbb{Z})$ such that the map $\omega_* : K_*(M_n(A \rtimes \mathbb{Z})) \to K_*(M_n(A \rtimes \mathbb{Z}))$ induced by ω is multiplication by -1 in both even and odd degrees.

Let now κ_n be as in Corollary 5.5, built using this ω. From Lemma 5.6, the map induced by κ_n on K-theory is the same as the canonical top-left corner inclusion $i_{2n+1} : A \rtimes \mathbb{Z} \to M_{2n+1}(A \rtimes \mathbb{Z})$, and in particular is an isomorphism on K-theory. Hence by the UCT (see [27, Proposition 7.5] for the precise consequence of the UCT being used here), κ_n is invertible in $KK(A \rtimes \mathbb{Z}, M_{2n+1}(A \rtimes \mathbb{Z}))$. Theorem 5.7,

[9] The references we give here are to a readable textbook exposition that explains the ideas, but does not quite contain complete proofs. For references with proofs that one can deduce the results from, see [22, Theorem 4.2.1] or [15, Theorem A] for parts (i) and (ii), and [22, Theorem 4.1.3] or [15, Theorem B] for part (iii).

[10] This assumption forces A to be non-unital, whence $A \rtimes \mathbb{Z}$ is stable by Zhang's dichotomy: see [25, Proposition 4.1.3], or [38, Theorem 1.2] for the original reference.

part (i) thus gives a $*$-isomorphism $\phi_n : A \rtimes \mathbb{Z} \to M_{2n+1}(A \rtimes \mathbb{Z})$ whose class in $KK(M_{2n+1}(A \rtimes \mathbb{Z}), A \rtimes Z)$ is the inverse of the class of κ_n.

The fact that $\psi_n \circ \kappa_n$ equals the class of the identity in $KK(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$ and Theorem 5.7 part (iii) implies that there is a sequence $(u_m)_{m=1}^{\infty}$ of unitaries in the multiplier algebra of $A \rtimes \mathbb{Z}$ such that $u_m(\psi_n \kappa_n(a))u_m^* \to a$ as $n \to \infty$ for all $a \in A \rtimes \mathbb{Z}$.

Now, let $q := \kappa_n(p)$, and let $h_n \in q(M_{2n+1}(A \rtimes \mathbb{Z}))q$, $qC_nq, qD_nq \subseteq M_{2n+1}(A \rtimes \mathbb{Z})$ be as Corollary 5.5. Define $p_m := u_m \psi_n(q) u_m^*$ which is a projection in $A \rtimes \mathbb{Z}$ such that $p_m \to p$ as $m \to \infty$. Hence by Lemma 2.10 (applied to the multiplier algebra $M(A \rtimes \mathbb{Z})$ of $A \rtimes \mathbb{Z}$) for all suitably large m there is a unitary $v_m \in M(A \rtimes \mathbb{Z})$ such that $v_m p_m v_m^* = p$, and such that $v_m \to 1_{M(A \rtimes \mathbb{Z})}$ as $m \to \infty$.

Direct checks now show that for sufficiently large m, the element $h := v_m u_m \psi_n(h_n) u_m^* v_m^*$ and C^*-subalgebras $C := v_m u_m \psi_n(C_n) u_m^* v_m^*$, and $D := v_m u_m \psi_n(D_n) u_m^* v_m^*$ have the properties in the statement. $\square$

The next corollary follows directly from the above and the definition of complexity rank (see Definition 2.3 above).

Corollary 5.9. *Let A be an AF algebra equipped with an action of $\mathbb{Z}$ so that the associated crossed product $A \rtimes \mathbb{Z}$ is a Kirchberg algebra, and let $p \in A \subseteq A \rtimes \mathbb{Z}$ be a projection. Then $p(A \rtimes \mathbb{Z})p$ has complexity rank at most one.* $\square$

We are finally ready to complete the proof of Theorem 5.2. We will use corner endomorphisms and the associated crossed products by $\mathbb{N}$: see [24, Section 2] for background on this.

Proof of Theorem 5.2. Let B be a unital Kirchberg algebra that satisfies the UCT. Using [24, Theorem 3.6] there is a simple, unital AF algebra A_0 with unique trace and a proper corner endomorphism ρ of A_0 such that the associated crossed product $A_0 \rtimes \mathbb{N}$ is a UCT Kirchberg algebra with the same K-theory invariant as B. Hence by the Kirchberg-Phillips classification theorem (see for example [25, Theorem 8.4.1] for an appropriate version) B is isomorphic to $A_0 \rtimes \mathbb{N}$. Hence it suffices to prove that $A_0 \rtimes \mathbb{N}$ has complexity rank at most one.

Define now A to be the direct limit of the sequence

$$A_0 \xrightarrow{\rho} A_0 \xrightarrow{\rho} A_0 \xrightarrow{\rho} \cdots .$$

Then A is a direct limit of AF algebras so itself an AF algebra, and as discussed in [25, pages 75-76, and also pages 72-73], A is equipped with a $\mathbb{Z}$-action and a projection $p \in A$ such that $p(A \rtimes \mathbb{Z})p \cong A_0 \rtimes \mathbb{N}$. Thanks to Corollary 5.9, we are done. $\square$

5.2 The general case

In this subsection, we finish the proof of Theorem 5.1 by computing the complexity rank of general unital UCT Kirchberg algebras. We will need existence of a good class of "models", i.e. a collection of C^*-algebras with well-understood structure so that every UCT Kirchberg algebra is isomorphic to one in the collection. Our models will be built from Cuntz algebras, and one other Kirchberg algebra with special K-theory. We need some notation. For $n \in \{2, 3, 4, ...\} \cup \{\infty\}$, we let $\mathcal{O}_n$ denote the Cuntz algebra. We also let $\mathcal{O}_{1,\infty}$ be a unital UCT Kirchberg algebra with $K_0(\mathcal{O}_{1,\infty}) = 0$ and $K_1(\mathcal{O}_{1,\infty}) = \mathbb{Z}$; such exists by [25, Proposition 4.3.3] (and is unique up to isomorphism by the Kirchberg-Phillips classification theorem).

The next proposition gives the models we will use. Variants of this are very well-known: see for example [25, Proposition 8.4.11] (and the erratum on the author's webpage).

Proposition 5.10. *Any unital Kirchberg algebra in the UCT class can be written as an inductive limit of C^*-algebras of the form*

$$B_0 \oplus (B_1 \otimes \mathcal{O}_{1,\infty}), \tag{27}$$

where B_0 and B_1 are both of the form

$$\bigoplus_{j=1}^{N} M_{n_j}(\mathcal{O}_{m_j})$$

with $N \in \mathbb{N}$, each $n_j \in \mathbb{N}$, and each $m_j \in \{2, 3, ...\} \cup \{\infty\}$.

To establish this, we will need another well-known variant of the Kirchberg-Phillips classification theorem, due to Kirchberg[11]. For the statement, recall that a $*$-homomorphism $\phi : A \to B$ is *full* if for any non-zero $a \in A$, $\phi(a)$ generates B as a two-sided ideal.

Theorem 5.11 (Kirchberg)**.** *Let A be a separable, nuclear, unital C^*-algebra that satisfies the UCT, and let B be a unital, properly infinite C^*-algebra. Then for any (graded) homomorphism $\alpha : K_*(A) \to K_*(B)$ such that $\alpha[1_A] = [1_B]$ there exists a full, unital $*$-homomorphism $\phi : A \to B$ inducing α.*

[11] As far as we are aware, Kirchberg's proof has not been published: the reader can consult [15, Theorem A] for a proof (which is independent of Kirchberg's).

Proof. Let A be a separable, nuclear, unital C^*-algebra, and let B be a unital, properly infinite C^*-algebra. Then [15, Theorem A] implies that for any $x \in KK(A,B)$ such that the map $x_* : K_0(A) \to K_0(B)$ induced on K-theory takes $[1_A]$ to $[1_B]$, there exists a full unital $*$-homomorphism $\phi : A \to B$ such that the class $[\phi]$ in $KK(A,B)$ equals x: precisely, the given reference has strictly weaker assumptions on A and B (in particular, only that A is exact), and works with classes in $KK_{nuc}(A,B)$ rather than $KK(A,B)$. However, we assume above that A is nuclear, which implies that $KK_{nuc}(A,B) = KK(A,B)$.

On the other hand, as A satisfies the UCT, the canonical map

$$KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$$

is surjective. The result follows from this and the comments above on lifting α to some $x \in KK(A,B)$. $\square$

Proof of Proposition 5.10. Let $(K_0(A), [1_A], K_1(A))$ be the K-theory invariant of A. Choose a sequence $(G_{n,0}, G_{n,1})$ such that $G_{n,0}$ is a finitely generated subgroup of $K_0(A)$ containing $[1_A]$, $G_{n,1}$ is a finitely generated subgroup of $K_1(A)$, and such that $K_i(A) = \bigcup_{n\in\mathbb{N}} G_{n,i}$ for $i \in \{0,1\}$. Using the Künneth formula (see [28, page 443] or [3, Theorem 23.1.3]) and the well-known K-theory of the Cuntz algebras (see for example [25, page 74]) it is straightforward to see that for each n, there is a C^*-algebra C_n of the form in line (27) such that $(K_0(C_n), [1_{C_n}], K_1(C_n)) \cong (G_{0,n}, [1_A], G_{1,n})$. Identifying these groups via a fixed isomorphism, Corollary 5.11 implies that for each n the inclusion map

$$(G_{0,n}, [1_A], G_{1,n}) \to (G_{0,n+1}, [1_A], G_{1,n+1})$$

is induced by a full unital $*$-homomorphism $\phi_n : C_n \to C_{n+1}$. We claim that A is isomorphic to the inductive limit C of the system (C_n, ϕ_n). Indeed, first note that C is unital, and that by continuity of K-theory, $(K_0(C), [1_C], K_1(C)) \cong (K_0(A), [1_A], K_1(A))$. As each ϕ_n is unital and full, C is unital and simple. As each C_n is nuclear, C is nuclear. As each C_n is a finite direct sum of purely infinite C^*-algebras, C is purely infinite: it is straightforward to check this using the condition in [24, Proposition 4.1.8 (iv)], for example. Hence by the Kirchberg-Phillips classification theorem (for example, [22, Theorem 4.3.4]), A is isomorphic to C as claimed. $\square$

Theorem 5.12. *Any unital UCT Kirchberg algebra A has complexity rank at most two.*

Proof. Proposition 5.10 writes A as an inductive limit of C^*-algebras of the form $B_0 \oplus B_1 \otimes \mathcal{O}_{1,\infty}$ with B_0 and B_1 a finite direct sum of matrix algebras over Cuntz algebras. Using Theorem 5.2, any unital UCT Kirchberg algebra with torsion free K_1-group has complexity rank one. Using this and Lemma 2.5, each of B_0, B_1 and $\mathcal{O}_{1,\infty}$ has complexity rank at most one. Hence Proposition 2.22 implies that $B_1 \otimes \mathcal{O}_{1,\infty}$ has complexity rank at most two, and thus so does $B_0 \oplus B_1 \otimes \mathcal{O}_{1,\infty}$ using Lemma 2.5 again. As complexity rank is non-increasing under taking inductive limits (Lemma 2.6), the complexity rank of A is at most 2. □

We finish this section by recording a proof of Theorem 5.1.

Proof of Theorem 5.1. Let A be a unital UCT Kirchberg algebra. Then A has complexity rank at most two by Theorem 5.12. As A is not locally finite-dimensional, it does not have complexity rank zero.

If A has complexity rank one, then it has torsion-free K_1-group by Theorem 4.1. Conversely, if A has torsion-free K_1 group, then it has complexity rank one by Theorem 5.2. □

6 Questions

We conclude the paper with some open questions that seem interesting to us.

The first question is important (and probably difficult) as it is equivalent to the UCT for all nuclear C^*-algebras.

Question 6.1. Do all (unital) Kirchberg algebras have finite complexity?

Even knowing finite complexity for Kirchberg algebras with trivial K-theory would imply the UCT for all nuclear C^*-algebras.

The next question is about the most interesting example that we do not currently know the complexity rank of.

Question 6.2. What is the complexity rank of an irrational rotation algebra?

We conjecture the answer is always one; more generally, we conjecture that the complexity rank of a separable $A\mathbb{T}$-algebra of real rank zero (and which is not AF) is always one.

Question 6.3. What is the complexity rank of (classifiable) AH (or even ASH) algebras of real rank zero?

It would also be interesting to give non-trivial upper bounds, maybe in terms of the dimensions of the spectra of (sub)homogeneous algebras appearing in a directed system for the given A(S)H algebra.

The following question is very natural. We know too little to hazard a reasonable guess at the moment.

Question 6.4. Which ordinal numbers can be the complexity rank of a C^*-algebra?

We did not seriously attempt to address this question, but at the moment, the only values we know can be taken are 0, 1, and 2. It is conceivable that for the uniform Roe algebras $C_u^*(X)$ associated to a space X, the complexity rank of $C_u^*(X)$ and the complexity rank of X in the sense of [17, Definition 2.9] (at present we know only that the complexity rank of the C^*-algebra is bounded above by that of the space). If these ranks were equal, it would follow for example from [17, Sections 4 and 5] and [11] that many complexity ranks are possible for C^*-algebras.

Question 6.5. Does (weak) complexity rank at most one imply real rank zero in general?

There are some interesting connections of this question to other problems: compare Remark 3.15 above. The question seems more likely to have a positive answer in the simple case, which would also be interesting.

The following question seems basic (we tried to find an answer and were not able to).

Question 6.6. Does having complexity rank at most α pass to corners?

This would be interesting to know even for $\alpha = 1$. The answer is 'yes' for *weak* complexity rank at most one: one can see this by adapting the proof of [19, Proposition 3.8], for example.

Our last question is a little vague, but would be useful to have, particularly with regards permanence properties.

Question 6.7. Is there a 'good' definition of decomposability in the non-unital case?

Many of the results in this paper have reasonably natural variants in the non-unital case, but we were not able to come up with a really clean and natural definition, so in the end opted to write the paper entirely in the unital setting for the sake of simplicity. Certainly having a notion that applied equally in the unital case would be very interesting, however.

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