Cartan subalgebras in uniform Roe algebras

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Abstract
Cartan subalgebras of $C^*$-algebras were introduced by Renault. They are of interest as a Cartan subalgebra $B \subseteq A$ gives rise to a (twisted) groupoid $G$ such that $A$ is the associated twisted groupoid $C^*$-algebra; thus Cartan subalgebras give a $C^*$-algebraic way of encoding when a $C^*$-algebra comes from a dynamical system, in some sense. On the other hand, the uniform Roe algebra $C_u^*(X)$ is a $C^*$-algebra associated to a discrete metric space $X$ that captures aspects of the large scale geometry of $X$. A uniform Roe algebra $C_u^*(X)$ contains a ‘diagonal’ copy of $\ell^\infty(X)$ as a Cartan subalgebra; this corresponds to the description of $C_u^*(X)$ as a groupoid $C^*$-algebra due to Skandalis, Tu, and Yu.

In this paper, we study general Cartan subalgebras of uniform Roe algebras. We first show that the structure of Cartan subalgebras in uniform Roe algebras is quite restricted, but that ‘exotic’ examples exist, even in simple situations. We then give an abstract characterization of when an inclusion $B \subseteq A$ of a Cartan subalgebra is isomorphic to the canonical inclusion $\ell^\infty(X) \subseteq C_u^*(X)$ coming from a uniform Roe algebra. Our main result, which uses the above as ingredients, shows that is $X$ is a space with finite decomposition complexity in the sense of Guentner, Tessera, and Yu, and $B$ is a Cartan subalgebra of $C_u^*(X)$ that is $*$-isomorphic to $\ell^\infty(X)$ and satisfies an additional separability condition, then there is a unitary $u \in C_u^*(X)$ such that $uBu^* = \ell^\infty(X)$.

1 Introduction
The aim of this paper is to study Cartan subalgebras in uniform Roe algebras, and in particular to what extent the ‘standard’ Cartan sub-
algebra is unique. We first recall the definitions of bounded geometry metric spaces and the associated uniform Roe algebras.

**Definition 1.1.** A metric space $X$ has **bounded geometry** if for all $r > 0$ there is $n_r \in \mathbb{N}$ such that all balls in $X$ of radius $r$ have at most $n_r$ elements.

A good example to bear in mind is $\mathbb{Z}$ equipped with its standard metric, or more generally a finitely generated group equipped with a choice of word metric.

Uniform Roe algebras as in the following definition were originally introduced for their connections to (higher) index theory and the associated applications to manifold topology and geometry. They have since been fairly extensively studied for their own sake, and provide an interesting bridge between coarse geometry and $C^*$-algebra theory.

**Definition 1.2.** Let $X$ be a bounded geometry metric space, and let $a$ be a bounded operator on $\ell^2(X)$, which we think of as an $X$-by-$X$ matrix $a_{x,y}$. The **propagation** of $a$ is

$$\text{prop}(a) := \sup \{ d(x,y) \mid a_{x,y} \neq 0 \} \in [0, \infty].$$

Let $C_u[X]$ denote the collection of bounded operators on $\ell^2(X)$ with finite propagation; this is a $*$-algebra. The **uniform Roe algebra** of $X$, denoted $C^*_u(X)$, is the closure of $C_u[X]$ for the operator norm.

As a special case, note that if $X$ is a finitely generated group $\Gamma$ equipped with some word metric, then $C^*_u(X)$ is naturally $*$-isomorphic to $\ell^\infty(\Gamma) \rtimes_r \Gamma$; this is proved for example in [3, Proposition 5.1.3].

The uniform Roe algebra of a bounded geometry metric space always contains two subalgebras that will play a crucial role in this paper:

(i) the compact operators $\mathcal{K}(\ell^2(X))$, as an essential ideal (note that $\mathcal{K}(\ell^2(X))$ is also the unique minimal $C^*$-ideal);

(ii) the subalgebra $\ell^\infty(X)$ of multiplication operators, which is a ‘Cartan subalgebra’ in the sense of Renault [18, Definition 5.1], as in the next definition.

**Definition 1.3.** Let $A$ be a $C^*$-algebra. A **Cartan subalgebra** of $A$ is a $C^*$-subalgebra $B \subseteq A$ such that:

(i) $B$ is a maximal abelian self-adjoint subalgebra (MASA) of $A$;
(ii) $B$ contains an approximate unit for $A$;

(iii) the normalizer of $B$ in $A$, defined as

$$N_A(B) := \{ a \in A \mid aBa^* \cup a^*Ba \subseteq B \}$$

generates $A$ as a C*-algebra;

(iv) there is a faithful conditional expectation $E : A \to B$.

A Cartan pair is a nested pair $B \subseteq A$ of C*-algebras such that $B$ is a Cartan subalgebra of $A$.

This is a generalization of a well-studied notion in the theory of von Neumann algebras [9]: the main interest of the C*-algebra notion is due to its close connection with the theory of étale groupoids and their C*-algebras as shown in [18]. See [15] for a recent collection of results on existence and uniqueness of Cartan subalgebras in C*-algebras; without going into details here, we note that Cartan subalgebras of C*-algebras are very rarely unique in any reasonable sense. This contrasts to the von Neumann algebra case, where (for example) classical results of Feldman and Moore give uniqueness of the Cartan subalgebra in the hyperfinite $II_1$ factor up to automorphism [10]; and much more recent work starting with Ozawa-Popa [17] gives uniqueness of Cartan subalgebras in certain $II_1$ factors up to inner automorphisms.

As already mentioned above, the ‘diagonal’ copy of $\ell^2(X)$ that acts on $\ell^2(X)$ by multiplication operators is always a Cartan subalgebra inside the uniform Roe algebra $C^*_u(X)$ (for a proof of this, see Proposition 3.11 below). Our aim in this paper is to study the following questions.

- What form can Cartan subalgebras in $C^*_u(X)$ can take? – this could mean what isomorphism type as an abstract C*-algebra, or it could mean the more refined spatial theory of how a Cartan subalgebra can be represented on $\ell^2(X)$.
- When does an abstract Cartan pair $B \subseteq A$ come from a uniform Roe algebra?
- Is the canonical Cartan subalgebra $\ell^2(X) \subseteq C^*_u(X)$ unique? – here uniqueness might mean up to automorphism of $C^*_u(X)$, or more strongly up to inner automorphism of $C^*_u(X)$, and might refer to uniqueness among some class of Cartan subalgebras satisfying additional conditions.

1We will mainly be interested in the case that $A$ is unital, in which case condition (ii) is automatic: indeed condition (i) implies that $B$ contains the unit of $A$. 

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We are able to give at least partial answers to these questions; we now discuss each in turn.

**What form can Cartan subalgebras in $C^*_u(X)$ can take?**

We start with a fairly general result about Cartan subalgebras in $C^*$-algebras that contain the compact operators. To state it, for a subset $S$ of the bounded operators $\mathcal{B}(H)$ on a Hilbert space, let $C^*(S)$ (respectively $vN(S)$) be the $C^*$-algebra (respectively von Neumann algebra) generated by $S$.

**Theorem 1.4.** Let $A \subseteq \mathcal{B}(H)$ be a concrete $C^*$-algebra that contains the compact operators $\mathcal{K}(H)$, and let $B \subseteq A$ be a Cartan subalgebra. Then there exists a complete orthogonal set of rank one projections $\{p_i\}_{i \in I}$ on $H$ such that

$$C^*(\{p_i\}_{i \in I}) \subseteq B \subseteq vN(\{p_i\}_{i \in I}).$$

In words, the result says that $H$ admits an orthonormal basis such that every operator in $B$ is diagonal with respect to that basis, and moreover that $B$ contains all compact diagonal operators.

Let us now specialize to the case that $A = C^*_u(X)$ is the uniform Roe algebra of a discrete, bounded geometry metric space, and $B \subseteq A$ is some arbitrary Cartan subalgebra. Theorem 1.4 applies, so $B$ identifies with a (unital) $C^*$-subalgebra of $\ell^\infty(I)$. The spectrum of $B$ is thus a compact space of the form $I \sqcup \hat{B}_\infty$, with $I$ an open and dense subset with the discrete topology, and $\hat{B}_\infty$ a ‘corona’ of $I$; thus in particular, every point in $\hat{B}_\infty$ is the limit of some net from $I$.

**Proposition 1.5.** Let $C^*_u(X)$ be the uniform Roe algebra of a bounded geometry metric space\(^2\) and let $B \subseteq C^*_u(X)$ be a Cartan subalgebra, with spectrum $I \sqcup \hat{B}_\infty$ as above. Then no point in $\hat{B}_\infty$ is the limit of a sequence of points from $I$.

In particular, the spectrum of $B$ fails to be first countable, so $B$ must be fairly ‘large’. It would be even better if one could show that the inclusion $B \subseteq vN(\{p_i\})$ of Theorem 1.4 is an equality; unfortunately this is impossible.

\(^2\)This result holds in more generality: see Proposition 2.4 below.
Example 1.6. Let $X = \{n^2 \mid n \in \mathbb{N}\}$, equipped with the metric it inherits as a subset of $\mathbb{N}$. Then there exists a Cartan subalgebra $B \subseteq C^*_u(X)$ that is not $\ast$-isomorphic to $\ell^\infty(X)$.

Thus in particular uniform Roe algebras can admit ‘exotic’ Cartan subalgebras that are not $\ast$-isomorphic (so in particular, not conjugate in any reasonable sense) to the standard Cartan subalgebra.

When does an abstract Cartan pair $B \subseteq A$ come from a uniform Roe algebra?

We now ask for an abstract characterization of those Cartan pairs $B \subseteq A$ that are isomorphic to the standard Cartan pair $\ell^\infty(X) \subseteq C^*_u(X)$ coming from a uniform Roe algebra.

Definition 1.7. Let $B \subseteq A$ be a Cartan subalgebra in a $C^*$-algebra. We say that $B$ is co-separable if there is a countable subset $S$ of $A$ such that $A = C^*(S, B)$, and that $B$ is an $\ell^\infty$-Cartan if it is $\ast$-isomorphic to $\ell^\infty(\mathbb{N})$.

A Cartan pair $B \subseteq A$ of is Roe-like if:

(i) $A$ is unital;

(ii) $A$ contains the $C^*$-algebra of compact operators on a separable infinite dimensional Hilbert space as an essential ideal;

(iii) $B$ is a co-separable $\ell^\infty$-Cartan in $A$.

The conditions above are easily seen to be necessary for $A$ to be isomorphic to $C^*_u(X)$ for some bounded geometry metric space $X$. The next result says that they are also sufficient. To state it in a precise form, let $\mathcal{K}$ be the essential ideal in $A$ isomorphic to the compact operators, and choose an irreducible (faithful) representation of $\mathcal{K}$ on a separable Hilbert space $H$. This representation extends uniquely to a representation of $A$, which is also irreducible, and is faithful as $\mathcal{K}$ is essential. We may thus identify $A$ with its image in this representation.

Theorem 1.8. Let $B \subseteq A$ be a Roe-like Cartan pair. Then there exists a bounded geometry metric space $Y$ such that for any irreducible and faithful representation of $A$ on $H$ as above there is a unitary isomorphism $v : \ell^2(Y) \to H$ such that

$$v^* B v = \ell^\infty(Y) \quad \text{and} \quad v^* A v = C^*_u(Y).$$

$^3$Equivalently, of $\mathcal{N}_A(B)$.

$^4$and therefore as its unique minimal ideal.
It is interesting to ask to what extent $Y$ as above is unique, and to what extent $\ell^\infty(Y)$ and $B$ are really ‘the same’ in some sense. Combining the above with the main results of [25], we can answer these questions in some cases. First we need to recall the definition of what it means for two metric spaces to be coarsely equivalent.

**Definition 1.9.** A function $f : X \to Y$ between metric spaces is uniformly expansive if for all $r > 0$ we have that

$$\sup_{x,y \in X, \, d_X(x,y) \leq r} d_Y(f(x), f(y)) < \infty.$$  

A function $f : X \to Y$ is a coarse equivalence if it is uniformly expansive, and if there is a uniformly expansive function $g : Y \to X$ such that

$$\sup_{x \in X} d_X(x, g(f(x))) < \infty \quad \text{and} \quad \sup_{y \in Y} d_Y(y, f(g(y))) < \infty.$$  

**Theorem 1.10.** Let $A = C_u^*(X)$ be the uniform Roe algebra of a bounded geometry metric space, and let $B \subseteq A$ be a Roe-like Cartan pair (with the same $A$). Assume moreover that $A$ is nuclear. Then if $Y$ is as in Theorem 1.8, we have that $X$ and $Y$ are coarsely equivalent.

The hypotheses of the above theorem apply fairly broadly. Say for example $X$ is a finitely generated discrete group $\Gamma$ equipped with some choice of word metric. Then $C_u^*(X)$ is nuclear if and only if $\Gamma$ is exact in the sense of Kircherg and Wassermann [13], if and only if $\Gamma$ has Yu’s property A [28, Section 2], as shown in [16]. The class of exact groups is very large, including for example all linear groups, all groups with finite asymptotic dimension, and all amenable groups: see [27] for a survey.

**Is the canonical Cartan subalgebra $\ell^\infty(X) \subseteq C_u^*(X)$ unique?**

Reasonable uniqueness results for a Cartan subalgebra might give uniqueness up to automorphism, or more strongly up to inner automorphism.

The first result we have is of the weaker kind: it is a fairly direct consequence of Theorem 1.7 above and results of Špakula-Willett [25]. To state it, we make an ad-hoc definition, which may be of some interest in its own right.
**Definition 1.11.** A bounded geometry metric space $X$ is **bijectively rigid** if whenever there is a coarse equivalence $f : X \to Y$ to another bounded geometry metric space $Y$, then there is a bijective coarse equivalence $f' : X \to Y$.

An important theorem of Whyte [26] (more precisely, a slight variation, with the same proof) shows that any non-amenable (in the sense, for example, of Block and Weinberger [2]) bounded geometry metric space is bijectively rigid. It is also elementary to see that $\mathbb{Z}$ is bijectively rigid. On the other hand, locally finite groups tend not to be, as follows for example from [14, Corollary 4.10 and Theorem 4.11].

**Theorem 1.12.** Let $X$ be a bounded geometry, bijectively rigid metric space with Yu’s property A. If $B \subseteq A$ is a co-separable $\ell^\infty$-Cartan subalgebra, then there is a $\ast$-automorphism $\alpha : A \to A$ such that $\alpha(\ell^\infty(X)) = B$.

The theorem admits a corollary with more purely $C^\ast$-algebraic hypotheses.

**Corollary 1.13.** Say $A = C^\ast_u(X)$ is the uniform Roe algebra of a bounded geometry metric space, and that $A$ is nuclear and has no tracial states. If $B \subseteq A$ is a co-separable $\ell^\infty$-Cartan subalgebra, then there is a $\ast$-automorphism $\alpha : A \to A$ such that $\alpha(\ell^\infty(X)) = B$.

**Proof.** Nucularity of $C^\ast_u(X)$ is equivalent to $X$ having Yu’s property A by [23, Theorem 5.3] (see also [3, Theorem 5.5.7]). On the other hand, $C^\ast_u(X)$ having no tracial states is equivalent to $X$ being non-amenable (see for example [1, Section 4]). Thus Corollary 1.13 follows from Theorem 1.12 and the aforementioned result of Whyte [26].

The next result is of the stronger sort, giving uniqueness up to inner automorphism. This is the central result of this paper; it uses both Theorems 1.4 and 1.8 above as ingredients in its proof, and is much more substantial than Theorem 1.12.

**Theorem 1.14.** Let $X$ be a bounded geometry metric space with finite decomposition complexity. If $B \subseteq C^\ast_u(X)$ is a co-separable $\ell^\infty$-Cartan subalgebra then there is a unitary $u \in C^\ast_u(X)$ such that $uBu^* = \ell^\infty(X)$.

Finite decomposition complexity (FDC) was introduced by Guentner, Tessera, and Yu in the course of their work on the stable Borel conjecture [11]. We will not define it here as the definition is a little long, but at least comment that it is a fairly general condition:
spaces with FDC include all spaces of finite asymptotic dimension, all word hyperbolic groups, all elementary amenable groups, and all linear groups (see [12]). The only groups that are known not to have FDC are again the Gromov monsters.

Remark 1.15. Let us compare this result to Theorem 1.12 above. There are no known examples of spaces X with property A where X does not have FDC. Probably the simplest examples of spaces which property A is known, but FDC is open are certain non-elementary amenable groups, such as the first Grigorchuk group [9, Chapter VIII]. Thus it is reasonable to think of Theorem 1.14 as the stronger of the two: there are many examples where 1.14 applies but 1.12 does not (e.g. many elementary amenable groups), and it gives a stronger conclusion (uniqueness up to inner automorphism rather than just automorphism). Nonetheless, it is plausible that there are examples where Theorem 1.12 applies and Theorem 1.14 does not; moreover, Theorem 1.12 is easier to prove.

Theorem 1.14 has the following ‘rigidity’ corollary.

Corollary 1.16. Say X and Y are bounded geometry metric spaces. Then the following are equivalent:

(i) there is a bijective coarse equivalence between X and Y;
(ii) the coarse groupoids associated to X and Y (see [23] or [19, Chapter 10]) are isomorphic;
(iii) there is a ∗-isomorphism from C_u^*(X) to C_u^*(Y) that takes ℓ^∞(X) to ℓ^∞(Y).

Moreover, if X has FDC, then these statements statements are equivalent to
(iv) there is a ∗-isomorphism from C_u^*(X) to C_u^*(Y).

The equivalence of (i), (ii), and (iii) in the above is well-known and not difficult to prove (although we are not sure if it has explicitly appeared in the literature before). The content of the corollary is the equivalence of these with (iv) when X has FDC.

Remark 1.17. In [15, Theorem 6.2], Li and Renault show (among other things) that the following are equivalent whenever X and Y are non-amenable, finitely generated, exact groups:

(i) there is a ∗-isomorphism from C_u^*(X) to C_u^*(Y) that takes ℓ^∞(X) to ℓ^∞(Y);
(ii) there is a $*$-isomorphism from $C^*_u(X)$ to $C^*_u(Y)$;

(iii) there are projections $p \in \ell^\infty(X)$ and $q \in \ell^\infty(Y)$ that are full in $C^*_u(X)$ and $C^*_u(Y)$ respectively, and a $*$-isomorphism from $pC^*_u(X)p$ to $qC^*_u(Y)q$ that takes $p\ell^\infty(X)p$ to $q\ell^\infty(Y)q$.

In [15, Remark 6.3], Li and Renault point out that (iii) implies (i) fails for some of the lamplighter type groups studied by Dymarz in [8]; this is because (i) is equivalent to $X$ and $Y$ being bi-Lipschitz equivalent, (ii) is equivalent to $X$ and $Y$ being quasi-isometric, and Dymarz shows that there are groups in the class she studies which are quasi-isometric, but not bi-Lipschitz equivalent. Now, it follows that one of the implications (iii) implies (ii), or (ii) implies (i) must fail in this setting, and Li and Renault ask which. As the groups studied by Dymarz are elementarily amenable, they have FDC, and thus Corollary 1.16 shows (i) that (ii) are equivalent for groups from this class; hence it is (iii) implies (ii) that fails for Dymarz’s lamplighters.

Let us conclude this subsection with a few remarks on the necessity of the assumptions used for Theorem 1.14. Note first that if $B \subseteq C^*_u(X)$ is a Cartan subalgebra that is unitarily conjugate to $\ell^\infty(X)$ (or just conjugate by a non-inner automorphism), then $B$ must certainly be a co-separable $\ell^\infty$-Cartan, as these assumptions are true for $\ell^\infty(X)$ itself. In other words, these assumptions are necessary for the conclusion to hold. Moreover, the condition that $B$ be an $\ell^\infty$-Cartan is not automatic, as shown by Example 1.6. We do not know, however, if the assumption of co-separability is automatic: note that a Cartan subalgebra $B \subseteq A$ in a unital $C^*$-algebra is co-separable if and only if the underlying groupoid is $\sigma$-compact (we leave this as an exercise for the reader).

The assumption of FDC in Theorem 1.14 comes about due to our reliance on a recent and striking result due to Špakula and Tikuisis [24] characterizing when certain operators are in the uniform Roe algebra; FDC is a key ingredient in their work. We also need to use the operator norm localization property (ONL) for $X$ as introduced by Chen, Tessera, Wang, and Yu [4]; the fact that FDC implies ONL is due to Chen, Wang, and Wang [5], and also follows from Sako’s result that ONL for $X$ is equivalent to Yu’s property A [22]. If the assumptions needed for the work of Špakula and Tikuisis could be weakened to just $C^*_u(X)$ being nuclear (equivalently, $X$ having property A), then Theorem 1.14 and Corollary 1.16 would also be true under this assumption.
It seems plausible to us that Theorem 1.14 will fail without some assumption on $X$, due to the well-known exotic analytic properties of uniform Roe algebras outside of the property A setting: see for example [21] and [20]. We would be very interested in any progress towards the construction of exotic examples, or in showing that they cannot exist.

Outline of the paper

Section 2 discusses the general structure of Cartan subalgebras in concrete $C^*$-algebras containing the compact operators, says a little more in the case of uniform Roe algebras, and then gives an example showing that not every Cartan subalgebra of a uniform Roe algebra can be isomorphic to the standard one (even as an abstract $C^*$-algebra). Section 3 shows that Roe-like Cartan pairs in the sense of Definition 1.7 above are essentially the same thing as uniform Roe algebras; as it might be a little more conceptually clear, we actually do this in the more general setting of abstract coarse structures, which has the advantage of obviating the need for the co-separability assumption. Section 4 proves Theorem 1.14 and Corollary 1.16.

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2 The structure of Cartan subalgebras in uniform Roe algebras

Our first aim in this section is to prove some general structural results about Cartan subalgebras in $C^*$-algebras that contain the compact operators. We then further restrict the structure of Cartan subalgebras under some additional assumptions that are satisfied by uniform Roe algebras. We conclude the section with a ‘negative’ result: an example showing that a Cartan subalgebra in a uniform Roe algebra $C^*_u(X)$ need not be isomorphic to $\ell^\infty(X)$.

Lemma 2.1. Let $A \subseteq \mathcal{B}(H)$ be a concrete $C^*$-algebra, and assume that $A$ contains the compact operators on $H$. Let $B \subseteq A$ be a maximal
abelian subalgebra, equipped with a conditional expectation $E : A \to B$. Then for any compact operator $a \in A$, $E(a)$ is also compact.

Proof. It suffices to show that $E(e)$ is compact for any rank one projection $e$ on $H$, which we fix from now on. First, we establish the following claim, called $(\dagger)$ in the rest of the proof: there cannot exist $\lambda > 0$ such that for any $N \in \mathbb{N}$ there are positive and mutually orthogonal contractions $b_1, \ldots, b_N$ in $B$ such that $\|b_i E(e) b_i\| \geq \lambda$ for each $i$. Indeed, say such a $\lambda > 0$ exists, and find $b_1, \ldots, b_N$ with the properties above. Let $\text{Tr} : \mathcal{B}(H)_+ \to [0, \infty]$ be the canonical unbounded trace. Then we have that

$$\text{Tr} \left( \sum_{i=1}^{N} b_i e b_i \right) = \left| \text{Tr} \left( \sum_{i=1}^{N} b_i^2 e \right) \right| \leq \left\| \sum_{i=1}^{N} b_i^2 \right\| \text{Tr}(e) = 1, \quad (1)$$

where the last inequality follows as mutual orthogonality of the $b_i$ gives $\| \sum_{i=1}^{N} b_i^2 \| = \sup_{i=1}^{N} \| b_i^2 \|$, and this is at most one as each $b_i$ is a contraction. On the other hand, using that $E$ is a conditional expectation (so in particular contractive) and that the $b_i$ are in $B$, we have that

$$\| b_i e b_i \| \geq \| E(b_i e b_i) \| = \| b_i E(e) b_i \| \geq \lambda$$

for each $i$, and so combining this with line (1) and using $\| \cdot \|_1$ for the trace norm

$$1 \geq \text{Tr} \left( \sum_{i=1}^{N} b_i e b_i \right) = \sum_{i=1}^{N} \| b_i e b_i \|_1 \geq \sum_{i=1}^{N} \| b_i e b_i \| \geq N \lambda;$$

as $N$ was arbitrary, this is impossible, so the proof of claim $(\dagger)$ is complete.

We next claim that for any $\lambda > 0$, the intersection of the spectrum of $E(e)$ and $[\lambda, \infty)$ must be finite. Indeed, if not, then fix $\lambda > 0$ such that the intersection of the spectrum of $E(e)$ with $[\lambda, \infty)$ is infinite. For any $N$, there are continuous functions $\phi_1, \ldots, \phi_N : \mathbb{R} \to [0, 1]$ supported on $[\lambda, \infty)$, with mutually disjoint supports, and with the property that each $\phi_i$ attains the value 1 somewhere on the intersection of the spectrum of $E(e)$ and $[\lambda, \infty)$. Setting $b_i := \phi_i(E(e))$ the functional calculus gives us that the $b_i$ are positive, mutually orthogonal contractions with $\| b_i E(e) b_i \| \geq \lambda$ for each $i$ and so we have contradicted claim $(\dagger)$.

At this point, we know that the spectrum of $E(e)$ is a countable subset of $[0, \infty)$, and the only possible limit point is 0. To complete
the proof, it suffices to show that each positive spectral value of $E(e)$, which is isolated and thus an eigenvalue, necessarily has finite dimensional eigenspace. Assume for contradiction that this is not the case, and let $\lambda > 0$ be an eigenvalue such that the associated spectral projection $p := \chi_{\{\lambda\}}(E(e)) \in B$ has infinite rank.

We claim first that there is some non-zero $b \in B_+$ such that $b \leq p$, and such that $b$ is not a scalar multiple of $p$. Indeed, if not then for any $b \in B$ we have that $pb = \lambda bp$ for some $\lambda_p \in \mathbb{C}$. This implies that any rank one subprojection $r$ of $p$ (which is in $A$, as $A$ contains the compacts) commutes with all elements of $B$. Hence as $B$ is a MASA in $A$, $r$ is in $B$, contradicting the claim.

Let then $b \in B_+$ be non-zero, such that $b \leq p$, and such that $b$ is not a scalar multiple of $b$. If there exists $\mu > 0$ such that the spectrum of $b$ has infinite intersection with $[\mu, \infty)$, then for any $N$ we may again find continuous $\phi_1, \ldots, \phi_N : \mathbb{R} \to [0, 1]$ supported on $[\mu, \infty)$, with mutually disjoint supports, and such that each attains the value one somewhere on the spectrum of $b$. Set $b_i := \phi_i(b)$, and note that for any $i$,

$$
\|b_i E(e) b_i\| \geq \frac{1}{\lambda} \|b_i p b_i\| \geq \frac{1}{\lambda} \|b_i b b_i\| \geq \frac{\mu}{\lambda} > 0.
$$

This contradicts the claim $(\ast)$ at the start of the proof. Hence $b$ must have countable spectrum whose only possible limit point is 0. In particular, $b$ has a non-zero spectral projection $p_1 \in B$ which is thus a non-zero proper subprojection of $p$. Replacing $p_1$ with $(1 - p_1)(1 - p)$ if necessary, we may assume that $p_1$ also has infinite rank.

We may now repeat the above process, giving a strictly decreasing infinite sequence $p \supseteq p_1 \supseteq p_2 \supseteq \cdots$ of infinite rank projections in $B$. Set $b_i := p_i - p_{i-1}$. Then for any $i$, we have

$$
\|b_i E(e) b_i\| \geq \frac{1}{\lambda} \|b_i p b_i\| = \frac{1}{\lambda}.
$$

Again, this contradicts claim $(\ast)$ from the start of the lemma. At this point, we may conclude that all spectral projections of $E(e)$ are finite rank, and this completes the proof.

**Lemma 2.2.** Say $A \subseteq B(H)$ is a concrete $C^*$-algebra containing the compact operators on $H$. Let $B \subseteq A$ be a Cartan subalgebra. Then $B$ contains a complete orthogonal set of rank one projections.

**Proof.** Write $E : A \to B$ for the faithful conditional expectation that comes with the fact that $B$ is Cartan in $A$, and let $e$ be a rank one
projection. Then $E(e)$ is compact by Lemma 2.1 and non-zero as $E$ is faithful. It follows from the spectral theorem that $B$ contains a non-zero finite rank projection, and thus a minimal non-zero finite rank projection, say $q$. We claim that $q$ must be rank one. Indeed, if not there is a non-zero subprojection of $q$, say $q_0$. As $q$ is minimal and $B$ is commutative, for any $b \in B$ there exists a scalar $\lambda_b \in \mathbb{C}$ with $qb = qbq = \lambda bq$; otherwise the spectral theorem would give us a subprojection of $qbq$ in $B$. It follows that for any $b \in B$, $q_0$ commutes with $b = qb + (1 - q)b$. Hence $q_0$ is in $B$ as $B$ is a MASA, giving us our contradiction.

Let now $S$ be the collection of all rank one projections in $B$, which is non-empty by the above argument. Note that as $B$ is commutative, the projections in $S$ are all mutually orthogonal, and thus the sum $p := \sum_{q \in S} q$ converges strongly to a projection. Note that as $p$ is a strong limit of operators in $B$, it commutes with everything in $B$. We claim that in fact $p$ commutes with everything in the normalizer of $B$ in $A$. Indeed, if not, there exists $a \in \mathcal{N}_A(B)$ such that $pa(1 - p) \neq 0$. The definition of $p$ thus gives a rank one projection $q$ in $B$ such that $qa(1 - p) \neq 0$. Hence $(1 - p)a*qa(1 - p) \neq 0$; note that this operator is positive and rank one, so a non-zero scalar multiple of a projection, say $r$. Note that as $a$ normalizes $B$, the element $r$ is in the cut-down $(1 - p)B(1 - p)$, which is a commutative $C^*$-algebra as $p$ commutes with $B$. Now, $r$ is in $A$ as it is rank one and $A$ contains the compacts. Hence it is in $B$ as this $C^*$-algebra is maximal abelian in $A$ and as $r$ commutes with everything in $B \subseteq pBp \oplus (1 - p)B(1 - p)$. However, $r$ is orthogonal to $p$, and thus orthogonal to everything in $S$, which is impossible and we have completed the proof that $p$ commutes with everything in $\mathcal{N}_A(B)$.

Finally, note that as $B \subseteq A$ is a Cartan subalgebra, $\mathcal{N}_A(B)$ generates $A$ as a $C^*$-algebra, and thus $p$ commutes with everything in $A$. As $A$ contains the compacts, this forces $p = 0$ or $p = 1$. As we already know $p \neq 0$, $p$ is the identity operator; this completes the proof.

Recall that if $S$ is a subset of $B(H)$, then $C^*(S)$ denotes the $C^*$-algebra generated by $S$, and $\nu N(S)$ the von Neumann algebra generated by $S$. We are now ready to prove Theorem 1.4 which is essentially just a restatement of the previous lemma.

**Proof of Theorem 1.4** Let $\{p_i\}_{i \in I}$ be the complete set of orthogonal rank one projections in $B$ given by Lemma 2.2. As $B$ is a $C^*$-algebra,
it contains $C^*([p_i])$. As $vN([p_i])$ is the maximal abelian $*$-subalgebra of $B(H)$ that contains $C^*([p_i])$, $B$ is contained in $vN([p_i])$. \hfill \Box

Note that the conclusion of Theorem 1.4 on the structure of $B$ is best possible with those assumptions. Indeed, if $\{p_i\}_{i\in I}$ is a complete orthogonal set of rank one projections on $H$, and $B$ is a $C^*$-subalgebra of $B(H)$ with

$$C^*([p_i]) \subseteq B \subseteq vN([p_i])$$

then $A := B + K(H)$ clearly contains $B$ as a Cartan subalgebra.

On the other hand, we have the following corollary giving some sufficient conditions for $B$ to equal $vN([p_i])$, which will be important later in the paper.

**Corollary 2.3.** Say $A \subseteq B(H)$ is a concrete $C^*$-algebra containing the compact operators on $H$. Let $B \subseteq A$ be a Cartan subalgebra. Then there is a complete orthogonal set $\{p_i\}_{i\in I}$ of rank one projections such that

$$C^*([p_i]) \subseteq B \subseteq vN([p_i]).$$

Assume moreover that either:

(i) $B$ is closed in the strong topology\footnote{As $A$ contains the finite rank operators as its unique minimal algebraic ideal, this condition can be defined in a representation-independent way using that $b_n \to b$ strongly if and only if $b_n f \to bf$ in norm for each finite rank $f$.}; or

(ii) $B$ is abstractly $*$-isomorphic to $\ell^\infty(X)$ for some set $X$.

Then $B$ equals $vN([p_i])$.

**Proof.** As the strong closure of $C^*([p_i])$ equals $vN([p_i])$, part (i) is clear. For part (ii), let $\phi : B \to \ell^\infty(X)$ be an abstract $*$-isomorphism. As $\phi$ must take the family $\{p_i\}_{i\in I}$ of minimal projections in $B$ bijectively to the family $\{q_x\}_{x\in X}$ of minimal projections in $\ell^\infty(X)$, it induces a bijection $f : I \to X$. Now, note that if $S \subseteq I$ and $q_{f(S)} := \sum_{i\in S} q_{f(i)}$ is the corresponding projection in $\ell^\infty(X)$, then $\phi^{-1}(q_{f(S)})$ is a projection on $H$ that commutes with the set $\{p_i\}_{i\in I}$, and that satisfies

$$\phi^{-1}(q_{f(S)})p_i = \begin{cases} p_i & i \in S \\ 0 & i \notin S \end{cases}.$$

This is only possible if $\phi^{-1}(q_{f(S)})$ equals the projection $p_S := \sum_{i\in S} p_i$ on $H$. Hence $p_S$ is in $B$, and as $S$ was arbitrary, $B$ contains all
projections in \( vN(\{p_i\}) \). The projections in \( vN(\{p_i\}) \) span a norm-dense subset, however, so this gives us \( B = vN(\{p_i\}) \).

The next lemma adds another assumption on \( A \) (that is satisfied by uniform Roe algebras) in order to limit the structure of \( B \) a little more. In order to state it, we introduce a little more notation. Let notation be as in Theorem 1.4 and let \( \{p_i\}_{i \in I} \) be as in the conclusion, so in particular

\[
C^*(\{p_i\}) \subseteq B \subseteq vN(\{p_i\}).
\]

Assume moreover \( A \) is unital, whence \( B \) is too. Then the spectrum of \( B \) is a compact Hausdorff set \( \hat{p}_B \) that contains a copy of \( I \) as an open, dense, discrete subset; indeed, this follows as \( C^*(\{p_i\}) \) is an essential ideal in \( B \), and the spectrum of \( C^*(\{p_i\}) \) identifies with \( I \).

Write \( \hat{B}_\infty := \hat{B} \setminus I \), so that \( \hat{B}_\infty \) is a closed subset of \( \hat{B} \); by density of \( I \) in \( \hat{B} \), note that every point in \( \hat{B}_\infty \) is a limit of a net from \( I \). The next result is a slightly more general version of Theorem 1.5 from the introduction.

\textbf{Proposition 2.4.} Say \( A \subseteq \mathcal{B}(H) \) is a concrete unital \( C^* \)-algebra containing the compact operators, and assume that \( H \) is infinite dimensional and separable. Let \( B \subseteq A \) be a Cartan subalgebra, with

\[
C^*(\{p_i\}) \subseteq B \subseteq vN(\{p_i\}).
\]

as above. Assume moreover that there is another complete orthogonal set of projections \( \{q_j\}_{j \in J} \) for \( H \) such that \( A \) contains the corresponding \( C^* \)-algebra \( vN(\{q_j\}) \) of multiplication operators. Then no element of \( \hat{B}_\infty \) is the limit of a sequence from \( I \).

Note that the assumptions of the lemma certainly apply if \( A \) is the uniform Roe algebra of a bounded geometry metric space, and \( B \) any Cartan subalgebra of \( A \). One can think of the lemma as saying that the topology of the spectrum of \( B \) must be fairly complicated, and in particular \( B \) cannot be separable. Unfortunately, it does not imply that \( B \) is all of \( \ell^\infty \) as we will see in Example 2.5 below.

\textbf{Proof.} As \( H \) is separable and infinite dimensional, we may identify \( I \) and \( J \) with \( \mathbb{N} \) and will do so for the rest of the proof (for definiteness, let us say that \( \mathbb{N} \) starts at 1). In particular, we will identify \( \hat{B} = \mathbb{N} \sqcup \hat{B}_\infty \), and write \( \{p_n\}_{n \in \mathbb{N}} \) for the complete orthogonal set of projections that we started with; in terms of the spectrum \( \hat{B} = \mathbb{N} \sqcup \hat{B}_\infty \) of \( B \), \( p_n \)
can be thought of as the characteristic function of the singleton \( \{ n \} \).

For each \( r \in \mathbb{N} \), let \( Q_r \in \ell^\infty(\mathcal{W}) \) be defined by

\[
Q_r := q_1 + \cdots q_r
\]

and set \( Q_0 = 0 \).

Assume for contradiction that there is some point \( x_\infty \in \hat{B}_\infty \) and a sequence in \( \mathbb{N} \) that converges to it. We claim that we can iteratively construct strictly increasing subsequences \( (n_k)_{k=1}^\infty \) and \( (m_k)_{k=1}^\infty \) of the given sequence converging to \( x_\infty \), a strictly increasing sequence \( (r_k)_{k=1}^\infty \) in \( \mathbb{N} \cup \{0\} \), and a sequence \( (e_k)_{k=1}^\infty \) of mutually orthogonal finite rank projections in \( vN(\{q_j\}) \) with the following properties:

1. \( \| p_{n_k} e_k p_{n_k} \| > 3/4 \) for all \( k \);
2. \( \| p_{m_k} e_j p_{m_k} \| < (1/4) 2^{-j} \) for all \( k \) and all \( j \in \{1, \ldots, k\} \);
3. \( e_j \leq Q_{r_k} \) for all \( k \) and all \( j \in \{1, \ldots, k-1\} \);
4. \( e_j \leq 1 - Q_{r_k} \) for all \( k \) and all \( j \geq k \);
5. \( \| p_{m_j} Q_{r_k} p_{m_j} \| > 3/4 \) for all \( j \in \{1, \ldots, k-1\} \).

Indeed, to start the process off with \( k = 1 \), set \( r_1 = 0 \), so \( Q_0 = 0 \). Let \( n_1 \) be the first element of the given sequence that converges to \( x_\infty \), and choose \( e_1 = Q_r \) where \( r \) is large enough that (1) holds. Now choose \( m_1 \) large enough in the given sequence that (2) holds. Note that (3) and (5) are vacuous, and that (4) will hold whatever we eventually do. Now, say we have constructed the desired elements up to stage \( k \). Choose \( r_{k+1} > r_k \) large enough so that (3) and (5) both hold. Choose \( n_{k+1} > n_k \) far enough along the sequence converging to \( x_\infty \) so that \( \| p_{n_{k+1}} Q_{r_k} p_{n_{k+1}} \| < 1/4 \); in particular, this implies that we can choose \( e_{k+1} \) satisfying both (1) and (4) (the latter in the special case \( e_{k+1} \leq 1 - Q_{r_{k+1}} \)). Finally, choose \( m_{k+1} > m_k \) far enough along the given sequence so that (2) holds. It is not too difficult to show that the resulting sequences have the claimed properties.

Now, given the above, set \( e := \sum_{k=1}^\infty e_k \), which converges strongly to an element of \( vN(\{q_j\}) \). Let \( E : A \to B \) be the conditional expectation. Thinking of elements of \( B \) as functions on \( \mathbb{N} \), we have that \( E(e) \) is the function \( f : n \mapsto \| p_n e p_n \| \). On the one hand, note that

\[
\| p_{n_k} e p_{n_k} \| \geq \| p_{n_k} e_k p_{n_k} \| > 3/4
\]

for each \( k \), and on the other that

\[
\| p_{m_k} e p_{m_k} \| \leq \sum_{k=1}^\infty \| p_{m_k} e p_{m_k} \| + \| p_{m_k} (1 - Q_{r_k}) p_{m_k} \| < 1/2.
\]
Now, as both sequences \( (n_k) \) and \( (m_k) \) converge to \( x_\infty \), we have that

\[
f(x_\infty) = \lim_{k \to \infty} f(n_k) = \lim_{k \to \infty} \| p_{n_k} e p_{n_k} \| \geq 3/4
\]

from line (2), and that

\[
f(x_\infty) = \lim_{k \to \infty} f(m_k) = \lim_{k \to \infty} \| p_{m_k} e p_{m_k} \| \leq 1/2
\]

from line (3), giving us the desired contradiction.

We conclude this section with an example showing that if \( A = C_u^*(X) \), and \( B \subseteq A \) is a Cartan subalgebra, then, with notation as above, one cannot conclude that \( B \) equals \( vN(\{p_i\}) \), or therefore by Corollary 2.3 that \( B \) is not even abstractly \( \ast \)-isomorphic to \( \ell^\infty(\mathbb{N}) \).

**Example 2.5.** Let

\[
X = \{ n^2 \mid n \in \mathbb{N} \}
\]

be the space of square numbers equipped with the metric it inherits as a subspace of \( \mathbb{N} \). For concreteness, let us assume that \( \mathbb{N} \) starts with 1. Note that we have

\[
C_u^*(X) = \ell^\infty(X) + K(\ell^2(X)). \tag{4}
\]

This follows as the points of \( X \) get more and more widely spaced, whence the only finite propagation operators are those of the form ‘diagonal plus finite rank’.

Now, for each \( n \in 2\mathbb{N} \), let \( \xi_n = \frac{1}{\sqrt{2}} (\delta_{n^2} + \delta_{(n+1)^2}) \) and \( \eta_n = \frac{1}{\sqrt{2}} (\delta_{n^2} - \delta_{(n+1)^2}) \), so the set

\[
S := \{ \xi_n, \eta_n \mid n \in 2\mathbb{N} \}
\]

is an orthonormal basis for \( X \). Let \( \ell^\infty(S) \) be the corresponding \( C^*_\ast \)-algebra of multiplication operators on \( \ell^2(X) \), and let \( B = C_u^*(X) \cap \ell^\infty(S) \). Thinking of \( X \) as decomposed in the following way

\[
X = \bigsqcup_{n \in 2\mathbb{N}} \{ n^2, (n+1)^2 \},
\]

and \( \ell^2(X) \) correspondingly decomposed into a direct sum of two dimensional subspaces

\[
\ell^2(X) = \bigoplus_{n \geq 1} \ell^2(\{ n^2, (n+1)^2 \})
\]

there is nothing particularly special about the sequence \( (n^2) \) here: any strictly increasing subsequence \( (a_n) \) of \( \mathbb{N} \) such that \( |a_{n+1} - a_n| \to \infty \) as \( n \to \infty \) would work just as well.
operators in $\ell^\infty(S)$ look like

$$\prod_{n \geq 1} \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix},$$

where $(a_n)$ and $(b_n)$ are arbitrary bounded sequences. Elements of $B$ look like this, except now we must also ask that $b_n \to 0$ as $n \to \infty$ (it is straightforward to check that this is a necessary and sufficient for such an operator to be in $C_u^*(X)$).

We claim the algebra $B$ is a Cartan subalgebra of $C_u^*(X)$. This follows from the computations below.

(i) It is maximal abelian. The algebra $B$ contains $C_0(S)$. The commutant of $C_0(S)$ in $B(\ell^2(X))$ is $\ell^\infty(S)$, and thus $B$ contains everything in $C_u^*(X)$ that commutes with $C_0(S)$, and in particular contains everything that commutes with $B$ itself.

(ii) The normalizer $\mathcal{N}_{C_u^*(X)}(B)$ of $B$ in $C_u^*(X)$ generates. Indeed, thinking of operators in $B$ as matrices as in line (5) above, we see that the normalizer of $B$ in $C_u^*(X)$ contains all products of matrices of the form

$$\prod_{n \in 2\mathbb{N}} \begin{pmatrix} c_n & 0 \\ 0 & c_n \end{pmatrix} \quad \text{and} \quad \prod_{n \in 2\mathbb{N}} \begin{pmatrix} d_n & 0 \\ 0 & -d_n \end{pmatrix},$$

where $(c_n), (d_n)$ are arbitrary bounded sequences. Clearly then the $C^*$-algebra generated by the normalizer $\mathcal{N}_{C_u^*(X)}(B)$ contains $\ell^\infty(X)$. It also straightforward to see that it contains $\mathcal{K}(\ell^2(X))$, and so by line (i) is all of $C_u^*(X)$.

(iii) There is a faithful conditional expectation $C_u^*(X) \to B$. Let $E : B(\ell^2(X)) \to \ell^\infty(S)$ be the canonical conditional expectation, which is faithful. We need to check that $E$ takes $C_u^*(X)$ onto $B$ (and not onto some larger subalgebra of $\ell^\infty(S)$). Looking at line (i) above, $E$ takes $\mathcal{K}(\ell^2(X))$ to $C_0(S) \subseteq B$, so it suffices to check that $E(\ell^\infty(X)) \subseteq B$. With respect to a matrix decomposition as in line (5) above, an arbitrary element of $\ell^\infty(X)$ looks like

$$\prod_{n \in 2\mathbb{N}} \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix}$$

for some bounded sequences $(a_n)$ and $(b_n)$. Computing the image of this under the conditional expectation $E$, we may compute one
matrix at a time. Doing this, with \( E_n \) the restriction of \( E \) to the bounded operators on \( \ell^2(\{n^2, (n+1)^2\}) \), we see that

\[
E\left( \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \\
\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_n & 0 \\ 0 & b_n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
= \frac{1}{2} \begin{pmatrix} a_n + b_n & 0 \\ 0 & a_n + b_n \end{pmatrix}
\]

and this is certainly in \( B \).

**Remark 2.6.** The Cartan subalgebra \( B \) above is co-separable in the sense of Definition [17], and indeed we do not know if it is possible for the uniform Roe algebra of a bounded geometry metric space to admit a Cartan subalgebra that is not co-separable. To see coseparability of \( B \), let \( S_0 \) be a countable subset of \( \mathcal{N}_{C^*_u(X)}(B) \) that generates \( \mathcal{K}(\ell^2(X)) \), and with our usual matrix conventions, let \( s \) be the element

\[
s := \prod_{n \in 2\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

of \( \ell^\infty(X) \), which normalizes \( B \). Set \( S = S_0 \cup \{s\} \). We claim that \( S \) and \( B \) together generate \( C^*_u(X) \). By assumption on \( S_0 \) and line [4], it suffices to show that the \( C^*-\)algebra generated by \( s \) and \( B \) contains \( \ell^\infty(X) \). Let then

\[
\prod_{n \in 2\mathbb{N}} \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix}
\]

be an arbitrary element of \( \ell^\infty(X) \), and note that

\[
\prod_{n \in 2\mathbb{N}} \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix} = \frac{1}{2} \prod_{n \in 2\mathbb{N}} \begin{pmatrix} a_n + b_n & 0 \\ 0 & a_n + b_n \end{pmatrix} + s \frac{1}{2} \prod_{n \in 2\mathbb{N}} \begin{pmatrix} a_n - b_n & 0 \\ 0 & a_n - b_n \end{pmatrix};
\]

as the two products of matrices on the right hand side are in \( B \), we are done.
3 Abstract coarse structures and Roe-like Cartan subalgebras

Our goal in this section is to prove that $C^*$-algebras containing a co-separable $\ell^\infty$-Cartan in the sense of Definition 1.7 are essentially the same thing as bounded geometry metric spaces (considered up to bijective coarse equivalence). Actually, we work a little more generally than this, using the language of abstract coarse structures as this seems to give slightly cleaner results.

The following definition is due to Roe [19, Chapter 2].

Definition 3.1. Let $X$ be a set. A coarse structure on $X$ is a collection $\mathcal{E}$ of subsets of $X \times X$ such that:

(i) for all $E, F \in \mathcal{E}$ the union $E \cup F$ is in $\mathcal{E}$;

(ii) for all $E, F \in \mathcal{E}$, the composition

$$E \circ F := \{(x, z) \in X \times X \mid \text{there is } y \in X \text{ with } (x, y) \in E \text{ and } (y, z) \in F\}$$

is in $\mathcal{E}$;

(iii) for all $E \in \mathcal{E}$, the inverse

$$E^{-1} := \{(x, y) \in X \times X \mid (y, x) \in E\}$$

is in $\mathcal{E}$;

(iv) for all $E \in \mathcal{E}$, if $F \subseteq E$, then $F \in \mathcal{E}$;

(v) $\mathcal{E}$ contains the diagonal $\{(x, x) \in X \times X \mid x \in X\}$.

A pair $(X, \mathcal{E})$ of a set $X$ and a coarse structure on $X$ is called a coarse space; when it is unlikely to cause confusion, we will leave $\mathcal{E}$ implicit, and just says that $X$ is a coarse space.

A coarse space $(X, \mathcal{E})$ is:

(a) of bounded geometry if for all $E \in \mathcal{E}$, the cardinalities of the ‘slices’

$$E_x := \{(y, x) \in E \mid y \in X\} \text{ and } E^x := \{(x, y) \in E \mid y \in X\}$$

are bounded independently of $x$;

(b) connected if for every $x, y \in X$, $\mathcal{E}$ contains $\{(x, y)\}$;

(c) countably generated if there is a countable collection $S$ of subsets of $X \times X$ such that $\mathcal{E}$ is generated by $S$ (i.e. such that $\mathcal{E}$ is the intersection of all coarse structures containing $S$).
The basic example of a coarse structure arises as follows.

**Definition 3.2.** Let \((X, d)\) be a metric space. The bounded coarse structure on \(X\) is defined by

\[ E_d := \{E \subseteq X \times X \mid d|_E \text{ is bounded}\} \]

(it is straightforward to check that this is a coarse structure). A coarse space \((X, E)\) is *metrizable* if there exists a metric on \(d\) on \(X\) such that \(E\) is the associated bounded coarse structure.

Note that the bounded coarse structure associated to a metric has bounded geometry if and only if the metric does in the usual sense of Definition 1.1 above. Assuming (as we will) that a metric \(d\) on \(X\) only takes finite values, the associated bounded coarse structure is connected and generated by the countably many sets

\[ E_n := \{(x, y) \in X \times X \mid d(x, y) \leq n\}. \]

Conversely, one has the following result: see [19, Theorem 2.55] for a proof.

**Theorem 3.3.** A coarse space \(X\) is metrizable if and only if it is connected and countably generated.

The following combinatorial lemma (a standard ‘greedy algorithm’ argument) will be used several times below.

**Lemma 3.4.** Let \((X, E)\) be a bounded geometry coarse space and \(E\) be an element of \(E\). Then there exists \(N \in \mathbb{N}\) and a decomposition

\[ E = \bigsqcup_{n=1}^{N} E_n \]

of \(E\) into disjoint subsets such that for each \(x \in X\) and each \(n\), there is at most one element in each set

\[ E_n \cap \{(x, y) \mid y \in X\} \quad \text{and} \quad E_n \cap \{(y, x) \mid y \in X\} \]

(in words, \(E_n\) intersects each row and column in \(X \times X\) at most once).

**Proof.** Set \(E_0\) to be the empty set. Having chosen subsets \(E_0, E_1, \ldots, E_n\) of \(E\), set \(E_{n+1}\) to be a maximal subset of \((X \times X) \setminus (E_1 \cup \cdots \cup E_n)\) that intersects each row and column at most once. We claim that for
some $N$, $E_n$ is empty for all $n \geq N$. Indeed, if not, then for every $N$, there is some element $(x_N, y_N)$ in $E_N$, and in particular that has not appeared in any of $E_1, \ldots, E_{N-1}$. Maximality of these sets implies that for each $n \in \{1, \ldots, N-1\}$ there is either $x_n$ such that $(x_n, y_N)$ is in $E_n$, or $y_n$ such that $(x_N, y_n)$ is in $E_n$. This implies that at least one of the sets

$$\{(x_N, y) \in E \mid y \in X\} \text{ or } \{(x, y_N) \in E \mid x \in X\}$$

has cardinality at least $|(N-1)/2|$. As this happens for all $N$, this contradicts that $(X, E)$ has bounded geometry. $\Box$

We now turn to bounded operators. We start with a basic class of operators.

**Definition 3.5.** Let $V = \{\xi_i\}_{i \in I}$ be an orthonormal basis for a Hilbert space $H$. For any bounded operator $a$ on $H$, let $a_{ij} = \langle \xi_i, a \xi_j \rangle$ be the corresponding matrix entries. We will say that a matrix $(a_{ij})$, or the operator defining it (if one exists) is supported on a single diagonal if for each $i$ there is at most one $j$ such that $a_{ij} \neq 0$, and at most one $k$ such that $a_{ki} \neq 0$ (in words, $a$ has at most one non-zero matrix entry in each row and column).

The following elementary lemma is well-known.

**Lemma 3.6.** Let $V = \{\xi_i\}_{i \in I}$ be an orthonormal basis for $H$, and let $\{p_i\}_{i \in I}$ be the corresponding complete set of orthogonal rank one projections.

(i) Let $B \subseteq B(H)$ be a $C^*$-algebra such that

$$C^*(\{p_i\}) \subseteq B \subseteq vN(\{p_i\}).$$

Then if $a \in B(H)$ normalizes $B$, we have that $a$ is supported on a single diagonal with respect to the basis $V$.

(ii) Let $(a_{ij})_{i,j \in I}$ be a matrix supported on a single diagonal (not necessarily coming from a bounded operator). Then matrix multiplication by $(a_{ij})$ defines a bounded operator $a$ if and only if its matrix entries are uniformly bounded, and in this case, $\|a\| = \sup_{i,j} |a_{ij}|$.

**Proof.** For each $i \in I$ and $a \in B(H)$, the operators $ap_i a^*$ and $a^* p_i a$ have matrix entries given by

$$(ap_i a^*)_{jk} = a_{ji} a_{ki} \quad \text{and} \quad (a^* p_i a)_{jk} = a_{ij} a_{ik}$$
respectively. As $B \subseteq vN(\{p_i\})$, in order for these operators to be in $B$ for some fixed $i$ the entries can only be non-zero if $j = k$, which can only happen if $a$ has at most one non-zero entry in each row and column.

For part (ii), assume $a$ is supported on a single diagonal. Note that $a_i = a_{t(i)}v_{t(i)}$, where $t(i)$ is the unique element of $I$ such that $a_{t(i)} \neq 0$, or $a_i = 0$ if no such $t(i)$ exists. Moreover, we have that if $i \neq j$, then $a_i$ is orthogonal to $a_j$. Hence for any element $v = \sum_{i \in I} \lambda_i \xi_i$ of $H$,

\[
\|av\|^2 = \sum_{i \in I, t(i) \text{ exists}} \|a_{t(i)}\lambda_i v_{t(i)}\|^2 \leq \sup_{i \in I, t(i) \text{ exists}} \|a_{t(i)}\|^2 \sum_{i \in I} |\lambda_i|^2 = \sup_{i,j} |a_{ij}|^2 \|v\|^2.
\]

This gives $\|a\| \leq \sup_{i,j} |a_{ij}|$; the opposite inequality follows as $\|a\| \geq |\langle \xi_i, a\xi_j \rangle|$ for any $i, j$.

As a special case of Definition 3.5, we equip $\ell^2(X)$ with its canonical orthonormal basis $\{\delta_x\}_{x \in X}$, so the matrix entries of a bounded operator $a$ on $\ell^2(X)$ are $a_{xy} := \langle \delta_x, a\delta_y \rangle$.

**Definition 3.7.** Let $(X, E)$ be a coarse space. An operator $a \in \mathcal{B}(\ell^2(X))$ has finite propagation if

\[
\{(x, y) \in X \times X \mid a_{xy} \neq 0\} \in E.
\]

From the axioms for a coarse structure, it is not difficult to check that the collection of finite propagation operators is a $*$-algebra, and thus we can make the following definition.

**Definition 3.8.** Let $(X, E)$ be a coarse space. Let $\mathbb{C}_u[X; E]$ denote the $*$-algebra of finite propagation operators on $\mathcal{B}(\ell^2(X))$ as in Definition 3.7. The uniform Roe algebra of $X$, denoted $C^*_u(X; E)$, is the norm closure of $\mathbb{C}_u[X; E]$.

The following special class of operators in $\mathbb{C}_u[X; E]$ are useful enough to be worth isolating.
**Definition 3.9.** Let \((X, \mathcal{E})\) be a bounded geometry coarse space. For any \(E \in \mathcal{E}\) with at most one entry in each row and column, define a matrix \((v^E_{xy})\) by the formula

\[
v^E_{xy} := \begin{cases} 
1 & (x, y) \in E \\
0 & (x, y) \notin E 
\end{cases}.
\]

Let \(v^E\) denote the unique bounded operator on \(\ell^2(X)\) associated to this matrix by Lemma 3.5.

We now have the following useful structure lemma for \(C^*_u(X; \mathcal{E})\) that holds whenever \(X\) has bounded geometry.

**Lemma 3.10.** With notation as in Definition 3.9, \(v^E\) is a partial isometry in \(C^*_u[X; \mathcal{E}]\) that normalizes \(\ell^2(X)\). Moreover, if \(S \subseteq \mathcal{E}\) is a collection of subsets of \(X \times X\), each with at most one entry in each row and column, and that generates the coarse structure, then the collection

\[
\{v^E \mid E \in S\} \cup \ell^2(X)
\]

generates \(C^*_u[X; \mathcal{E}]\) as a \(*\)-algebra (and therefore generates \(C^*_u(X; \mathcal{E})\) as a \(C^\ast\)-algebra).

**Proof.** That each \(v^E\) is a partial isometry in \(C^*_u[X; \mathcal{E}]\) follows from straightforward computations, and each normalizes \(\ell^2(X)\) by Lemma 3.6 part (i). Let now \(a\) be an arbitrary element of \(C^*_u[X; \mathcal{E}]\), so the set \(E = \{(x, y) \in X \times X \mid a_{xy} \neq 0\}\) is in \(\mathcal{E}\). Bounded geometry and Lemma 3.4 give us a decomposition

\[
E = \bigcup_{n=1}^N E_n
\]

with the property that each \(E_n\) has at most one element in each row and column. For each \(n \in \{1, ..., N\}\) define \(f_n \in \ell^\infty(X)\) by

\[
f_n(x) = \begin{cases} 
a_{xy} & (x, y) \in E_n \\
0 & (x, y) \notin E_n 
\end{cases}
\]

Then one checks that the operator

\[
\sum_{n=1}^N f_nv^E_n
\]

has the same matrix entries as \(a\), and thus the two are equal. \(\square\)
We now get to one of our main goals of this section; this is presumably known, but does not seem to have appeared in the literature before.

**Proposition 3.11.** Let \((X, \mathcal{E})\) be a bounded geometry coarse structure. Then \(\ell^\infty(X)\) is a Cartan subalgebra in \(C^*_u(X; \mathcal{E})\). Moreover, if \(X\) is connected, then \(C^*_u(X; \mathcal{E})\) contains the compact operators.

**Proof.** It is well-known that \(\ell^\infty(X)\) is a unital, maximal abelian \(C^*\)-subalgebra of \(B(\ell^2(X))\) that is the image of a faithful conditional expectation \(B(\ell^2(X)) \to \ell^\infty(X)\), so it certainly also has these properties when considered as a \(C^*\)-subalgebra of \(C^*_u(X; \mathcal{E})\). The normalizer of \(\ell^\infty(X)\) generates \(C^*_u(X; \mathcal{E})\) by Lemma 3.10, completing the proof that \(\ell^\infty(X)\) is a Cartan subalgebra. Assuming that \(X\) is connected, then with notation as in Definition 3.9 we get that for any \((x, y) \in X \times X\) the operator \(v_{\langle x, y \rangle}\) is in \(C_u[X; \mathcal{E}]\). These operators generate the compact operators, so we are done.

To summarize, Definition 3.8 gives us a recipe for taking a connected bounded geometry coarse structure on a set \(X\), and producing a \(C^*\)-subalgebra of \(B(\ell^2(X))\) that contains \(\ell^\infty(X)\) as a Cartan subalgebra, and contains the compact operators. Our next goal is to prove a sort of converse.

**Definition 3.12.** Let \(A\) be a unital \(C^*\)-algebra containing a copy \(\mathcal{K}\) of the compact operators on some Hilbert space as an essential ideal, and let \(B \subseteq A\) be a Cartan subalgebra. Let \(X = \{p_x\}_{x \in X}\) be the set of minimal projections in \(B\), and for each \(a \in \mathcal{N}_B(A)\) and each \(\epsilon > 0\), define

\[
E_{a, \epsilon} := \{(x, y) \in X \times X \mid \|p_x a p_y\| \geq \epsilon\}
\]

Define \(\mathcal{E}_A\) to be the coarse structure on \(X\) generated by the collection

\[
\{E_{a, \epsilon} \mid a \in \mathcal{N}_B(A) \text{ and } \epsilon > 0\}.
\]

**Remark 3.13.** With notation as in Definition 3.12 fix a faithful irreducible representation of \(\mathcal{K}\) on some Hilbert space \(H\); such a representation exists, is unique up to unitary equivalence, and necessarily consists of an isomorphism \(\mathcal{K} \cong \mathcal{K}(H)\) of \(\mathcal{K}\) with the compact operators on \(H\) (see for example [7, Section 4.1]). As \(\mathcal{K}\) is an ideal in \(A\), this

\[\text{We treat } X \text{ as its own index set; apologies for this abuse of notation. It is non-empty, as we will see in Remark 3.13.}\]
representation extends uniquely to a representation of $A$ on $H$, which is also irreducible, and which is necessarily faithful as $K$ is essential in $A$. Identify $A$ with its image under this representation. We may now apply Theorem 1.4: this implies in particular that the set $\{p_x\}_{x \in X}$ of minimal projections in $B$ identifies with a complete collection of orthogonal rank one projections on $H$, and that

$$C^*\{\{p_x\}\} \subseteq B \subseteq vN(\{p_x\}).$$  \hfill (6)

**Lemma 3.14.** With notation as in Definition 3.12, the coarse space $(X, \mathcal{E}_A)$ is connected and has bounded geometry.

**Proof.** Fix a representation $H$ of $A$ with the properties in Remark 3.13. For each $x$, choose a unit vector $\xi_x$ in the range of $p_x$, so the collection $\{\xi_x\}_{x \in X}$ is an orthonormal basis for $H$. Use this basis to write operators on $H$ as matrices $(a_{xy})_{x,y \in X}$ as in Definition 3.5. Note that $|a_{xy}| = \|p_x a p_y\|$ for any $x, y \in X$.

Now, as $A$ contains the compact operators, for any $p_{x,y} \in \mathcal{E}_A$ the operator $v(p_{x,y})$ whose matrix has a single entry equal to one in the $(x,y)^{th}$ position and zeros elsewhere is in $A$, and is moreover in $\mathcal{N}_H(A)$ by Lemma 3.6; this implies that $\{p_{x,y}\}$ is in $\mathcal{E}_A$, and thus the coarse space $X$ is connected.

Let $S$ be the collection of all elements $E$ of $\mathcal{E}_A$ such that $E$ has at most one element in each row and column. Then Lemma 3.6 implies that each $E_{a,\epsilon}$ is in $S$ as $a$ ranges over $\mathcal{N}_H(A)$ and $\epsilon$ over $(0, \infty)$, whence $S$ generates $\mathcal{E}_A$. Note that $S$ is closed under all the operations defining a coarse structure, except (possibly) unions. It follows that $\mathcal{E}_A$ consists precisely of finite unions of sets from $S$, and thus has bounded geometry.

**Lemma 3.15.** With notation as in Definition 3.12, identify $A$ with image in some representation on a Hilbert space $H$ with the properties in Remark 3.13. For each $x \in X$, choose a unit vector $\xi_x$ in the range of $p_x$, so $\{\xi_x\}_{x \in X}$ is an orthonormal basis of $H$, and define a unitary isomorphism

$$v: \ell^2(X) \to H, \quad \delta_x \mapsto \xi_x.$$  

Consider $C^*_v(X; \mathcal{E}_A)$ and its Cartan subalgebra $\ell^\infty(X)$ as represented on $\ell^2(X)$ in the canonical way. Then $v^*Bv$ is contained in $\ell^\infty(X)$, and $v^*Av$ is contained in $C^*_v(X; \mathcal{E}_A)$.

**Proof.** Note that $v^*pv$ is the orthogonal projection onto the span of $\delta_x$, whence $v^*(vN(\{p_x\}))v = \ell^\infty(X)$. Hence by line (6) above,
To see that \( v^* A v^* \subseteq \ell^\infty(X) \), it suffices to show that \( v^* N_B(A) v \) is contained in \( C^*_u(X; \mathcal{E}_A) \). Let then \( a \) be an element of \( N_B(A) \) and let \( \epsilon > 0 \). Then as the matrix associated to \( a \) has at most one non-zero entry in each row and column, Lemma 3.6 implies that the operator \( a^{(\epsilon)} \) with matrix entries

\[
a^{(\epsilon)}_{xy} := \begin{cases} a_{xy} & |a_{xy}| \geq \epsilon \\ 0 & |a_{xy}| < \epsilon \end{cases}
\]

is well-defined, bounded, and that the collection \( (a^{(\epsilon)}_{\epsilon})_{\epsilon > 0} \) satisfies \( \|a^{(\epsilon)} - a\| \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). Clearly each conjugate \( v^* a^{(\epsilon)} v \) is in \( C^*_u(X; \mathcal{E}_A) \), however, so we are done.

If \( B \) is abstractly isomorphic to some \( \ell^\infty(I) \) for some set \( I \), then we can do better.

**Proposition 3.16.** With notation as in Lemma 3.15, assume moreover that \( B \) is abstractly isomorphic to \( \ell^\infty(I) \) for some set \( I \). Then the inclusions \( v^* B v \subseteq \ell^\infty(X) \), and \( v^* A v \subseteq C^*_u(X; \mathcal{E}_A) \) are equalities.

**Proof.** The fact that \( v^* B v = \ell^\infty(X) \) follows from part (ii) of Lemma 2.3. To see that \( v^* A v = C^*_u(X; \mathcal{E}) \), Lemma 3.10 implies that it suffices to show that for each \( a \in N_B(A) \) and each \( \epsilon > 0 \), if \( E = E_{a,\epsilon} \), then the partial isometry \( vE \) is in \( v^* A v \). Define \( f \in \ell^\infty(X) \) by

\[
f(x) := \begin{cases} (a_{xy})^{-1} & |a_{xy}| \geq \epsilon \\ 0 & |a_{xy}| < \epsilon \end{cases};
\]

this definition makes sense as Lemma 3.6 implies that the matrix underlying \( a \) has at most one non-zero entry in each row. Noting that \( f \in \ell^\infty(X) = v^* B v \subseteq v^* A v \), we get that \( vE = f v^* a v \) is in \( v^* A v \) and the proof is complete.

The next definition and theorem formalizes much of the above discussion.

**Definition 3.17.** Let \( \mathcal{A} \) be the collection\(^8\) of triples \( (A, B, K) \), where \( A \) is a unital \( C^*-\)algebra, \( B \subseteq A \) is a Cartan subalgebra abstractly \(*\)-isomorphic to \( \ell^\infty(I) \) for some set \( I \), and \( K \) is an essential ideal of \( A \) that is abstractly \(*\)-isomorphic to the compact operators on some Hilbert space. Let \( \mathcal{X} \) be the collection of connected, bounded geometry coarse spaces \( (X, \mathcal{E}) \).

\(^8\)It is not a set; as usual, there are ways around this difficulty.

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Define correspondences
\[ \Phi : \mathcal{X} \to A, \quad (X, \mathcal{E}) \mapsto (C_u^*(X; \mathcal{E}), \ell^\infty(X), K(\ell^2(X))) \]
(notation on the right as in Definition 3.8) and
\[ \Psi : A \to \mathcal{X}, \quad (A, B, K) \mapsto (X, \mathcal{E}_A) \]
(notation on the right as in Definition 3.12).

The following theorem is essentially a more general version of Theorem 1.8 from the introduction.

**Theorem 3.18.** The two correspondences \( \Phi : \mathcal{X} \to A \) and \( \Psi : A \to \mathcal{X} \) are well-defined. Moreover, the compositions \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \) are both ‘isomorphic to the identity’ in the following precise senses.

For \( \Phi \circ \Psi \), for any triple \((A, B, K)\), let \( H \) be a representation as in Remark 3.13; then there is a unitary isomorphism \( v : \ell^2(X) \to H \) such that
\[
\begin{align*}
v^* Av &= C_u^*(X; \mathcal{E}_A), \\
v^* Bv &= \ell^\infty(X), \quad \text{and} \\
v^* K v &= K(\ell^2(X)).
\end{align*}
\]

For \( \Psi \circ \Phi \): for any \((X, \mathcal{E})\), letting \( A = C_u^*(X; \mathcal{E}) \) and identifying the set of minimal projections in \( \ell^\infty(X) \) with \( X \), we have that \( \mathcal{E} = \mathcal{E}_A \).

**Proof.** The correspondence \( \Phi : \mathcal{X} \to A \) takes values in \( A \) by Proposition 3.11. The correspondence \( \Psi : A \to \mathcal{X} \) takes values in \( \mathcal{X} \) by Lemma 3.14.

The statement about the composition \( \Phi \circ \Psi \) follows immediately from Proposition 3.16.

To see the stated property for \( \Psi \circ \Phi \), we first show that \( \mathcal{E}_A \subseteq \mathcal{E} \). For this it suffices to show that each set \( E_{a, \epsilon} \) from Definition 3.12 is contained in \( \mathcal{E} \). If not, then there is \( a \in \mathcal{N}_{C_u^*(X; \mathcal{E})}(\ell^\infty(X)) \) and \( \epsilon > 0 \) such that the set \( \{(x, y) \in X \times X \mid |a_{xy}| \geq \epsilon\} \) is not in \( \mathcal{E} \). This however implies that for any \( b \in C_u[X; \mathcal{E}] \), the matrix representation of \( b - a \) will contain non-zero entries of size at least \( \epsilon \), and hence has norm at least \( \epsilon \), contradicting that \( C_u[X; \mathcal{E}] \) is dense in \( C_u^*(X; \mathcal{E}_A) \).

For the reverse inclusion \( \mathcal{E} \subseteq \mathcal{E}_A \), let \( E \) be an arbitrary element of \( \mathcal{E} \). Lemma 3.4 gives us a decomposition
\[
E = \bigcup_{n=1}^{N} E_n
\]
of $E$ into sets $E_n$ whose intersection with each row and column contains at most one element. Then with the notation of Definition 3.9, $v^E_n$ is a well-defined partial isometry for each $n$. With the notation of Definition 3.12, we have that $E_{v,n,1/2} = E_n$, and thus $E_n$ is contained in $E_A$. As this is true for each $n$, $E$ is contained in $E_A$, and we are done.

We need one more result before proving Theorems 1.8 and 1.10 from the introduction.

**Lemma 3.19.** Let $(A, B, K)$ and $(X, E)$ correspond to each other under the constructions of Definition 3.17 and Theorem 3.18. Then $B$ is co-separable in $A$ if and only if $E_A$ is countably generated. In particular, $B$ is co-separable in $A$ if and only if $E_A$ is metrizable.

**Proof.** Say first that $E_A$ is countably generated, say by $E_1, E_2, \ldots$. Then Lemma 3.4 allows us to decompose each $E_m$ into finitely many parts

$$E_m = \bigcup_{n=1}^{N_m} E_n^m$$

such that each $E_n^m$ only intersects each row and column at most once. Lemma 3.10 then gives us a countable set of operators $\{v^{E_n^m} \mid m \geq 1, 1 \leq n \leq N_m\}$ that together with $B \cong \ell^\infty(X)$ generate $A \cong C^*(X, E)$. Hence $B$ is co-separable in $A$.

Conversely, say $B$ is co-separable in $A$. Let $\{a_1, a_2, \ldots\}$ be a countable subset of $\mathcal{N}_B(A)$ such that $S := B \cup \{a_1, a_2, \ldots\}$ generates $A$; we may assume $S$ is closed under taking adjoints. Let $D = \{(x, x) \mid x \in X\}$ be the diagonal in $X \times X$. We claim that the coarse structure $\mathcal{D}$ generated by the countable collection $\{E_{a_n,1/m} \mid n, m \geq 1\} \cup \{D\}$ of sets in $\mathcal{E}$ is actually equal to $\mathcal{E}$. Let $S$ equal the collection $\{a_1, a_2, \ldots\} \cup B$, which generates $A$ by assumption.

We first claim by induction on $n$ that if $b = b_1 \ldots b_n$ is a finite product of elements from $S$, and $\epsilon > 0$ then the set $\{(x, y) \in X \times X \mid |b_{xy}| \geq \epsilon\}$ is in $\mathcal{D}$. Indeed, this is clear if $n = 1$. Given the claim for products of $n - 1$ elements of $S$, say we are given $b = b_0b_n$, where $b_0$ is a product of at most $n - 1$ elements from $S$; we may assume $b_0$ and $b_n$ are non-zero. Then using that $b_0$ and $b_n$ both have at most one non-zero matrix entry in each row and column, we have that

$$\{(x, y) \mid |b_{xy}| \geq \epsilon\} \subseteq \{(x, y) \mid |(b_0)_{xy}| \geq \epsilon/\|b_n\|\} \cap \{(x, y) \mid |(b_n)_{xy}| \geq \epsilon/\|b_0\|\},$$
and thus is in $D$. Now, assume that $c \in A$ is such that $c = b_1 + \cdots + b_n$, where each $b_i$ is a product of finitely many elements of $S$. Then for any $\epsilon > 0$ we have that

$$\{(x, y) \mid |c_{xy}| \geq \epsilon\} \subset \bigcup_{i=1}^{n} \{(x, y) \mid |(b_i)_{xy}| \geq \epsilon/n\},$$

and so the set on the left is in $D$ too.

Having got through these preliminaries, we will now use them to show that for any $a \in N_B(A)$ and $\epsilon > 0$ the set $E_{a, \epsilon}$ is in $D$; this suffices to prove that $D = E$. As $a$ is in $A$ and our set $S$ generates, we can find an element $c$ which is a finite sum of finite products of elements from $S$ such that $\|c - a\| < \epsilon/2$. This implies that the difference between the $(x, y)^{\text{th}}$ matrix coefficients of $a$ and $c$ is at most $\epsilon/2$ for any $(x, y)$, so in particular

$$E_{a, \epsilon} = \{(x, y) \mid |a_{xy}| \geq \epsilon\} \subset \{(x, y) \mid |c_{xy}| \geq \epsilon/2\}.$$

As we have already shown that the set on the right is in $D$, we are done.

The remaining comment about metrizability is immediate from Theorem 3.3.

\[\square\]

**Proof of Theorem 1.8.** Immediate from Theorem 3.18 and Lemma 3.19.

**Proof of Theorem 1.10.** With notation as in the corollary, we have that $C_u^*(X) \cong C_u^*(Y)$. As $C_u^*(X)$ is nuclear, $X$ has Yu’s property A by [3, Theorem 5.5.7]. The result now follows from [25, Theorem 1.4].

\[\square\]

### 4 Uniqueness of Cartan subalgebras

We start this section with a proof of Theorem 1.12, which is a reasonably straightforward consequence of the results of the previous section combined with the main results of [25] and a theorem of Whyte from [26].

**Proof of Theorem 1.12.** Let $Y$ be as in Theorem 1.8, and identify $B$ with $\ell^\infty(Y)$, and $A$ with $C_u^*(Y)$. Choose an orthonormal basis $\{\xi_y\}_{y \in Y}$ that is compatible with the identification $B \cong \ell^\infty(Y)$: precisely, if the
minimal projection in $B$ corresponding to the characteristic function of \{y\} is $p_y$, then choose $\xi_y$ to be a unit vector in the image of $p_y$. Applying \cite{25} Theorem 1.4 gives a coarse equivalence from $X$ to $Y$, and out assumption that $X$ is bijectively rigid gives us a bijective coarse equivalence $f : X \rightarrow Y$.

Now, define a map $u : \ell^2(X) \rightarrow \ell^2(X)$ by $u \delta_x = v_{f(x)}$, which is a unitary isomorphism, as $f$ is a bijection. Using that $f$ is a coarse equivalence, it follows that conjugation by $u$ takes $A$ to $C_u^*(Y)$, or in other words, $u$ conjugates $A$ to itself. Define $\alpha : A \rightarrow A$ by $\alpha(a) = uau^*$; we then have $\alpha(\ell^\infty(X)) = u\ell^\infty(X)u^* = B$ as required.

\begin{remark}
It does not seem to be clear if the unitary $u$ produced by the above proof is actually in $A$ (or can be chosen to be in $A$), so we cannot conclude that the automorphism $\alpha$ in the statement of Theorem \ref{thm:1.12} is inner. Note that while any automorphism of a uniform Roe algebra $C_u^*(X)$ is induced by a unitary $u : \ell^2(X) \rightarrow \ell^2(X)$ (see \cite{25} Lemma 3.1) there are typically many non-inner automorphisms of $C_u^*(X)$: for example, take $X = \mathbb{Z}$ with its usual metric, and the automorphism given by conjugation by the unitary $u : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$, $\delta_n \mapsto \delta_{-n}$.
\end{remark}

In the remainder of this section, we prove our main result, Theorem \ref{thm:1.14}; notation is as in the statement of that theorem for the rest of the section.

We start by stating a general result of Špakula-Tikuisis from \cite{24} (technically, a uniform version of it, but this follows from the same proof).

\begin{thm}
Let $Z$ be a bounded geometry metric space with FDC, and let $E$ be a subset of $B(\ell^2(Z))$. Assume that $E$ has the following property: for all $\epsilon > 0$ there exists $s > 0$ such that if $A, B \subseteq Z$ satisfy $d(A, B) > s$ and $a \in E$, then $\|\chi_A a \chi_B\| < \epsilon \|a\|$.

Then for any $\epsilon > 0$ there exists $s > 0$ such that for each $a \in E$, there is an operator $b \in C_u^*(Z)$ with propagation at most $s$ such that $\|a - b\| \leq \epsilon \|a\|$.
\end{thm}

Let now $B \subseteq C_u^*(X)$ satisfy the assumptions of Theorem \ref{thm:1.14}, so $B$ is a co-separable $\ell^\infty$-Cartan in $A$ as in Definition \ref{def:1.7}. Theorem \ref{thm:1.7} (with $H = \ell^2(X)$) implies that there is a bounded geometry metric space $Y$, and a unitary isomorphism $v : \ell^2(Y) \rightarrow \ell^2(X)$ such that $v\ell^\infty(Y)v^* = B$ and $vC_u^*(Y)v^* = C_u^*(X)$. 31
Lemma 4.3. With notation as above, the space $Y$ has FDC.

Proof. As $X$ has FDC, it has Yu’s property A by [11, Theorem 4.4]. Hence $C_u^*(X)$ is nuclear by [23, Theorem 5.3] or [3, Theorem 5.5.7]. Theorem 1.10 then implies that $X$ and $Y$ are coarsely equivalent. In particular as $X$ has FDC, and as FDC is invariant under coarse equivalences [11, 3.3], this implies that $Y$ has FDC too. 

Note that at this point, our set up has become symmetric in some sense. Precisely, we now have two spaces $X$ and $Y$ with FDC, and a unitary isomorphism $v : \ell^2(Y) \rightarrow \ell^2(X)$ that conjugates $C_u^*(Y)$ to $C_u^*(X)$. Our task is to show that there is some $u \in C_u^*(X)$ that conjugates $v \ell^2(Y) v^*$ to $\ell^2(X)$, but we could just as well do the same thing with the roles of $X$ and $Y$ reversed.

We will need some notation that will be used throughout the rest of this section. For each $y \in Y$, let $q_y \in B(\ell^2(Y))$ denote the orthogonal projection onto the span of $\delta_y$, and let $p_y = v q_y v^* \in B$. Similarly, for each $x \in X$, let $p_x \in B(\ell^2(X))$ be the orthogonal projection onto the span of $\delta_x$, and set $q_x := v^* p_x v$. Thus the families $\{q_y\}_{y \in Y}$ and $\{q_x\}_{x \in X}$ (respectively, $\{p_y\}_{y \in Y}$ and $\{p_x\}_{x \in X}$) are complete sets of orthogonal rank one projections on $\ell^2(Y)$ (respectively on $\ell^2(X)$). For a subset $A$ of $X$ (respectively, of $Y$) define

$$\chi_A := \sum_{x \in A} p_x \left( \text{respectively, } \chi_A := \sum_{y \in Y} q_y \right)$$

for the corresponding multiplication operator on $\ell^2(X)$ (respectively, on $\ell^2(Y)$).

We now give some technical lemmas that reduce the proof of Theorem 1.14 to more concrete problems.

Lemma 4.4. There is a unitary $u \in C_u^*(X)$ such that $u B u^* = \ell^2(X)$ if and only if there is a bijection $h : X \rightarrow Y$ such that for all $\epsilon > 0$ there exists $s > 0$ such that if $A, B \subseteq X$ satisfy $d(A, B) > s$ then

$$\left\| \sum_{x \in A} p_{h(x)} \chi_B \right\|_{B(\ell^2(X))} < \epsilon.$$ 

Proof. Let $\xi_y = v \delta_y$ for each $y \in Y$, so $\{\xi_y\}_{y \in Y}$ is an orthonormal basis for $\ell^2(Y)$. Any unitary $u \in B(\ell^2(X))$ with $u B u^* = \ell^2(X)$ is necessarily of the form

$$u : \xi \mapsto \sum_{y \in Y} \lambda_y \langle \xi_y, \xi \rangle \delta_{h^{-1}(y)}$$

(7)
for some bijection \( h : X \to Y \) and collection \( \{\lambda_y\}_{y \in Y} \) of complex numbers of unit modulus. For any \( \xi \in \ell^2(X) \) and \( A, B \subseteq X \), consider

\[
\|\chi_A u \chi_B\|\|^2 = \left\| \sum_{y \in Y} z_y \langle \xi_y, \chi_B \xi \rangle \chi_A \delta_{h^{-1}(y)} \right\|^2 = \sum_{y \in h(A)} \left| \langle \xi_y, \chi_B \xi \rangle \right|^2
\]

\[
= \left\| \sum_{x \in A} p_{h(x)} \chi_B \xi \right\|^2.
\]

Taking the supremum over all unit vectors \( \xi \in \ell^2(X) \), we see that if \( u \) and \( h \) are connected via the formula in line (7) then for any \( A, B \subseteq X \) we have

\[
\|\chi_A u \chi_B\| = \left\| \sum_{x \in A} p_{h(x)} \chi_B \right\|.
\] (8)

Now, assume that we have a bijection \( h \) satisfying the assumptions in the statement, and define \( u \) by line (7) above with \( z_y = 1 \) for all \( y \in Y \). Line (8), the assumption on \( h \) in the statement, and Theorem 1.2 applied to the set \( E = \{u\} \) imply that \( u \) is in \( C^*_u(X) \).

Conversely, if \( u \in C^*_u(X) \) is a unitary such that \( uBu^* = \ell^\infty(X) \) let \( h : X \to Y \) be the bijection from line (7) above. Then for any \( \epsilon > 0 \) there exists an \( s > 0 \) and an operator \( u_s \in C_u[X] \) of propagation at most \( s \) such that \( \|u - u_s\| < \epsilon \). Hence if \( d(A, B) > s \) then

\[
\|\chi_A u \chi_B\| \leq \|\chi_A (u - u_s) \chi_B\| + \|\chi_A u_s \chi_B\| < \epsilon + 0.
\]

Comparing this to line (8) above implies that \( h \) satisfies the condition in the statement of the lemma. \( \square \)

The next lemma is a variant on [25, Lemma 3.2].

**Lemma 4.5.** Let \( (a_n)_{n \in \mathbb{N}} \) be a sequence of mutually orthogonal operators in \( C^*_u(X) \), and assume that for any \( I \subseteq \mathbb{N} \), the sum \( \sum_{n \in I} a_n \) converges strongly to an operator in \( C^*_u(X) \). Then for any \( \epsilon > 0 \) there is \( s > 0 \) such that if \( A, B \subseteq X \) satisfy \( d(A, B) > s \), then for any \( I \subseteq \mathbb{N} \),

\[
\|\chi_A \left( \sum_{n \in I} a_n \right) \chi_B\| < \epsilon.
\]

**Proof.** A special case of [25, Lemma 3.2] states that the following holds: “If \( (b_m)_{m \in \mathbb{N}} \) is any sequence of mutually orthogonal operators in \( C^*_u(X) \), and if for any \( I \subseteq \mathbb{N} \), the sum \( \sum_{m \in I} b_m \) converges strongly
to an operator in $C^*_u(X)$, then for any $\epsilon > 0$ there is $s > 0$ such that
if $A, B \subseteq X$ satisfy $d(A, B) > s$, then for any $m \in \mathbb{N}$
\[ \| \chi_{A^m} b_m \chi_{B} \| < \epsilon. \]

Now, assume for contradiction that the statement of the lemma does not hold, so there exists an $\epsilon > 0$ such that for any $r > 0$ there is a subset $I_r$ of $\mathbb{N}$ and $A_r, B_r \subseteq X$ with $d(A, B) > r$ and
\[ \| \chi_{A_r} \left( \sum_{n \in I_r} a_n \right) \chi_{B_r} \| \geq \epsilon. \] (9)

We will iteratively construct a sequence $(r_m)$ of positive real numbers tending to infinity, finite disjoint subsets $(F_m)$ of $\mathbb{N}$, and subsets $A_m, B_m \subseteq X$ with $d(A_m, B_m) > r_n$ such that
\[ \| \chi_{A_m} \left( \sum_{n \in F_m} a_n \right) \chi_{B_m} \| \geq \epsilon/2. \]

This will suffice to complete the proof: indeed, on setting $b_m := \sum_{n \in F_m} a_n$, we get a contradiction to the statement above from [25, Lemma 3.2].

Set then $r_1 = 1$, and let $I_1$ and $A_1, B_1$ be as in line (9) above. As the sum
\[ \chi_{A_1} \left( \sum_{n \in F_1} a_n \right) \chi_{B_1} \]

is strongly convergent, there must exist a finite subset $F_1$ of $I_1$ such that
\[ \| \chi_{A_1} \left( \sum_{n \in F_1} a_n \right) \chi_{B_1} \| \geq \epsilon/2. \]

Now say that $r_1, \ldots, r_m, F_1, \ldots, F_m, A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ have all been chosen. Set $F = \bigcup_{n=1}^{m} F_m$. Then for each $F' \subseteq F$ the operator
\[ b_{F'} = \sum_{n \in F'} a_n \]

is in $C^*_u(X)$; as the set $\{ b_{F'} \mid F' \subseteq F \}$ is finite, there exists $r \geq m + 1$ such that whenever $A, B \subseteq X$ are such that $d(A, B) > r$ we have
\[ \| \chi_{A^m} b_{F'} \chi_{B} \| < \epsilon/4. \] (10)
Set $r_{m+1} := r$. Let now $I_r$, $A_r$ and $B_r$ be as in line (9) above. Then by strong convergence of the sum in line (9) there exists a finite subset $G$ of $I_r$ with

$$\left\| \chi_{A_r} \left( \sum_{n \in G} a_n \right) \chi_{B_r} \right\| \geq 3\epsilon/4.$$  \hfill (11)

Set $A_{m+1} = A_r$, $B_{m+1} = B_r$ and $F_{m+1} = G \setminus F$ (so in particular, $F_{m+1}$ is disjoint from $F_1, ..., F_m$). Then

$$
\left\| \chi_{A_{m+1}} \left( \sum_{n \in F_{m+1}} a_n \right) \chi_{B_{m+1}} \right\|
\geq\left\| \chi_{A_{m+1}} \left( \sum_{n \in G} a_n \right) \chi_{B_{m+1}} \right\| - \left\| \chi_{A_{m+1}} \left( \sum_{n \in G \cap F} a_n \right) \chi_{B_{m+1}} \right\|.
$$

Lines (10) and (11) imply this is at least $\epsilon/2$, which completes the iterative construction, and the proof.

\[\square\]

**Corollary 4.6.** For any $\epsilon > 0$ there exists $s > 0$ such that for any $A \subseteq X$ and any $B \subseteq Y$ there is a $e \in C^*_u(X)$ with propagation at most $s$, and

$$\left\| v\chi_B v^* \chi_A - a \right\| \leq \epsilon \left\| v\chi_B v^* \chi_A \right\|.$$

The content of the corollary is the order of the quantifiers: the analogous statement with order of quantifiers $\forall \epsilon \forall A \forall B \exists s$ is immediate.

**Proof.** Note that $v\chi_B v^* = \sum_{y \in B} p_y$. Applying Lemma 4.5 to the mutually orthogonal family $\{p_y\}_{y \in Y}$ we get that for all $\epsilon > 0$ there exists $s_0 > 0$ such that for any $B \subseteq Y$, if $C, D \subseteq X$ satisfy $d(C, D) > s_0$, then

$$\left\| \chi_C v\chi_B v^* \chi_D \right\| < \epsilon.$$

For any $B \subseteq Y$ and any $A \subseteq X$ we have that $d(C, D) > s_0$ implies that $d(C, A \cap D) > s_0$ and thus the above also gives that $d(C, D) > s_0$ implies

$$\left\| \chi_C v\chi_B v^* \chi_{A \cap D} \right\| = \left\| \chi_C v\chi_B v^* \chi_A \right\| < \epsilon.$$

Theorem 4.2 then gives the desired approximants. \[\square\]

We now recall the definition of the operator localization property (ONL) from [4, Definition 2.2]. The version we give below is equivalent to the usual one by [4, Proposition 2.4].
Definition 4.7. A bounded geometry metric space $X$ has the operator norm localization property (ONL) if for any $\epsilon \in (0, 1)$ and any $s > 0$ there is $r > 0$ such that for any operator $a \in C_0^*(X)$ with propagation at most $s$ there exists a unit vector $\xi \in \ell^2(X)$ with

$$\|a\xi\| \geq (1 - \epsilon)\|a\|$$

and with $\xi$ supported in a set of diameter at most $r$.

Again, the point is order of quantifiers: with the order $'\forall\epsilon\forall s\forall a\exists r'$, the analogous statement is automatic.

Lemma 4.8. The spaces $X$ and $Y$ have ONL.

Proof. Recall that $X$ has FDC by assumption, and that $Y$ has FDC by Lemma 4.3. FDC implies ONL by [5, Corollary 3.4], so our spaces $X$ and $Y$ have ONL (alternatively, one can instead use [22, Theorem 4.1], which shows that ONL is equivalent to property A, and [11, Theorem 4.4], which shows that FDC implies property A). \qed

Lemma 4.9. For any $\epsilon \in (0, 1)$ there exists $r > 0$ such that for any $A \subseteq X$ and $B \subseteq Y$, there is $C \subseteq X$ with diam$(C) < r$ and

$$\|v\chi_Bv^*\chi_{A\cap C}\| \geq (1 - \epsilon)\|v\chi_Bv^*\chi_{A}\|.$$  

Proof. Fix $\delta > 0$ for the moment. Using Lemma 4.6 there exists $s > 0$ (depending only on $\delta$) such that for any $A \subseteq X$ and any $B \subseteq Y$, there is $a \in C_0^*(X)$ with propagation less than $s$ and

$$\|v\chi_Bv^*\chi_{A} - a\| < \delta\|v\chi_Bv^*\chi_{A}\|. \quad (12)$$

Using the operator norm localization property, there exists $r > 0$ (depending only on $s$ and $\delta$) such that there is a unit vector $\xi \in \ell^2(X)$ with support in a set of diameter at most $r$ such that

$$\|a\xi\| \geq (1 - \delta)\|a\|. \quad (13)$$

Write $b = v\chi_Bv^*\chi_{A}$. Then applying line (12)

$$\|(a - b)\xi\| \leq \|a - b\| < \delta\|b\|.$$ 

As moreover $\|a\xi - b\xi\| \geq \|b\xi\| - \|a\xi\|$, we get

$$\|b\xi\| > \|a\xi\| - \delta\|b\| > (1 - \delta)\|a\| - \delta\|b\|.$$
On the other hand, line (12) again implies that
\[ \|a\| \geq (1 - \delta)\|b\|, \]
and combining this with the line above gives
\[ \|b\xi\| > (1 - \delta^2 - \delta)\|b\|. \]
Letting \( C \) be the support of \( \xi \), this implies that
\[ v\chi_B v^* \chi_{A \cap C} \geq (1 - \delta^2 - \delta)\|v\chi_B v^* \chi_A\|. \]
Taking \( \delta \) small enough that \((1 - \delta^2 - \delta) > 1 - \epsilon \) completes the proof.

Now, we need some more notation. For each \( y \in Y \) and \( \delta > 0 \), define
\[ X_{y,\delta} := \{ x \in X \mid \|p^* y\|^2 \geq \delta \}. \]
Analogously, define
\[ Y_{x,\delta} := \{ y \in Y \mid \|q^* y\|^2 \geq \delta \}, \]
and note that as
\[ \|p^* y\|^2 = \|p^* v_q v\|^2 = \|vp^* v_q\|^2 = \|q^* y\|^2, \]
we have that \( x \) is in \( X_{y,\delta} \) if and only if \( y \) is in \( Y_{x,\delta} \).

**Lemma 4.10.** With notation as above:

(i) for each \( \epsilon > 0 \), there is \( \delta > 0 \) such that for all \( y \in Y \), \( \|p_y \chi_{X_{y,\delta}}\|^2 \geq 1 - \epsilon \);

(ii) for each \( \delta > 0 \) there exists \( r > 0 \) such that for all \( y \in Y \), the diameter of \( X_{y,\delta} \) is at most \( r \).

**Proof.** Look first at part (i). Applying Lemma 4.9 with \( B = \{ y \} \) and \( A = X \), there is \( r > 0 \) depending only on \( \epsilon \) such that for each \( y \in Y \) there is \( C \subseteq X \) with \( \text{diam}(C) < r \) and \( \|p_y \chi_C\|^2 > (1 - \epsilon/2)\|p_y\|^2 \), or in other words so that
\[ \|p_y \chi_C\|^2 > 1 - \epsilon/2 \]
Let \( \xi_y \) be any unit vector in the range of \( p_y \), and note that \( \|p_y \chi_C\|^2 = \|\chi_C p_y\|^2 = \|\chi_C \xi_y\|^2 \), so this says that
\[ \sum_{x \in C} |\xi_y(x)|^2 > 1 - \epsilon/2. \] (14)
Let \( N \in \mathbb{N} \) be an absolute bound on the cardinalities of all balls of radius \( r \) in \( X \), and let \( \delta < \frac{r}{2N} \) (which only depends on \( r \) and \( \epsilon \), so only on \( \epsilon \)). Then
\[
\| p_y \chi_{X_{y,\delta}} \|^2 = \sum_{x \in X_{y,\delta}} |\xi_y(x)|^2 \\
\geq \sum_{x \in C \cap X_{y,\delta}} |\xi_y(x)|^2 \\
= \sum_{x \in C} |\xi_y(x)|^2 - \sum_{C \cap X_{y,\delta}} |\xi_y(x)|^2.
\]
Now, on the one hand line (14) gives \( \| p_y \chi_{X_{y,\delta}} \|^2 > 1 - \epsilon/2 \), and on the other hand \( |\xi_y(x)|^2 = \| p^* p_y \|^2 < \delta \) for all \( x \notin X_{y,\delta} \) and \( |C| \leq N \) whence \( \sum_{C \cap X_{y,\delta}} |\xi_y(x)|^2 < N \delta \). The above inequality thus implies
\[
\| p_y \chi_{X_{y,\delta}} \|^2 > 1 - \epsilon - N \delta,
\]
and this is at least \( 1 - \epsilon \) by choice of \( \delta \).

For part (ii), let \( \delta > 0 \). As in part (i) let \( \xi_y \) be any unit vector in the image of \( p_y \). Again, just as in part (i) we can apply Lemma 4.9 with \( B = \{ y \} \) and \( A = X \) to find \( r > 0 \) depending only on \( \delta \) such that for all \( y \in Y \) there exists \( C \subseteq X \) with \( \text{diam}(C) < r \) and such that
\[
\sum_{x \in C} |\xi_y(x)|^2 > 1 - \delta.
\]
This implies that \( X_{y,\delta} \subseteq C \), however, as if not the sum above differs from \( \sum_{x \in X} |\xi_y(x)|^2 = 1 \) by a term of size at least \( \delta \), which is impossible. \( \square \)

At this point, recall that our set-up is symmetric, so that all the lemmas above hold with the roles of \( X \) and \( Y \) interchanged. Looking at Lemma 4.9 this gives us the following.

**Lemma 4.11.** For any \( \epsilon \in (0,1) \) there exists \( r > 0 \) such that for any \( A \subseteq X \) and any \( B \subseteq Y \), there is \( D \subseteq Y \) with \( \text{diam}(D) < r \) and
\[
\| v \chi_{B \cap D} v^* \chi_A \| > (1 - \epsilon) \| v \chi_B v^* \chi_A \|. \qed
\]

**Lemma 4.12.** For any \( \epsilon > 0 \), there is \( \delta > 0 \) such that for any subset \( B \) of \( Y \), if \( X_{B,\delta} := \bigcup_{y \in B} X_{y,\delta} \), then
\[
\| v \chi_B v^* (1 - \chi_{X_{B,\delta}}) \| < \epsilon.
\]
In particular, for all suitably small \( \delta \), the cardinality of \( X_{B,\delta} \) is at least as large as that of \( B \).

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Proof. Fix $\gamma > 0$, to be chosen later in a way depending only on $\epsilon$. Using Lemma 4.11 there is $r > 0$ such that for any $A \subseteq X$ and $B \subseteq Y$ we have $D \subseteq Y$ with $\text{diam}(D) < r$ such that

$$\|v\chi_{B \cap D}^*\chi_A\| \geq (1 - \gamma)\|v\chi_{B}^*\chi_A\|.$$ 

Hence

$$\|v\chi_{B}^*\chi_A\| \leq \frac{1}{1 - \gamma}\|v\chi_{B \cap D}^*\chi_A\|. \quad (15)$$

Let $M$ be some large positive number, to be chosen later (in a way that depends only on $r$, so only on $\gamma$, so only on $\epsilon$). Applying Lemma 4.10 gives $\delta > 0$ such that $\|p_y\chi_{X_{y,\delta}}\| \geq 1 - \frac{1}{M}$ for all $y \in Y$, whence in particular for any $y \in B$

$$\|p_y(1 - \chi_{X_{y,\delta}})\| \leq \|p_y(1 - \chi_{X_{y,\delta}})\| \leq 1/M. \quad (16)$$

Now, apply line (15) with $A = X \setminus X_{B,\delta}$ and line (16) to get

$$\|v\chi_{B}^*\chi_A\| \leq \frac{1}{1 - \gamma}|D|\sup_{y \in B}\|p_y(1 - \chi_{X_{y,\delta}})\| \leq \frac{1}{1 - \gamma}|D|\frac{1}{M}.$$ 

Bounded geometry of $Y$ tells us that $|D|$ is bounded by some number that depends only on $r$, so this implies the desired inequality for suitably large $M$ and small $\gamma$, both depending only on $\epsilon$.

To see that the inequality implies the cardinality statement, let $\epsilon = 1/2$, and $\delta$ be any number satisfying the condition in the statement. We need to see that the rank of $\chi_{X_{y,\delta}}$ is at least as big as that of $v\chi_{B}^*$. If not, then the rank of $v\chi_{B}^*$ is strictly larger than that of $\chi_{X_{y,\delta}}$; this forces the images of $v\chi_{B}^*$ and $1 - \chi_{X_{y,\delta}}$ to have non-trivial intersection and thus $\|v\chi_{B}^*(1 - \chi_{X_{y,\delta}})\| \geq 1$, contradicting the inequality.

Lemma 4.13. Let $\epsilon \in (0,1)$, and let $\delta > 0$ be any number with the property in Lemma 4.12 for this $\epsilon$. There exists an injection $f : Y \rightarrow X$ such that $f(y) \subseteq X_{y,\delta}$ for all $y \in Y$.

Proof. Consider the function $\phi : Y \rightarrow \mathcal{P}(X)$ defined by $\phi(y) = X_{y,\delta}$. Using Hall’s marriage theorem, it suffices to show that for any finite subset $F$ of $Y$,

$$\left| \bigcup_{y \in F} \phi(y) \right| \geq |F|.$$ 

In the notation of Lemma 4.12, $\bigcup_{y \in F} \phi(y)$ is just $X_{F,\delta}$. The desired inequality is thus the cardinality statement from Lemma 4.12. \qed
Now, again everything is symmetric, so we get the following lemma.

**Lemma 4.14.** For any suitably small $\delta > 0$, there exists an injection $g : Y \to X$ such that $g(x) \subseteq Y_{x, \delta}$ for all $x \in X$. \hfill \Box

Finally, we are ready to finish the proof of Theorem 1.14.

**Proof of Theorem 1.14.** Let $\delta > 0$ be small enough so that Lemmas 4.13 and 4.14 give us injections $f : Y \to X$ and $g : X \to Y$ with the properties stated there. König’s proof of the Cantor-Schröder-Bernstein theorem gives us a bijection $h : X \to Y$ with the property that either $h(x) = g(x)$ or $x$ is in the image of $f$, and $h(x) = f^{-1}(x)$ for every $x \in X$.

We claim that there exists $r > 0$ such that for every $x \in X$, $X_{h(x), \delta}$ is contained in the ball $B(x; r)$ centered at $x$ with radius $r$. Indeed, let $r$ equal the supremum of the diameters of the sets $X_{y, \delta}$ as $y$ ranges over $Y$; $r$ is finite by part (ii) of Lemma 4.10. Note first that if $x \in X$ is such that $h(x) = f^{-1}(x)$ for some $x \in X$, then $f(h(x)) = x$ is an element of $X_{h(x), \delta}$ by the properties of $f$. This implies that $X_{h(x), \delta}$ is contained in $B(x; r)$ by choice of $r$. On the other hand, say $x \in X$ is such that $h(x) = g(x)$. Then by the defining property of $g$, $g(x)$ is in $Y_{x, \delta}$, from which it follows that $\|v^* p^* v q_g(x)\|^2 \geq \delta$, where $q_g(x)$ is the rank one projection on $\ell^2(Y)$ with image the span of $\delta_g(x)$, and $v : \ell^2(X) \to \ell^2(Y)$ is our original unitary implementing the given $*$-isomorphism. Hence $\|p^* v q_g(x) v^*\|^2 \geq \delta$, which says exactly that $x$ is in $X_{g(x), \delta}$. The result follows by assumption on the diameter of all of the $X_{y, \delta}$.

To complete the proof, we claim that $h$ has the property in Lemma 4.4 above, which will suffice to complete the proof. Let $\epsilon > 0$ be given. It suffices to show that there is $t > 0$ such that for any subset $A \subseteq X$ we have

$$\|v \chi_{h(A)} v^*: (1 - \chi_{N_t(A)})\| < \epsilon.$$ 

Applying Lemma 4.12 with $B = h(A)$ gives us $\gamma \in (0, \delta)$ such that

$$\|v \chi_{h(A)} v^*: (1 - \chi_{X_{h(A), \gamma}})\| < \epsilon.$$ 

Now, as $\gamma < \delta$, we have that $X_{h(x), \gamma} \supseteq X_{h(x), \delta}$ for all $x \in A$. Let $r$ be such that $X_{h(x), \delta}$ is contained in $B(x; r)$ as above, and let $s$ be such that $X_{h(x), \gamma}$ has diameter at most $s$ for all $x \in A$ (such an $s$ exists by Lemma 4.10 part (ii)). Hence each $X_{h(x), \gamma}$ is contained in $B(x; s + r)$. The result follows with $t = s + r$. \hfill \Box
Proof of Corollary 1.16. The fact that (i) implies (ii) implies (iii) is straightforward. The implication (iii) implies (i) follows as such a \(\ast\)-isomorphism induces a bijection between the minimal projections in \(\ell^\infty(X)\) and those in \(\ell^\infty(Y)\), and thus a bijection \(f : X \to Y\). We claim that \(f\) is uniformly expansive in the sense of Definition 1.9. Indeed, if not, then there is \(r > 0\) and a sequence \(((x^n_1, x^n_2))_{n \in \mathbb{N}}\) of pairs in \(X \times X\) such that \(d(x^n_1, x^n_2) \leq r\) for all \(n\), but such that \(d_Y(f(x^n_1), f(x^n_2)) \to \infty\) as \(n \to \infty\). Passing to a subsequence and using bounded geometry, we may assume that no point of \(X\) appears twice in the set \(\{x^n_1, x^n_2 \mid n \in \mathbb{N}\}\). Now, consider the \(X\)-matrix defined by the condition that \(a_{x^n_1, x^n_2} = 1\) for all \(n\), and all other matrix entries zero. Our assumptions that no element appears twice in \(\{x^n_1, x^n_2 \mid n \in \mathbb{N}\}\) implies that this matrix is supported on a single diagonal, and thus defined a bounded operator \(a\) on \(\ell^2(X)\) by Lemma 3.6. Moreover, the fact that \(d(x^n_1, x^n_2) \leq r\) for all \(n\) implies that \(a\) is in \(C_u[X]\). On the other hand, our isomorphism takes \(a\) to an operator in \(C^\ast_u(Y)\) whose \((f(x^n_1), f(x^n_2))^\text{th}\) matrix entry is one for all \(n\). The assumption that \(d_Y(f(x^n_1), f(x^n_2)) \to \infty\) implies that this is impossible, however. A precisely analogous argument now shows that \(f^{-1}\) is also uniformly expansive, so \(f\) is a coarse equivalence as required.

As (iii) implies (iv) is trivial, it will now suffice to prove (iv) implies (i). Assume that \(C^\ast_u(X)\) and \(C^\ast_u(Y)\) are \(\ast\)-isomorphic. As in [25, Lemma 3.1], there is a unitary isomorphism \(v : \ell^2(X) \to \ell^2(Y)\) such that \(vC^\ast_u(X)v^* = C^\ast_u(Y)\). Let \(B = v\ell^\infty(X)v^*\), which satisfies the assumptions of Theorem 1.14. This needs that \(Y\) has FDC, which follows from [25, Theorem 1.4] and the fact that FDC is a coarse invariant. Hence there is \(u \in C^\ast_u(Y)\) with \(uBu^* = \ell^\infty(Y)\). Now, we have that \(uv\ell^\infty(X)v^*u^* = \ell^\infty(Y)\), whence there is a bijection \(f : X \to Y\) and a family of unit modulus complex numbers \(\{z_x\}_{x \in X}\) with

\[
uv : \delta_x \mapsto z_x \delta_{f(y)}
\]

for all \(x \in X\). We also have that \(uvC^\ast_u(X)v^*u^* = C^\ast_u(Y)\). From this, an analogous argument to that used for (iii) implies (i) shows that \(f\) is a coarse equivalence, so we are done.

References


