UNIFORM ROE ALGEBRAS OF UNIFORMLY LOCALLY FINITE METRIC SPACES ARE RIGID

FLORENT P. BAUDIER, BRUNO M. BRAGA, ILIJAS FARAH, ANA KHUKHRO, ALESSANDRO VIGNATI, AND RUFUS WILLETT

Abstract. We show that if X and Y are uniformly locally finite metric spaces whose uniform Roe algebras, $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$, are isomorphic as C^*-algebras, then X and Y are coarsely equivalent metric spaces. Moreover, we show that coarse equivalence between X and Y is equivalent to Morita equivalence between $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$. As an application, we obtain that if Γ and Λ are finitely generated groups, then the crossed products $\ell_\infty(\Gamma) \rtimes_r \Gamma$ and $\ell_\infty(\Lambda) \rtimes_r \Lambda$ are isomorphic if and only if Γ and Λ are bi-Lipshitz equivalent.

1. Introduction

Coarse geometry is the study of metric spaces when one forgets about the small scale structure and focuses only on large scales. For example, this philosophy underlies much of geometric group theory. As the local structure of a space is irrelevant, one typically assumes that the spaces one is working with are discrete: we will focus here on *uniformly locally finite*[1] metric spaces (X, d_X), meaning that $\sup_{x\in X} |B_r(x)| < \infty$ for all $r > 0$, where $|B_r(x)|$ is the cardinality of the closed ball in X of radius r centered at x. Typical examples that are important for applications are finitely generated groups with word metrics, and discretizations of non-discrete spaces such as Riemannian manifolds. There is a natural *coarse category* of metric spaces considered from a large-scale point of view, and the isomorphisms in this category are called *coarse equivalences*.

Here is the formal definition. Given metric spaces (X, d_X) and (Y, d_Y), a map $f\colon X \to Y$ is *coarse* if for all $r > 0$ there is $s > 0$ so that

$$d_X(x, x') \leqslant r \quad \text{implies} \quad d_Y(f(x), f(x')) \leqslant s$$

for all $x, x' \in X$. If $f : X \to Y$ and $g : Y \to X$ are coarse and

$$\sup_{x\in X} d_X(x, g(f(x))) < \infty \quad \text{and} \quad \sup_{y\in Y} d_Y(y, f(g(y))) < \infty$$

then f and g are called *mutual coarse inverses*, each of f and g is called a *coarse equivalence*, and X and Y are said to be *coarsely equivalent*.

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[1]Also called *bounded geometry* metric spaces in the literature.

Associated to the large-scale structure of a uniformly locally finite metric space is a C*-algebra, i.e., a norm-closed and adjoint-closed algebra of bounded operators on a complex Hilbert space, called the *uniform Roe algebra of* X and denoted by $\mathrm{C}^*_u(X)$. Prototypical versions of this C*-algebra were introduced by Roe [31] for index-theoretic purposes. The theory was consolidated in the 1990s by Roe, Yu and others, and uniform Roe algebras have since found applications in index theory (for example, [36, 16]), C*-algebra theory (for example, [34, 26]), single operator theory (for example, [30, 38]), topological dynamics (for example, [22, 6]), and mathematical physics (for example, [10, 17]).

Here is the formal definition. For a metric space (X, d_X), the *propagation* of an X-by-X matrix $a = [a_{xy}]$ of complex numbers is

$$\mathrm{prop}(a) \coloneqq \sup\{d_X(x, y) \mid a_{xy} \neq 0\} \in [0, \infty].$$

If $a = [a_{xy}]$ has finite propagation and uniformly bounded entries, then a canonically induces a bounded operator on the Hilbert space $\ell_2(X)$ as long as (X, d_X) is uniformly locally finite. For any such (X, d_X), the operators with finite propagation form a $*$-algebra, and $\mathrm{C}^*_u(X)$ is the C*-algebra defined as the norm closure of this $*$-algebra.

For many applications of uniform Roe algebras, one wants to know how much of the underlying metric geometry is remembered by $\mathrm{C}^*_u(X)$. This leads to the foundational question below.

Problem 1.1 (Rigidity of uniform Roe algebras)**.** *If the uniform Roe algebras of uniformly locally finite metric spaces are $*$-isomorphic, are the underlying metric spaces coarsely equivalent?*

Recently, the rigidity problem for uniform Roe algebras has been extensively studied. For example: [37] started this study; [2] introduced several new ideas that are relevant for this paper; and [25] represents the most recent developments before this paper. All of these papers (and others) give positive answers to Problem 1.1 in the presence of additional geometric conditions on the underlying metric spaces.

1.1. Main results.

In this paper, we give an *unconditional* positive answer to the rigidity problem.

Theorem 1.2. *Let X and Y be uniformly locally finite metric spaces. If $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are $*$-isomorphic, then X and Y are coarsely equivalent.*

We will discuss an outline of the proof in Section 1.2 below. For now, let us focus on some applications and elaborations.

A first application of Theorem 1.2 regards groups. Associated to an action of a group Γ on a compact topological space X, there is a C*-algebra *crossed product* $C(X) \rtimes_r \Gamma$ that models the underlying dynamics. In particular, one can do this when $X = \beta\Gamma$ is the Čech-Stone compactification of Γ, which is the universal compact Γ-space in some sense. If Γ is discrete, $C(\beta\Gamma)$ naturally identifies with $\ell_\infty(\Gamma)$, so we get the crossed product $\ell_\infty(\Gamma) \rtimes_r \Gamma$.

If Γ is a finitely generated group, then it becomes a uniformly locally finite metric space when equipped with a word metric. The uniform Roe algebra of Γ then identifies with the C*-algebra crossed product $\ell_\infty(\Gamma)\rtimes_r\Gamma$ discussed above, i.e., there is a canonical $*$-isomorphism $\mathrm{C}^*_u(\Gamma)\cong\ell_\infty(\Gamma)\rtimes_r\Gamma$ (see [9, Proposition 5.1.3]).

The following result is of interest in pure C*-algebra theory and topological dynamics (see Corollary 3.5 below for a more general statement).

Corollary 1.3. *Let Γ and Λ be finitely generated groups. The following are equivalent:*

1. *With any choice of word metrics, Γ and Λ are bi-Lipschitz equivalent.*[2]
2. *The* C**-algebras* $\ell_\infty(\Gamma)\rtimes_r\Gamma$ *and* $\ell_\infty(\Lambda)\rtimes_r\Lambda$ *are $*$-isomorphic.*

Our next main result concerns *Morita equivalence*. This is a notion of isomorphism for C*-algebras that is a little weaker than $*$-isomorphism. Roughly, it says that the C*-algebras involved are $*$-isomorphic 'up to multiplicity', and is typically considered the 'correct' notion of isomorphism for C*-algebras in noncommutative geometry. That Morita equivalence of uniform Roe algebras is connected to coarse equivalence of the underlying spaces seems to have been guessed at by Gromov in the early 90s [19, page 263]. Brodzki, Niblo, and Wright [7, Theorem 4] subsequently showed that coarse equivalence of uniformly locally finite metric spaces implies Morita equivalence of their uniform Roe algebras. Our methods allow us to obtain that the converse also holds.

Theorem 1.4. *Let X and Y be uniformly locally finite metric spaces. The following are equivalent:*

1. *X and Y are coarsely equivalent.*
2. *$\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are Morita equivalent.*

Our findings also allow us to remove the geometric assumptions from the main result of [3]. Precisely, if X is a uniformly locally finite metric space then the compact operators $\mathcal{K}(\ell_2(X))$ are an ideal in $\mathrm{C}^*_u(X)$. The associated quotient is the *uniform Roe corona of X*, denoted by $\mathrm{Q}^*_u(X)$. In [3], the authors investigate whether rigidity also holds given the weaker assumption of isomorphism between uniform Roe coronas. In this paper we obtain the following:

Theorem 1.5. *Let X and Y be uniformly locally finite metric spaces. If $\mathrm{Q}^*_u(X)$ and $\mathrm{Q}^*_u(Y)$ are $*$-isomorphic and one assumes appropriate forcing axioms*[3]*, then X and Y are coarsely equivalent.*

[2] Metric spaces (X,d_X) and (Y,d_Y) are bi-Lipschitz equivalent if there is a bijection $f\colon X\to Y$ such that f and f^{-1} are Lipschitz.

[3] For the set theorist reader, this result is a theorem in $\mathrm{ZFC}+\mathrm{OCA_T}+\mathrm{MA}_{\aleph_1}$.

1.2. **The road to rigidity.** We now discuss our methods of proof in more detail. If H is a Hilbert space then $\mathcal{B}(H)$ denotes the C*-algebra of all bounded operators on H. The strong operator topology on $\mathcal{B}(H)$ is the topology of pointwise convergence on $\mathcal{B}(H)$. We write "SOT" as an abbreviation for "strong operator topology" and "SOT-$\sum$" for a sum that converges in the strong operator topology.

As already noted above, the C*-algebra of compact operators $\mathcal{K}(\ell_2(X))$ is an ideal in $\mathrm{C}^*_u(X)$, and in fact is the unique minimal ideal. As a result, a $*$-isomorphism between uniform Roe algebras of uniformly locally finite metric spaces sends compact operators to compact operators. Isomorphisms of the compact operators must be "spatially implemented", i.e., given by conjugation by an isometric isomorphism between the corresponding Hilbert spaces (see for example [14, Corollary 4.1.8]). From this discussion, it is not difficult to deduce the following result.

Lemma 1.6 ([37, Lemma 3.1])**.** *Let X and Y be uniformly locally finite metric spaces and $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be a $*$-isomorphism. Then there is an isometric isomorphism $u\colon \ell_2(X) \to \ell_2(Y)$ so that $\Phi(a) = uau^*$ for all $a \in \mathrm{C}^*_u(X)$. In particular, Φ is rank-preserving and continuous for the strong operator topology.* □

The automatic SOT-continuity of isomorphisms between uniform Roe algebras will be very important for us. In addition to this basic observation, our proof of Theorem 1.2 has two main ingredients at its core:

(I) the "equi-approximability" of certain families of operators by operators with uniformly bounded propagation (see Lemma 1.8);
(II) a uniform lower bound on certain matrix coefficients.[4]

Let us first look at equi-approximability. We need a definition which quantifies how well a bounded operator can be approximated by a finite propagation operator.

Definition 1.7. Let X be a metric space, $\epsilon > 0$, and $r \geqslant 0$. An operator a in $\mathcal{B}(\ell_2(X))$ is *ϵ-r-approximable* if there exists $b \in \mathcal{B}(\ell_2(X))$ with propagation at most r such that $\|a - b\| \leqslant \epsilon$.

The key "equi-approximability lemma" was obtained as an application of the Baire category theorem and diagonalization methods in [2, Section 4] (a weaker version appeared earlier in [37, Lemma 3.2]).

Lemma 1.8 ([2, Lemma 4.9])**.** *Let X be a uniformly locally finite metric space and let $(a_n)_n$ be a sequence of operators so that SOT-$\sum_{n\in M} a_n$ converges to an element of $\mathrm{C}^*_u(X)$ for all $M \subseteq \mathbb{N}$. Then for all $\varepsilon > 0$ there is $r > 0$ so that SOT-$\sum_{n\in M} a_n$ is ϵ-r-approximable for all $M \subseteq \mathbb{N}$.* □

The second ingredient (II) is a uniform lower bound on certain matrix entries. Given a set X, $(\delta_x)_{x\in X}$ denotes the standard orthonormal basis of

[4]This was formalized as *rigidity* of a $*$-isomorphism in [2, page 1008].

$\ell_2(X)$ and, given $x, y \in X$, e_{xy} denotes the rank 1 partial isometry sending δ_y to δ_x. The current proofs of rigidity in the literature all follow a similar path: given a $*$-isomorphism $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$, one uses some geometric property of Y in order to ensure an inequality of the form

$$\inf_{x\in X} \sup_{y\in Y} \|\Phi(e_{xx})\delta_y\| > 0. \tag{1.1}$$

This inequality was first obtained in [37, Lemma 4.6] under the assumption of Yu's property A (see [45, Definition 2.1]), which is an amenability-like property of metric spaces.

The inequality in line (1.1) was then shown to hold under conditions on the absence of certain *ghost operators* in [2, Section 6]: an operator $a = [a_{xy}]$ on $\ell_2(X)$ is a *ghost* if $\lim_{x,y\to\infty} a_{xy} = 0$. Compact operators are easily seen to be ghosts, and we regard these as the trivial ghost operators. Property A is equivalent to the statement that all ghost operators are compact ([33, Theorem 1.3]), i.e., that there are no non-trivial ghosts. In [2, Theorem 6.2], the inequality in line (1.1) was established under the absence of certain families of non-trivial ghost *projections*, which is much weaker. Prior to this paper, the most general geometric condition that is sufficient to establish the inequality in line (1.1) also used ghostly ideas, and is due to Li, Špakula, and Zhang [25, Theorem A]. Nonetheless, there are many examples where non-trivial ghosts exist, and that do not satisfy the Li–Špakula–Zhang condition.

The reason the condition in line (1.1) is useful is that it shows the existence of a map $f\colon X \to Y$ so that

$$\inf_{x\in X} \|\Phi(e_{xx})\delta_{f(x)}\| > 0.$$

The situation is symmetric, so that one also gets a map $g : Y \to X$ satisfying the same condition with the roles of X and Y reversed. Repeated use of the equi-approximability lemma (Lemma 1.8 above) implies that the maps f and g are both coarse, and in fact mutual coarse inverses. We isolate the key point in the following proposition, the proof of which is contained in the proof of [37, Theorem 4.1] (see also [2, Theorem 4.12]).

Proposition 1.9. *Let X and Y be uniformly locally finite metric spaces and $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be a $*$-isomorphism. If there are maps $f\colon X \to Y$ and $g\colon Y \to X$ so that $\inf_{x\in X} \|\Phi(e_{xx})\delta_{f(x)}\| > 0$ and $\inf_{y\in Y} \|\Phi^{-1}(e_{yy})\delta_{g(y)}\| > 0$, then f and g are mutual coarse inverses.* □

The key new idea in the current paper establishes the inequality in line (1.1) *unconditionally*. This is done by combining the equi-approximability lemma (Lemma 1.8) with a quantitative result on the approximate convexity of the range of a finite-dimensional, countably additive vector measure (see Lemma 2.1), which is in turn obtained as an application of the Shapley–Folkman theorem from economics.

1.3. **More rigidity.** We conclude this introduction with two other rigidity results.

For the first, it has already been noted above that the previous partial solutions to Problem 1.1 rely on conditions on the ideal of ghost operators in $\mathrm{C}^*_u(X)$. By its very definition, the "ghost-ness" of an operator is highly dependent on the choice of the orthonormal basis for $\ell_2(X)$. As such, it was unclear until now what happened to ghosts under $*$-isomorphisms. We solve this problem with the following result.

Theorem 1.10. *Every $*$-isomorphism between uniform Roe algebras of uniformly locally finite metric spaces sends ghost operators to ghost operators.*

For the second result, we look at possibly non-metrizable coarse spaces. Just as topological spaces abstract the small scale structure of metric spaces, the notion of *coarse spaces* abstracts their large scale structure, see Section 5 for precise definitions. The definition of uniform Roe algebras extends to coarse spaces naturally, and rigidity of uniform Roe algebras of non-metrizable coarse spaces has been studied in [2, 4]. The proofs of our main results do not immediately extend to coarse spaces, since Lemma 1.8 depends heavily on Baire categorical methods: these require coarse spaces to be metrizable (or at least small; see [2, Definition 4.2] and [18, §8.5] for more information on the role of the Baire Category theorem), which translates to a countability condition on the associated coarse structure.

In the earlier work on rigidity, property A plays a key role, typically via the *operator norm localization property* of Chen, Tessera, Wang, and Yu [11, Section 2], which was shown to be equivalent to property A by Sako [35]. Our vector measure approach together with a new lemma inspired by Sako's work implies that the operator norm localization property holds for certain operators regardless of the geometry of the spaces (Lemma 5.3). We are thus able to establish the result below for general coarse spaces (see Section 5 for the definition of a *coarse embedding*).

Theorem 1.11. *Let $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ be uniformly locally finite coarse spaces, and suppose $(X,\mathcal{E})$ is metrizable. If $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are $*$-isomorphic, then X coarsely embeds into Y.*

This result provides the first example of countable, coarse spaces without property A, whose uniform Roe algebras are not $*$-isomorphic to the uniform Roe algebra of any uniformly locally finite metric space. Indeed, any coarse space which contains no infinite metric space coarsely must satisfy this. In particular, this holds for $(\mathbb{N},\mathcal{E}_{\max})$, where $\mathcal{E}_{\max}$ is the maximal uniformly locally finite coarse structure on $\mathbb{N}$, i.e., $E\in\mathcal{E}$ if and only if the cardinality of the vertical and horizontal sections of E are uniformly bounded.

Corollary 1.12. *Let $\mathcal{E}_{\max}$ be the maximal uniformly locally finite coarse structure on $\mathbb{N}$. Then $\mathrm{C}^*_u(\mathbb{N},\mathcal{E}_{\max})$ is not $*$-isomorphic to the uniform Roe algebra of any uniformly locally finite metric space.* □

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2. Estimating the distance between the range of a vector measure and its convex hull

In this section, we prove a quantitative estimate on the distance between the range of a finite-dimensional vector measure on $\mathcal{P}(\mathbb{N})$ — the power set of $\mathbb{N}$ — and its convex hull which will be crucial in what follows. The results are related to Lyapunov's convexity theorem and a theorem of Elton-Hill: see Remark 2.4 for a discussion of these relationships.

A *vector measure* is a function μ from a σ-algebra Σ of sets into a Banach space which is countably additive, i.e., if $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in Σ, then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$, where the sum converges in norm.

The next lemma is the main result of this section. The norm in the statement, as well as in Lemma 2.3, is an arbitrary norm on $\mathbb{R}^n$.

Lemma 2.1. *Let $m \in \mathbb{N}$ and $\mu\colon \mathcal{P}(\mathbb{N}) \to (\mathbb{R}^m, \|\cdot\|)$ be a vector measure. Then, for all $v \in \mathrm{conv}(\mu[\mathcal{P}(\mathbb{N})])$ and $\epsilon > 0$, there exists a finite subset $F \subseteq \mathbb{N}$ such that*

$$\|\mu(F) - v\| \leqslant \sup\{\|\mu(C)\| \mid C \subseteq \mathbb{N}, |C| \leqslant m\} + \epsilon.$$

In particular, $\|\mu(F) - v\| \leqslant m \sup_{n\in\mathbb{N}} \|\mu(\{n\})\| + \epsilon$.

Lemma 2.1 will be obtained as an application of the Shapley–Folkman theorem. Given a subset S of a vector space, its convex hull is denoted by $\mathrm{conv}(S)$. If $S_1, ..., S_n$ are subsets of a vector space, their *Minkowski sum* is

$$\sum_{i=1}^n S_i \coloneqq \{s_1 + \cdots + s_n \mid s_i \in S_i\}.$$

It is a well-known elementary fact that the convex hull of a Minkowski sum is the Minkowski sum of the convex hulls. Precisely, given subsets $(S_i)_{i=1}^n$ of a vector space, we have

$$\mathrm{conv}\left(\sum_{i=1}^n S_i\right) = \sum_{i=1}^n \mathrm{conv}(S_i). \tag{2.1}$$

The Shapley–Folkman theorem (see [39, Appendix 2], or [46] for a short proof) provides additional quantitative information about the nature of the decomposition in (2.1) when the subsets are drawn from a finite-dimensional vector space. Precisely:

Theorem 2.2 (Shapley–Folkman theorem)**.** *Let $m \in \mathbb{N}$, $(S_i)_{i=1}^n$ be nonempty subsets of $\mathbb{R}^m$. Then each $v \in \operatorname{conv}(\sum_{i=1}^n S_i)$ can be written as $v = \sum_{i=1}^n v_i$ where $v_i \in \operatorname{conv}(S_i)$ for all $i \in \{1, \dots, n\}$, and so that*

$$|\{i \in \{1, \dots, n\} \mid v_i \notin S_i\}| \leqslant m.$$

We now use the Shapley–Folkman theorem to prove Lemma 2.1 for finite sets.

Lemma 2.3. *Let $m \in \mathbb{N}$, X be a finite set, and $\mu\colon \mathcal{P}(X) \to (\mathbb{R}^m, \|\cdot\|)$ be a vector measure. Then, for all $v \in \operatorname{conv}(\mu[\mathcal{P}(X)])$, there exists a subset $F \subseteq X$ such that*

$$\|\mu(F) - v\| \leqslant \max\{\|\mu(C)\| \mid C \subseteq X, |C| \leqslant m\}.$$

In particular, $\|\mu(F) - v\| \leqslant m \max_{x \in X} \|\mu(\{x\})\|$.

Proof. By shrinking X, we may assume that $\mu(\{x\}) \neq 0$ for all $x \in X$. For each $x \in X$, let $S_x \coloneqq \{0, \mu(\{x\})\} \subseteq \mathbb{R}^m$. As $0 \in S_x$, we have

$$\mu[\mathcal{P}(X)] = \sum_{x \in X} S_x.$$

Hence, if $v \in \operatorname{conv}(\mu[\mathcal{P}(X)])$, then it follows from (2.1) that there are $v_x \in \operatorname{conv}(S_x)$, for $x \in X$, such that $v = \sum_{x \in X} v_x$. By the Shapley–Folkman theorem, we may assume that the set $C \coloneqq \{x \in X \mid v_x \notin S_x\}$ has cardinality at most m. From (2.1), $\sum_{x \in C} \operatorname{conv}(S_x)$ is equal to the convex hull of the set $\sum_{x \in C} S_x = \{\mu(D) \mid D \subseteq C\}$. Therefore

$$\left\| \sum_{x \in C} v_x \right\| \leqslant \max_{D \subseteq C} \|\mu(D)\| \leqslant \max\{\|\mu(D)\| \mid D \subseteq X, |D| \leqslant m\}.$$

Let

$$S \coloneqq \{x \in X \mid v_x = \mu(\{x\})\} \quad \text{and} \quad V \coloneqq \{x \in X \mid v_x = 0\}.$$

Note that $S \sqcup C \sqcup V = X$. Hence,

$$\begin{aligned}
\|\mu(S) - v\| &= \left\| \mu(S) - \sum_{x \in X} v_x \right\| \\
&\leqslant \left\| \mu(S) - \sum_{x \in S} v_x \right\| + \left\| \sum_{x \in C} v_x \right\| + \left\| \sum_{x \in V} v_x \right\| \\
&\leqslant \max\{\|\mu(D)\| \mid D \subseteq X, |D| \leqslant m\}.
\end{aligned}$$

To conclude, it remains to note that $\|\mu(D)\| \leqslant |D| \max_{x \in D} \|\mu(\{x\})\|$ for all $D \subseteq X$. ◻

The proof of Lemma 2.1 now follows by a simple approximation argument.

Proof of Lemma 2.1. If $\sup_{n \in \mathbb{N}} \|\mu(\{n\})\| \in \{0, \infty\}$, the result is trivial, so assume that $\sup_{n \in \mathbb{N}} \|\mu(\{n\})\| \in (0, \infty)$. Let $\epsilon > 0$ and $v \in \operatorname{conv}(\mu[\mathcal{P}(\mathbb{N})])$. Then $v = \sum_{i=1}^k \lambda_i \mu(N_i)$ for some $N_1, \dots, N_k \subseteq \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \geqslant 0$ such that $\sum_{i=1}^{k}\lambda_i = 1$. Pick finite subsets $A_1,\dots,A_k \subseteq \mathbb{N}$ so that $\|\mu(N_i) - \mu(A_i)\| < \epsilon$ for all $1 \leqslant i \leqslant k$. Let $A \coloneqq \bigcup_{i=1}^{k} A_i$ and μ_A be the restriction of μ to $\mathcal{P}(A)$. So, A is finite and $v_A \coloneqq \sum_{i=1}^{k}\lambda_i\mu(A_i)$ belongs to $\mathrm{conv}(\mu_A[\mathcal{P}(A)])$. By Lemma 2.3, there exists a (finite) subset $F \subseteq A$ such that

$$\begin{aligned}\|\mu_A(F) - v_A\| &\leqslant \max\{\|\mu_A(C)\| \mid C \subseteq A, |C| \leqslant m\}\\ &\leqslant \sup\{\|\mu(C)\| \mid C \subseteq \mathbb{N}, |C| \leqslant m\}.\end{aligned}$$

Since $\mu(F) = \mu_A(F)$, we have that

$$\begin{aligned}\|\mu(F) - v\| &\leqslant \|\mu_A(F) - v_A\| + \|v_A - v\|\\ &\leqslant \sup\{\|\mu(C)\| \mid C \subseteq \mathbb{N}, |C| \leqslant m\} + \sum_{i=1}^{k}\lambda_i\|\mu(A_i) - \mu(N_i)\|\\ &\leqslant \sup\{\|\mu(C)\| \mid C \subseteq \mathbb{N}, |C| \leqslant m\} + \epsilon,\end{aligned}$$

and the statement is proved. □

Remark 2.4. The celebrated Lyapunov convexity theorem [27] states that the range of a finite-dimensional atomless vector measure is closed and convex.[5] Our measures of interest are atomic, however, and the ranges of such measures are not necessarily convex. On the other hand, a theorem of Elton–Hill (see [15, Theorem 1.2]) quantifies the distance between the range of a finite-dimensional vector measure and its convex hull in terms of the size of the atoms of the one-dimensional coordinate measures. In a less elementary and self-contained way, Lemma 2.1 can also be obtained as an application of the Elton–Hill theorem.

3. Rigidity of uniform Roe algebras

This section contains the proofs of Theorems 1.2 and 1.10. The former could also be obtained as a corollary of Theorem 4.1. However, for expository reasons we chose to present the proof of Theorem 1.2 first.

The following lemma, our main technical tool, is of independent interest. The norm on all direct sums in its proof is the standard Hilbert direct sum norm.

Lemma 3.1. *Let $\epsilon, r > 0$ and X be a uniformly locally finite metric space, and let $N_r \coloneqq \sup_{x\in X}|B_r(x)|$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of projections in $\mathcal{B}(\ell_2(X))$ such that:*

1. SOT-$\sum_{n\in A} p_n$ *is ϵ-r-approximable for all $A \subseteq \mathbb{N}$, and*
2. SOT-$\sum_{n\in\mathbb{N}} p_n = 1_{\ell_2(X)}$.

[5] There are many proofs of the Lyapunov convexity theorem ([21], [13, Corollary IX.1.5], [23] just to name a few). A particularly elegant and concise one is due to Lindenstrauss [28].

For each $x \in X$ and $\delta > 0$, let $M(x,\delta) \coloneqq \{n \in \mathbb{N} \mid \|p_n\delta_x\| \geqslant \delta\}$. Then, if $\delta \leqslant \epsilon^2/(8N_r)$, we have

$$\inf_{x\in X}\Big\|\sum_{n\in M(x,\delta)} p_n\delta_x\Big\| \geqslant 1-3\epsilon.$$

Proof. Fix $\epsilon, r > 0$ and N_r as in the statement. For $A \subseteq \mathbb{N}$ let $p_A \coloneqq$ SOT-$\sum_{n\in A} p_n$. Notice that the condition $p_{\mathbb{N}} = 1_{\ell_2(X)}$ forces the projections $(p_n)_n$ to be mutually orthogonal, so each p_A is a projection. Suppose that for some $\delta \leqslant \epsilon^2/(8N_r)$ there is $x \in X$ such that

$$\|p_{M(x,\delta)}\delta_x\| < 1-3\epsilon. \tag{3.1}$$

Our goal is to exhibit a projection of the form p_A that is not ϵ-r-approximable, and thus get a contradiction. To simplify notation, let $M \coloneqq M(x,\delta)$ and $M' \coloneqq \mathbb{N}\setminus M$.

Writing $B \coloneqq B_r(x)\setminus\{x\}$, the partition $X = \{x\}\sqcup B\sqcup(X\setminus B_r(x))$ gives a direct sum decomposition

$$\ell_2(X) = \mathbb{C}\oplus H_B\oplus H_U,$$

where $H_B = \ell_2(B)$ and $H_U = \ell_2(X\setminus B_r(x))$ (here 'B' and 'U' stand for 'bounded' and 'unbounded', respectively).

Now comes the crucial vector measure argument. Let $\pi\colon \ell_2(X)\to\mathbb{C}\oplus H_B$ be the orthogonal projection, which we identify with $\chi_{B_r(x)}$.[6] We define a vector measure $\mu : \mathcal{P}(M') \to \mathbb{C}\oplus H_B$ by

$$\mu(A) \coloneqq \pi p_A\delta_x, \tag{3.2}$$

for all $A \subseteq M'$ (μ is clearly countably additive). As $\|\mu(\{n\})\| = \|\pi p_n\delta_x\| \leqslant \|p_n\delta_x\|$, it follows from the definition of M' that $\sup_{n\in M'}\|\mu(\{n\})\| < \delta$. Therefore, as $\mu(M')/2$ belongs to the convex hull of the range of μ and as $\dim_{\mathbb{R}}(\mathbb{C}\oplus H_B) \leqslant 2N_r$, Lemma 2.1 gives a finite set $A \subseteq M'$ such that

$$\Big\|\mu(A) - \frac{\mu(M')}{2}\Big\| < 2N_r\delta \leqslant \frac{\epsilon^2}{4}. \tag{3.3}$$

Fix such A from now, and let $s, \bar{s} \in \mathbb{C}$ and $b, \bar{b} \in H_B$ be so that $\mu(A) = (s,b)$ and $\mu(M')/2 = (\bar{s},\bar{b})$, i.e.,

$$s = \langle p_A\delta_x|\delta_x\rangle,\ \bar{s} = \frac{1}{2}\langle p_{M'}\delta_x|\delta_x\rangle,\ b = \chi_B p_A\delta_x,\ \text{and } \bar{b} = \frac{1}{2}\chi_B p_{M'}\delta_x.$$

Moreover, let $u \coloneqq p_A\delta_x - s\delta_x - b$, i.e., $u = \chi_{X\setminus B_r(x)}p_A\delta_x$. Our next goal is to show that $\|u\| > \epsilon$.

It follows from (3.3) that $|s-\bar{s}| < \epsilon^2/4$ and so

$$s > \bar{s} - \epsilon^2/4. \tag{3.4}$$

[6] Here we use the following standard notation: for $S \subseteq X$, we let $\chi_S \coloneqq$ SOT-$\sum_{x\in S} e_{xx}$, i.e., χ_S is the operator on $\ell_2(X)$ that projects onto the coordinates indexed by S.

Also, as $\bar{s} \in [0, 1/2]$, we have

$$1 - s > \frac{1}{2} - \epsilon^2/4. \tag{3.5}$$

We may assume $\epsilon < 1/3$ (otherwise the conclusion is vacuous), whence $1-s$ is positive. As s is positive automatically, we may multiply (3.4) and (3.5) and then use that $\bar{s} \leqslant 1/2$ to get

$$s - s^2 > \frac{\bar{s}}{2} - \frac{\epsilon^2}{8} - \frac{\bar{s}\epsilon^2}{4} + \frac{\epsilon^4}{16} \geqslant \frac{\bar{s}}{2} - \frac{\epsilon^2}{4} + \frac{\epsilon^4}{16}. \tag{3.6}$$

Using (3.3) again, we have that

$$\|b\| < \frac{\epsilon^2}{4} + \|\bar{b}\|. \tag{3.7}$$

Therefore, keeping in mind that $\|\bar{b}\| \leqslant 1/2$ and $s = \langle p_A\delta_x|\delta_x\rangle = \|p_A\delta_x\|^2$, we have

$$\begin{aligned} \|u\|^2 &= \|p_A\delta_x\|^2 - |s|^2 - \|b\|^2 \\ &= s - s^2 - \|b\|^2 \\ &\overset{(3.6)\wedge(3.7)}{>} \frac{\bar{s}}{2} - \frac{\epsilon^2}{4} + \frac{\epsilon^4}{16} - \left(\frac{\epsilon^4}{16} + \frac{\epsilon^2}{2}\|\bar{b}\| + \|\bar{b}\|^2\right) \\ &> \frac{\bar{s}}{2} - \frac{\epsilon^2}{2} - \|\bar{b}\|^2. \qquad (3.8) \end{aligned}$$

As $\|\mu(M')\| \leqslant \|p_{M'}\delta_x\| = \sqrt{2\bar{s}}$, we have that $\|\mu(M')/2\|^2 \leqslant \bar{s}/2$. Therefore, as $\bar{s}^2 + \|\bar{b}\|^2 = \|\mu(M')/2\|^2$, it follows that

$$\bar{s}/2 - \|\bar{b}\|^2 \geqslant \bar{s}^2. \tag{3.9}$$

As $\mathbb{N} = M \sqcup M'$, we have that

$$\bar{s} = \frac{1}{2}\langle p_{M'}\delta_x|\delta_x\rangle = \frac{1}{2}\big(\langle p_{\mathbb{N}}\delta_x|\delta_x\rangle - \langle p_M\delta_x|\delta_x\rangle\big).$$

Using assumption (3.1) and also that $p_{\mathbb{N}} = 1_{\ell_2(X)}$, this implies that

$$\bar{s} > \frac{3\epsilon}{2}. \tag{3.10}$$

A combination of (3.8), (3.9), and (3.10) gives

$$\|u\|^2 > -\frac{\epsilon^2}{2} + \frac{9\epsilon^2}{4},$$

and so in particular $\|u\| > \epsilon$.

On the other hand, as $u = \chi_{X\setminus B_r(x)}p_A\delta_x$ and $d(x, X \setminus B_r(x)) > r$, this contradicts that p_A is ϵ-r-approximable: indeed, if $\|p_A - b\| \leqslant \epsilon$ for some operator b with $\mathrm{prop}(b) \leqslant r$, then, as $\chi_{X\setminus B_r(x)}b\chi_{\{x\}} = 0$, we get

$$\|u\| = \|\chi_{X\setminus B_r(x)}(p_A - b)\chi_{\{x\}}\| \leqslant \epsilon,$$

which is a contradiction. $\square$

Lemma 1.8 and Lemma 3.1 imply the following corollary.

Corollary 3.2. *Let X be a uniformly locally finite metric space and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of projections in $\mathcal{B}(\ell_2(X))$ such that*

1. SOT-$\sum_{n\in A} p_n \in \mathrm{C}^*_u(X)$ *for all* $A \subseteq \mathbb{N}$, *and*
2. SOT-$\sum_{n\in\mathbb{N}} p_n = 1_{\ell_2(X)}$.

Then,

$$\inf_{x\in X}\sup_{n\in\mathbb{N}} \|p_n\delta_x\| > 0.$$

Proof. Let $\epsilon = 1/4$. Then Lemma 1.8 implies there is r so that SOT-$\sum_{n\in A} p_n$ is ϵ-r-approximable for all $A \subseteq \mathbb{N}$. Lemma 3.1 implies in particular that for $\delta = 1/(128N_r)$ and any $x \in X$, $M(x,\delta)$ is non-empty. Hence

$$\inf_{x\in X}\sup_{n\in\mathbb{N}} \|p_n\delta_x\| \geqslant 1/(128N_r). \qquad \square$$

Proof of Theorem 1.2. Fix a $*$-isomorphism $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$. By Lemma 1.6, Φ is strongly continuous, so $(\Phi(e_{xx}))_{x\in X}$ satisfies the conditions on the family $(p_n)_{n\in\mathbb{N}}$ from Corollary 3.2. Therefore, there are $\delta > 0$ and $g\colon Y \to X$ such that

$$\|\Phi^{-1}(e_{yy})\delta_{g(y)}\| = \|\Phi(e_{g(y)g(y)})e_{yy}\| = \|\Phi(e_{g(y)g(y)})\delta_y\| > \delta$$

for all $y \in Y$.

Replacing δ by a smaller positive real if necessary, an argument analogous to the one above applied to $\Phi^{-1}\colon \mathrm{C}^*_u(Y) \to \mathrm{C}^*_u(X)$ gives us a map $f\colon X \to Y$ such that

$$\|\Phi(e_{xx})\delta_{f(x)}\| > \delta,$$

for all $x \in X$. By Proposition 1.9, f is a coarse equivalence. $\square$

Remark 3.3. It might be of interest to experts that we can also establish the natural analog of Theorem 1.2 for C*-algebras of 'quasi-local' operators. To explain this, let X be a uniformly locally finite metric space. An operator a on $\ell_2(X)$ is *ϵ-r-quasi-local* if whenever $A, B \subseteq X$ satisfy $d(A,B) > r$, we have $\|\chi_A a \chi_B\| < \epsilon$, and is *quasi-local* if for all $\epsilon > 0$ there exists $r \geqslant 0$ such that a is ϵ-r-quasi-local. The collection of all quasi-local operators forms a C*-algebra, denoted $\mathrm{C}^*_{ql}(X)$. One has $\mathrm{C}^*_u(X) \subseteq \mathrm{C}^*_{ql}(X)$, and Špakula-Zhang [41, Theorem 3.3] (building on techniques from Špakula-Tikuisis [40]) have shown that this inclusion is the identity when X has property A. In general, it is not clear whether this inclusion can be strict.

We sketch a proof that if $\mathrm{C}^*_{ql}(X)$ is isomorphic to $\mathrm{C}^*_{ql}(Y)$, then X and Y are coarsely equivalent. First, one shows a quasi-local version of the equi-approximability lemma (Lemma 1.8). Precisely: "if $(a_n)_n$ is a sequence of orthogonal operators on $\ell_2(X)$ so that SOT-$\sum_{n\in M} a_n$ converges to an element of $\mathrm{C}^*_{ql}(X)$ for all $M \subseteq \mathbb{N}$, then for all $\epsilon > 0$ there is $r \geqslant 0$ such that for all $M \subseteq \mathbb{N}$, SOT-$\sum_{n\in M} a_n$ is ϵ-r-quasi-local." This quasi-local equi-approximability lemma follows from a slight adaptation of [37, Lemma 3.2]. Second, one notes that the proof of Lemma 3.1 goes through verbatim if condition 1 from the statement is replaced with "SOT-$\sum_{n\in A} p_n$ is ϵ-r-quasi-local for all $A \subseteq \mathbb{N}$". From these two observations, one deduces that Corollary 3.2 follows with "$\mathrm{C}^*_u(X)$" replaced by "$\mathrm{C}^*_{ql}(X)$". Finally, the proof of Theorem 1.2 goes through analogously in the quasi-local case: the point is that the quasi-local equi-approximability lemma is enough to establish the analogue of Proposition 1.9 for an isomorphism $\Phi : \mathrm{C}^*_{ql}(X) \to \mathrm{C}^*_{ql}(Y)$ (this is essentially what is done in the original reference [37, Theorem 4.1]).

At this point, we are also in position to prove Theorem 1.10.

Proof of Theorem 1.10. Let X and Y be uniformly locally finite metric spaces and $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be a $*$-isomorphism. We need to prove that Φ sends ghost operators to ghost operators. Proceeding as in the proof of Theorem 1.2, there are $\delta > 0$ and a coarse equivalence $f\colon X \to Y$ such that $\|\Phi(e_{xx})\delta_{f(x)}\| > \delta$ for all $x \in X$.

Suppose that $A \subseteq X$ is infinite. Since X and Y are uniformly locally finite, the set $f[A]$ is also infinite. As $\|\Phi(\chi_A)\delta_{f(a)}\| > \delta$ for all $a \in A$, $\Phi(\chi_A)$ cannot be a ghost.

Now fix an arbitrary nonghost $a \in \mathrm{C}^*_u(X)$. Pick $\epsilon > 0$ and sequences of distinct elements $(x_n)_n$ and $(z_n)_n$ in X, such that $|\langle a\delta_{x_n}, \delta_{z_n}\rangle| > \epsilon$ for all $n \in \mathbb{N}$. Passing to subsequences if necessary, letting $A = \{x_n \mid n \in \mathbb{N}\}$ and $B = \{z_n \mid n \in \mathbb{N}\}$, we can assume that $\chi_B a \chi_A - \text{SOT-}\sum_n e_{z_n z_n} a e_{x_n x_n}$ is compact. Therefore, as $\chi_B a \chi_A$ belongs to the ideal generated by a, so does SOT-$\sum_n e_{z_n z_n} a e_{x_n x_n}$. As $|\langle a\delta_{x_n}, \delta_{z_n}\rangle| > \epsilon$ for all $n \in \mathbb{N}$, it follows that $b \coloneqq \text{SOT-}\sum_n e_{z_n x_n}$ belongs to the ideal generated by a, and hence so does $\chi_A = b^*b$ (alternatively, [12, Lemma 3.4] also implies that χ_A belongs to the ideal generated by a, even without going to subsequences). Since Φ is a $*$-isomorphism, $\Phi(\chi_A)$ belongs to the ideal generated by $\Phi(a)$. Since ghosts form an ideal, if $\Phi(a)$ is a ghost, then so is $\Phi(\chi_A)$. Hence, by the previous paragraph, $\Phi(a)$ is not a ghost. A symmetric argument gives that nonghost operators in $\mathrm{C}^*_u(Y)$ are mapped by Φ^{-1} to nonghost operators in $\mathrm{C}^*_u(X)$, and the conclusion follows. □

For the next result, we use the notion of amenability for uniformly locally finite metric spaces, as introduced by Block and Weinberger [1, Section 3].

Corollary 3.4. *Let X and Y be uniformly locally finite metric spaces, and consider the following statements:*

(1) X and Y are coarsely equivalent via a bijective coarse equivalence.
(2) The C^*-algebras $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are isomorphic.

Then (1) implies (2) in general, and (2) implies (1) if X is non-amenable.

Proof. The implication from (1) to (2) is well-known (e.g., [2, Theorem 8.1]): if $f\colon X \to Y$ is a bijective coarse equivalence, then one defines a unitary isomorphism $u\colon \ell_2(X) \to \ell_2(Y)$ by letting $u\delta_x \coloneqq \delta_{f(x)}$ for all $x \in X$, and direct checks show that $u\mathrm{C}^*_u(X)u^* = \mathrm{C}^*_u(Y)$.

For the converse, suppose that $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are $*$-isomorphic, whence by Theorem 1.2, X and Y are coarsely equivalent. If X is not amenable, then X being coarsely equivalent to a uniformly locally finite space Y implies that X is bijectively coarsely equivalent to it as shown in [42, Theorem 5.1] (the key idea is from [43, Theorem 4.1]). $\square$

If X and Y are countable groups, we can do better. The following result gives Corollary 1.3.

Corollary 3.5. *Let Γ and Λ be countable discrete groups equipped with uniformly locally finite metrics that are invariant under left translation*[7]*. Then the following are equivalent:*

(1) Γ and Λ are coarsely equivalent via a bijective coarse equivalence.
(2) The C^*-algebras $\ell_\infty(\Gamma) \rtimes_r \Gamma$ and $\ell_\infty(\Lambda) \rtimes_r \Lambda$ are isomorphic.

Moreover, if Γ and Λ are finitely generated and equipped with word metrics, then one can replace "bijective coarse equivalence" in (1) with "bi-Lipschitz bijection".

Proof. We use the well-known identification $\mathrm{C}^*_u(\Gamma) \cong \ell^\infty(\Gamma) \rtimes_r \Gamma$ (see [9, Proposition 5.1.3]) to replace the crossed products in (2) with uniform Roe algebras.

As already noted in the proof of Corollary (3.4), (1) implies (2) in general, and (2) implies (1) when Γ is non-amenable. On the other hand, if Γ is amenable, then as it is a group it has property A by [44, Lemma 6.2]. A $*$-isomorphism between uniform Roe algebras of uniformly locally finite metric spaces, one of which has property A, gives a bijective coarse equivalence between the underlying metric spaces (this was proved in [42, Corollary 6.13] for metric spaces and in [4, Theorem 1.3] for arbitrary coarse spaces).

Assume now that Γ and Λ are finitely generated and equipped with word metrics, and assume that $\mathrm{C}^*_u(\Gamma) \cong \mathrm{C}^*_u(\Lambda)$. Using our discussion so far, there is a bijective coarse equivalence $f : \Gamma \to \Lambda$. As Γ and Λ are finitely generated, it is straightforward to check that they are quasi-geodesic in the sense of [29, Definition 1.4.10]. Hence, f is a quasi-isometry (cf. [20, Proposition A.3] or [29, Corollary 1.4.14]). As $\inf_{\gamma \neq \gamma' \in \Gamma} d_\Gamma(\gamma, \gamma') = 1$ and $\inf_{\lambda \neq \lambda' \in \Lambda} d_\Lambda(\lambda, \lambda') = 1$, a bijective quasi-isometry is automatically bi-Lipschitz. $\square$

We do not know whether (1) and (2) from Corollary 3.4 are equivalent for uniformly locally finite metric spaces in general: a counterexample, if it exists, would have to be a pair of amenable, uniformly locally finite metric spaces, neither of which has property A. Many such examples exist: for example, any expander defines an amenable, uniformly locally finite metric space without property A.

Proof of Theorem 1.5. Let $\Lambda : \mathrm{Q}^*_u(X) \to \mathrm{Q}^*_u(Y)$ be a $*$-isomorphism, and let $\pi_X : \mathrm{C}^*_u(X) \to \mathrm{Q}^*_u(X)$ and $\pi_Y : \mathrm{C}^*_u(Y) \to \mathrm{Q}^*_u(Y)$ be the canonical

[7]Any countable group admits such a metric, which is moreover unique up to bijective coarse equivalence (e.g., [44, Proposition 2.3.3])

projections. By [3, Theorem 1.5], Λ and Λ^{-1} are *liftable on the diagonals* in the sense of [3, Definition 1.4(2)], i.e., there are strongly continuous $*$-homomorphisms $\Phi : \ell_\infty(X) \to \mathrm{C}^*_u(Y)$ and $\Psi : \ell_\infty(Y) \to \mathrm{C}^*_u(X)$ such that $\Lambda(\pi_X(a)) = \pi_Y(\Phi(a))$ and $\Lambda^{-1}(\pi_Y(b)) = \pi_X(\Psi(b))$ for all $a \in \ell_\infty(X)$ and all $b \in \ell_\infty(Y)$.

Claim 3.6. There are cofinite subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that

$$\inf_{x\in X'} \sup_{y\in Y} \|\Phi(e_{xx})\delta_y\| > 0 \text{ and } \inf_{y\in Y'} \sup_{x\in X} \|\Psi(e_{yy})\delta_x\| > 0.$$

Proof. By symmetry, it is enough to show that the result holds for Φ. For that, let $p = 1_{\ell_2(X)} - \Psi(1_{\ell_2(Y)})$. Then, as $\Lambda(\pi_X(\Psi(1_{\ell_2(Y)}))) = \pi_Y(1_{\ell_2(Y)})$, it follows that $\Lambda(\pi_X(p)) = 0$. Hence, $\pi_X(p) = 0$ which means that p is compact. As p is a projection, p has finite-rank. As Ψ is strongly continuous, we have that

$$1_{\ell_2(X)} = p + \Psi(1_{\ell_2(Y)}) = p + \text{SOT-}\sum_{y\in Y} \Psi(e_{yy}).$$

Therefore, Corollary 3.2 gives a partition $X = X' \sqcup X''$ and a map $f : X' \to Y$ so that $\inf_{x\in X'} \|\Psi(e_{f(x)f(x)})\delta_x\| > 0$ and $\inf_{x\in X''} \|p\delta_x\| > 0$. As p has finite rank, X' must be cofinite. By [3, Lemma 6.3], replacing X' by a smaller cofinite subset of X if necessary, we can assume that $\inf_{x\in X'} \|\Phi(e_{xx})\delta_{f(x)}\| > 0$; so the claim follows. ☐

By the previous claim, it follows immediately from [3, Theorem 6.11] that X and Y are coarsely equivalent. ☐

4. Rigidity of stable Roe algebras and Morita equivalence

Given a uniformly locally finite metric space X and an infinite-dimensional separable Hilbert space H, the *stable Roe algebra of* X is given by

$$\mathrm{C}^*_s(X) \coloneqq \mathrm{C}^*_u(X) \otimes \mathcal{K}(H),$$

where the tensor product above is the minimal tensor product of C*-algebra theory. We can describe $\mathrm{C}^*_s(X)$ more concretely as follows. For $x \in X$, let $v_x \colon H \to \ell_2(\{x\}, H) \subseteq \ell_2(X, H)$ denote the canonical inclusion. For a bounded operator a on $\ell_2(X, H)$ and $x, y \in X$, define the matrix entries $a_{xy} \coloneqq v_x^* a v_y \in \mathcal{B}(H)$, and define the propagation of a to be

$$\mathrm{prop}(a) \coloneqq \sup\{d_X(x, y) \mid a_{xy} \neq 0\} \in [0, \infty].$$

Given a finite-dimensional vector space $H' \subseteq H$ and $r > 0$, let $\mathrm{C}^*_s[X, r, H']$ denote the subspace of all operators $a = [a_{xy}] \in \mathcal{B}(\ell_2(X, H))$ with propagation at most r and such that each a_{xy} is an operator in $\mathcal{B}(H')$ (where $\mathcal{B}(H')$ is identified with a C*-subalgebra of $\mathcal{B}(H)$ in the canonical way). Then, under the canonical identification $\ell_2(X) \otimes H = \ell_2(X, H)$, the stable Roe algebra $\mathrm{C}^*_s(X)$ is the norm-closure in $\mathcal{B}(\ell_2(X, H))$ of the union of all such $\mathrm{C}^*_s[X, r, H']$.

We will now show that stable uniform Roe algebras are also coarsely rigid (which in turn will give us Theorem 1.4).

Theorem 4.1. *Suppose X and Y are uniformly locally finite metric spaces such that $\mathrm{C}^*_s(X)$ and $\mathrm{C}^*_s(Y)$ are isomorphic. Then X and Y are coarsely equivalent.*

Before presenting the proof of Theorem 4.1, we need two lemmas.

Lemma 4.2. *Let X be a uniformly locally finite metric space, and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of orthogonal projections in $\mathrm{C}^*_s(X)$ such that $p_A \coloneqq \mathrm{SOT}\text{-}\sum_{n\in A} p_n \in \mathrm{C}^*_s(X)$, for all $A\subseteq\mathbb{N}$. Then, for all $\epsilon>0$, there is a finite-rank projection $p\in\mathcal{B}(H)$, such that $\|(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes p)p_A\|\leqslant\epsilon$, for all $A\subseteq\mathbb{N}$.*

Proof. If not, there are $\epsilon>0$, an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of finite-rank projections on H converging to 1_H strongly, a sequence $(w_n)_{n\in\mathbb{N}}$ of finite-rank projections on $\ell_2(X,H)$, and a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ such that

$$\|w_n(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes q_n)p_{A_n}w_n\|>\epsilon,$$

for all $n\in\mathbb{N}$. Since $1_{\ell_2(X,H)}=\mathrm{SOT}\text{-}\lim_m 1_{\ell_2(X)}\otimes q_m$ and each p_{A_n} has finite-rank, going to a subsequence if necessary, we can assume that

$$\|(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes q_m)p_{A_n}\|<\epsilon 2^{-n-1},$$

for all $m>n$. Also, as each w_m is finite-rank and $(p_{A_n})_{n\in\mathbb{N}}$ is an orthogonal sequence, going to a further subsequence, we can assume that $\|p_{A_n}w_m\|<\epsilon 2^{-n-1}$, for all $n>m$. Therefore, letting $A\coloneqq\bigsqcup_{n=1}^\infty A_n$, we have

$$\begin{aligned}\|(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes q_m)p_A\| &\geqslant \|w_m(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes q_m)p_A w_m\|\\ &\geqslant \|w_m(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes q_m)p_{A_m} w_m\|\\ &\quad -\sum_{n\neq m}\|w_m(1_{\ell_2(X,H)}-1_{\ell_2(X)}\otimes q_m)p_{A_n} w_m\|\\ &\geqslant \frac{\epsilon}{2},\end{aligned}$$

for all $m\in\mathbb{N}$. As $1_H=\mathrm{SOT}\text{-}\lim_m q_m$, this contradicts that $p_A\in\mathrm{C}^*_s(X)$. $\square$

Our next lemma is a stable version of Corollary 3.2.

Lemma 4.3. *Let X be a uniformly locally finite metric space and $(p_n)_{n\in\mathbb{N}}$ be a sequence of orthogonal projections in $\mathrm{C}^*_s(X)$ such that $p_A\coloneqq\mathrm{SOT}\text{-}\sum_{n\in A}p_n$ is in $\mathrm{C}^*_s(X)$, for all $A\subseteq\mathbb{N}$. Then, for all unit vectors $\xi\in H$ with*

$$\sup_{x\in X}\|(1_{\ell_2(X,H)}-p_{\mathbb{N}})(\delta_x\otimes\xi)\|\leqslant 1/32,$$

we have that

$$\inf_{x\in X}\sup_{n\in\mathbb{N}}\|p_n(\delta_x\otimes\xi)\|>0.$$

Proof. Fix a unit vector $\xi \in H$ with $\gamma \coloneqq \sup_{x\in X} \|(1_{\ell_2(X,H)} - p_{\mathbb{N}})(\delta_x \otimes \xi)\| \leqslant 1/32$. Fix a positive $\epsilon < 1/64$. Lemma 1.8 has a natural analog for stable Roe algebras (see [5, Lemma 3.14]), hence there is $r \geqslant 0$ such that each p_A is ϵ-r-approximable.[8] Let $p \in \mathcal{B}(H)$ be a finite-rank projection given by Lemma 4.2 for ϵ, i.e.,

$$\|(1_{\ell_2(X,H)} - 1_{\ell_2(X)} \otimes p)p_A\| \leqslant \epsilon,$$

for all $A \subseteq \mathbb{N}$. Expanding the image of p, we can assume that $\xi \in \mathrm{Im}(p)$. Let $N_r \coloneqq \sup_{x\in X} |B_r(x)|$ and fix $\delta > 0$ such that $2N_r \operatorname{rank}(p)\delta < 1/64$.

If the lemma fails for ξ, pick $x \in X$ such that

$$\sup_{n\in\mathbb{N}} \|p_n(\delta_x \otimes \xi)\| < \frac{\delta}{2}.$$

Let $\pi \coloneqq \chi_{B_r(x)} \otimes p$ and $\pi_\perp \coloneqq \chi_{B_r(x)} \otimes (1 - p)$. Let p_ξ be the projection onto $\mathbb{C}\xi$ and set $\theta \coloneqq \pi - \chi_{\{x\}} \otimes p_\xi$. Then, as $\xi \in \mathrm{Im}(p)$, $\ell_2(X,H)$ can be decomposed as

$$\ell_2(X,H) = \mathbb{C}(\delta_x \otimes \xi) \oplus H_B \oplus H_{B^\perp} \oplus H_U,$$

where $H_B \coloneqq \theta[\ell_2(X,H)]$, $H_{B^\perp} \coloneqq \pi_\perp[\ell_2(X,H)]$, and $H_U \coloneqq (\chi_{X\setminus B_r(x)} \otimes 1_H)[\ell_2(X,H)]$. Define a vector measure $\mu \coloneqq \mu_{x,\xi} \colon \mathcal{P}(\mathbb{N}) \to \mathbb{C}(\delta_x \otimes \xi) \oplus H_B$ by letting

$$\mu(A) \coloneqq \pi p_A(\delta_x \otimes \xi),$$

for all $A \in \mathcal{P}(\mathbb{N})$. By our choice of x, $\sup_{n\in\mathbb{N}} \|\mu(\{n\})\| < \delta$. Let $z \coloneqq \mu(\mathbb{N})/2$, and set $\bar{s} \coloneqq (\chi_{\{x\}} \otimes p_\xi)(z)$ and $\bar{b} \coloneqq \theta(z)$, so z decomposes as

$$z = (\bar{s}, \bar{b}) \in \mathbb{C}(\delta_x \otimes \xi) \oplus H_B.$$

Therefore, since $\frac{1}{2}\delta_x \otimes \xi$ decomposes as $(1/2, 0)$ in $\mathbb{C}(\delta_x \otimes \xi) \oplus H_B$, our choices of r, p, and γ give that

$$\begin{aligned}
\|(\bar{s}, \bar{b}) - (1/2, 0)\| &= \frac{1}{2}\|\pi p_{\mathbb{N}}(\delta_x \otimes \xi) - \delta_x \otimes \xi\| \\
&\leqslant \frac{1}{2}\Big(\|(\pi - \chi_{B_r(x)} \otimes 1_H)p_{\mathbb{N}}(\delta_x \otimes \xi)\| + \|(\chi_{X\setminus B_r(x)} \otimes 1_H)p_{\mathbb{N}}(\delta_x \otimes \xi)\| \\
&\qquad + \|p_{\mathbb{N}}(\delta_x \otimes \xi) - \delta_x \otimes \xi)\|\Big) \\
&\leqslant \epsilon + \frac{\gamma}{2}.
\end{aligned}$$

So, $|\bar{s} - 1/2| \leqslant \epsilon + \gamma/2$ and $\|\bar{b}\| \leqslant \epsilon + \gamma/2$.

As $\dim_{\mathbb{R}}(H') \leqslant 2N_r \operatorname{rank}(p)$ and $z \in \operatorname{conv}(\mu[\mathcal{P}(\mathbb{N})])$, Lemma 2.1 gives an $A \subseteq \mathbb{N}$ such that

$$\|\mu(A) - z\| \leqslant 2N_r \operatorname{rank}(p)\delta.$$

For simplicity, let $\delta_1 \coloneqq 2N_r \operatorname{rank}(p)\delta$.

[8]Definition 1.7 naturally extends to operators in $\mathcal{B}(\ell_2(X,H))$. We will actually only need that $\|(\chi_C \otimes 1_H)p_A(\chi_D \otimes 1_H)\| \leqslant \epsilon$ for all $C, D \subseteq X$ with $d(C,D) > r$ and all $A \subseteq \mathbb{N}$.

Let $s \coloneqq \langle p_A(\delta_x \otimes \xi) | (\delta_x \otimes \xi)\rangle$, $b \coloneqq \theta p_A(\delta_x \otimes \xi)$, $b_\perp = \pi_\perp p_A(\delta_x \otimes \xi)$, and $u \coloneqq (\chi_{X \setminus B_r(x)} \otimes 1_H) p_A(\delta_x \otimes \xi)$. So

$$p_A(\delta_x \otimes \xi) = s(\delta_x \otimes \xi) + b + b_\perp + u \in \mathbb{C}(\delta_x \otimes \xi) \oplus H_B \oplus H_{B^\perp} \oplus H_U.$$

Hence, $\mu(A) = (s, b)$ and our estimates for $|\bar{s} - 1/2|$ and $\|\bar{b}\|$ above, together with our choice of A, give us that $|s - 1/2| < \delta_1 + \epsilon + \gamma/2$ and $\|b\| < \delta_1 + \epsilon + \gamma/2$. Also, by our choice of p, we have that $\|b_\perp\| \leqslant \epsilon$. Therefore, as $\|p_A(\delta_x \otimes \xi)\|^2 = s$, $\epsilon < 1/64$, $\delta_1 < 1/64$, and $\gamma \leqslant 1/32$, a simple, albeit tedious, computation gives that

$$\|u\|^2 = \|p_A(\delta_x \otimes \xi)\|^2 - \|s(\delta_x \otimes \xi)\|^2 - \|b\|^2 - \|b_\perp\|^2 > \tfrac{1}{8} > \epsilon^2.$$

Just as in Lemma 3.1, this contradicts that p_A is ϵ-r-approximable. $\square$

Before presenting the proof of Theorem 4.1, we isolate a result in the proof of [37, Theorem 6.1], which is the analog of Proposition 1.9 in the setting of stable Roe algebras.

Proposition 4.4. *Let X and Y be uniformly locally finite metric spaces and $\Phi\colon \mathrm{C}^*_s(X) \to \mathrm{C}^*_s(Y)$ be a $*$-isomorphism. Suppose there are a finite-rank projection p on H, a unit vector $\xi \in H$, and maps $f\colon X \to Y$ and $g\colon Y \to X$ such that*

$$\inf_{x \in X} \|\Phi(\chi_{\{x\}} \otimes p_\xi)(\chi_{\{f(x)\}} \otimes p)\| > 0$$

and

$$\inf_{y \in Y} \|\Phi^{-1}(\chi_{\{y\}} \otimes p_\xi)(\chi_{\{g(y)\}} \otimes p)\| > 0,$$

where p_ξ is the projection on H onto $\mathbb{C}\xi$. Then, f is a coarse equivalence with coarse inverse g. $\square$

Proof of Theorem 4.1. Let $\Phi\colon \mathrm{C}^*_s(X) \to \mathrm{C}^*_s(Y)$ be a $*$-isomorphism and let $\Psi = \Phi^{-1}$. We need to prove that X and Y are coarsely equivalent. Fix a unit vector $\xi \in H$ and let p_ξ be the projection of H onto $\mathbb{C}\xi$. By Lemma 4.2, there is a finite-rank projection $p \in \mathcal{B}(H)$ such that

$$\|(1_{\ell_2(Y,H)} - 1_{\ell_2(Y)} \otimes p)\Phi(\chi_{\{x\}} \otimes p_\xi)\| < \frac{1}{32}$$

for all $x \in X$. For each $y \in Y$, let $p_y = \Psi(\chi_{\{y\}} \otimes p)$ and for each $A \subseteq Y$, set $p_A \coloneqq \text{SOT-}\sum_{y \in A} p_y$. Hence

$$\begin{aligned}
\|(1_{\ell_2(X,H)} - p_Y)(\delta_x \otimes \xi)\| &= \|\chi_{\{x\}} \otimes p_\xi - \Psi(1_{\ell_2(Y)} \otimes p)(\chi_{\{x\}} \otimes p_\xi)\| \\
&= \|\Phi(\chi_{\{x\}} \otimes p_\xi) - (1_{\ell_2(Y)} \otimes p)\Phi(\chi_{\{x\}} \otimes p_\xi)\| \\
&< \frac{1}{32}
\end{aligned}$$

for all $x \in X$. Hence, Lemma 4.3 gives $\delta > 0$ and $f\colon X \to Y$ such that

$$\|\Phi(\chi_{\{x\}} \otimes p_\xi)(\chi_{\{f(x)\}} \otimes p)\| = \|\Psi(\chi_{\{f(x)\}} \otimes p)(\delta_x \otimes \xi)\| > \delta,$$

for all $x \in X$.

By replacing δ by a smaller positive real if necessary and p by a larger finite-rank projection, a symmetric argument gives $g\colon Y \to X$ such that

$$\|\Psi(\chi_{\{y\}} \otimes p_\xi)(\chi_{\{g(y)\}} \otimes p)\| = \|\Phi(\chi_{\{g(y)\}} \otimes p)(\delta_y \otimes \xi)\| > \delta,$$

for all $x \in X$. By Proposition 4.4, f is a coarse equivalence with coarse inverse g. $\square$

Proof of Theorem 1.4. The implication 1. $\Rightarrow$ 2. was established in [7, Theorem 4]. Suppose now that $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are Morita equivalent. Then, the stable Roe algebras $\mathrm{C}^*_s(X)$ and $\mathrm{C}^*_s(Y)$ must be isomorphic as shown in [8, Theorem 1.2][9]. It then follows from Theorem 4.1 that X and Y must be coarsely equivalent. $\square$

5. Uniform Roe algebras of coarse spaces

Theorem 1.11 is proven in this section. For that, we recall the basics of coarse spaces — we refer the reader to the monograph [32] for a detailed treatment of coarse spaces. Given a set X and a family $\mathcal{E}$ of subsets of $X \times X$, $\mathcal{E}$ is a *coarse structure on* X if

1. $\Delta_X = \{(x,x) \mid x \in X\} \in \mathcal{E}$,
2. $E \in \mathcal{E}$ and $F \subseteq E$ implies $F \in \mathcal{E}$,
3. $E, F \in \mathcal{E}$ implies $E \cup F \in \mathcal{E}$,
4. $E \in \mathcal{E}$ implies $E^{-1} \coloneqq \{(y,x) \mid (x,y) \in E\} \in \mathcal{E}$, and
5. $E, F \in \mathcal{E}$ implies $E \circ F \coloneqq \{(x,y) \mid \exists z, (x,z) \in E \wedge (z,y) \in F\} \in \mathcal{E}$.

The pair $(X, \mathcal{E})$ is then called a *coarse space*, and the elements of $\mathcal{E}$ are called *controlled sets* (or *entourages*). The motivating examples of coarse spaces are metric spaces. Indeed, if (X,d) is a metric space, X is endowed with the coarse structure

$$\mathcal{E}_d \coloneqq \left\{ E \subseteq X \times X \;\middle|\; \sup_{(x,y)\in E} d(x,y) < \infty \right\}.$$

A coarse space $(X, \mathcal{E})$ is called *metrizable* if $\mathcal{E} = \mathcal{E}_d$ for some metric d on X. It is well-known that $(X, \mathcal{E})$ is metrizable if and only if $\mathcal{E}$ is countably generated and connected[10] (see [32, Theorem 2.55]). If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are coarse spaces and $f\colon X \to Y$ is a map, f is called a *coarse embedding* if $(f \times f)[E] \in \mathcal{F}$ for all $E \in \mathcal{E}$ and $(f \times f)^{-1}[F] \in \mathcal{E}$ for all $F \in \mathcal{F}$.[11]

The definition of uniform Roe algebras naturally extends to uniformly locally finite coarse spaces: a coarse space $(X, \mathcal{E})$ is *uniformly locally finite*

[9] See [24, Chapter 7] for a shorter proof.

[10] In this context, *countably generated* means that there is a countable collection S of subsets of $X \times X$ such that $\mathcal{E}$ is the intersection of all coarse structures containing S, and *connected* means that $\{(x,y)\} \in \mathcal{E}$ for all $x, y \in X$. The connectedness condition in the metric setting means that metrics are not allowed to take infinite values.

[11] A map satisfying the former condition is called *coarse* and one satisfying the latter is called *expanding*.

if for each $E \in \mathcal{E}$ the cardinality of the vertical and horizontal sections,

$$E^x \coloneqq \{(x,y) \in E \mid y \in X\} \quad \text{and} \quad E_y \coloneqq \{(x,y) \in E \mid x \in X\},$$

are uniformly bounded. We say then that $a = [a_{xy}] \in \mathcal{B}(\ell_2(X))$ has *controlled support* if $\mathrm{supp}(a) \coloneqq \{(x,y) \mid a_{xy} \neq 0\}$ is in $\mathcal{E}$ and the *uniform Roe algebra of* $(X,\mathcal{E})$, denoted by $\mathrm{C}^*_u(X,\mathcal{E})$, is the norm-closure of all operators on $\ell_2(X)$ with controlled support. For brevity, we often simply write $\mathrm{C}^*_u(X)$ for $\mathrm{C}^*_u(X,\mathcal{E})$.

Before proving Theorem 1.11, we need some preliminary results. The following notation will be used: given a $*$-isomorphism $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$, $x \in X$, $y \in Y$, and $\eta > 0$, let

- $X_{y,\eta} \coloneqq \{z \in X \mid \|\Phi^{-1}(e_{yy})\delta_z\| \geqslant \eta\}$, and
- $Y_{x,\eta} \coloneqq \{z \in Y \mid \|\Phi(e_{xx})\delta_z\| \geqslant \eta\}$.

The next lemma isolates a result in [4] which we will need later.

Lemma 5.1 ([4, Lemma 4.7])**.** *Let $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ be uniformly locally finite coarse spaces, $\Phi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be a $*$-isomorphism, and $f\colon X \to Y$ be such that $\inf_{x\in X} \|\Phi(e_{xx})\delta_{f(x)}\| > 0$. The following holds:*

1. *If for all $\varepsilon > 0$ there is $\eta > 0$ such that*
$$\|\Phi(e_{xx})(1_{\ell_2(Y)} - \chi_{Y_{x,\eta}})\| \leqslant \varepsilon,$$
for all $x \in X$, then f is expanding.
2. *If for all $\varepsilon > 0$ there is $\eta > 0$ such that*
$$\|\Phi^{-1}(e_{f(x)f(x)})(1_{\ell_2(X)} - \chi_{X_{f(x),\eta}})\| \leqslant \varepsilon,$$
for all $x \in X$, then f is coarse. □

A simple application of Lemma 3.1 gives:

Corollary 5.2. *Let X and Y be uniformly locally finite coarse spaces and $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be a $*$-isomorphism. If X is metrizable, then for all $\epsilon > 0$ there is $\eta > 0$ such that $\|\Phi(e_{xx})(1_{\ell_2(Y)} - \chi_{Y_{x,\eta}})\| \leqslant \varepsilon$ for all $x \in X$.*

Proof. Applying Lemma 3.1 to the projections $(\Phi^{-1}(e_{yy}))_{y\in Y}$, we have that

$$\lim_{\eta\to 0} \inf_{x\in X} \|\Phi(e_{xx})\chi_{Y_{x,\eta}}\| = \lim_{\eta\to 0} \inf_{x\in X} \|\Phi^{-1}(\chi_{Y_{x,\eta}})e_{xx}\| = 1.$$

So, for all $\epsilon > 0$ there is $\eta > 0$ such that $\|\Phi(e_{xx})\chi_{Y_{x,\eta}}\| > 1 - \epsilon$. As each $\Phi(e_{xx})$ is a rank 1 projection (remember that Φ is rank-preserving), we have

$$1 = \|\Phi(e_{xx})1_{\ell_2(Y)}\|^2 = \|\Phi(e_{xx})(1_{\ell_2(Y)} - \chi_{Y_{x,\eta}})\|^2 + \|\Phi(e_{xx})\chi_{Y_{x,\eta}}\|^2,$$

and the result follows. □

We now present a technical lemma whose proof is inspired by [35, Proposition 3.1] (cf. [11, Proposition 2.4]). In a sense, this lemma shows that a kind of operator norm localization property holds for arbitrary spaces (see [11, Section 2] for the definition of the operator norm localization property).

Lemma 5.3. *Given $\epsilon, \delta > 0$, there is $\gamma > 0$ such that for all $s, t > 0$ there is $r > 0$ for which the following holds: Let X be a uniformly locally finite metric space, and let $p, q, a \in \mathcal{B}(\ell_2(X))$, where p is a projection and q is a rank 1 projection. If $\mathrm{prop}(q) \leqslant t$, $\|p - a\| < \gamma$, $\mathrm{prop}(a) \leqslant s$, and $\|pq\| \geqslant \delta$, then there is $C \subseteq X$ with $\mathrm{diam}(C) \leqslant r$ such that $\|p\chi_C\| \geqslant 1 - \epsilon$.*

Proof. Fix $\epsilon, \delta > 0$. Pick $k \in \mathbb{N}$ such that $(\delta/2)^{1/k} > 1 - \epsilon$. Pick a positive $\gamma < (\delta/2)^{1/k} - 1 + \epsilon$ small enough such that $\|p - a\| \leqslant \gamma$ implies $\|p - a^k\| \leqslant \delta/2$ for any projection p and any operator a in $\mathcal{B}(\ell_2(X))$.

From now on, fix $s, t > 0$, and $p, q, a \in \mathcal{B}(\ell_2(X))$ as in the statement of the lemma. By our choice of γ, $\|p - a^k\| \leqslant \delta/2$. Hence, as $\|pq\| \geqslant \delta$, we have that $\|a^k q\| \geqslant \delta/2$. Therefore, the classic telescoping argument implies that

$$\prod_{i=0}^{k-1} \frac{\|a^{i+1}q\|}{\|a^i q\|} \geqslant \frac{\delta}{2}$$

(notice that $a^i q \neq 0$ for all i's above). So, there is $j \in \{0, \ldots, k-1\}$ with

$$\|aa^j q\| \geqslant (\delta/2)^{\frac{1}{k}} \|a^j q\|.$$

As q is a rank 1 projection, we can pick a unit vector $\zeta \in \ell_2(X)$ such that $q = \langle \cdot, \zeta \rangle \zeta$. As $\mathrm{prop}(q) \leqslant t$, we have that $\mathrm{diam}(\mathrm{supp}(\zeta)) \leqslant t$. Let $\xi = a^j \zeta / \|a^j \zeta\|$. As $\mathrm{prop}(a) \leqslant s$, it follows that

$$\mathrm{prop}(a^j) \leqslant 2js + 2s \leqslant 2ks.$$

Therefore, we must have that $\mathrm{diam}(\mathrm{supp}(\xi)) \leqslant 4ks + t$.

At last, as $\|a\xi\| \geqslant (\delta/2)^{1/k}$, it follows that $\|p\xi\| \geqslant (\delta/2)^{1/k} - \gamma$. By our choice of γ, this shows that $\|p\xi\| \geqslant 1 - \varepsilon$. The conclusion follows by letting $r = 4ks + t$ and $C = \mathrm{supp}(\xi)$. □

Proof of Theorem 1.11. Let $\Phi \colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be a $*$-isomorphism and, for simplicity, let $\Psi = \Phi^{-1}$. We need to prove that X coarsely embeds into Y. As (X, d) is a metric space, Corollary 3.2 gives $\delta > 0$ and $f \colon X \to Y$ such that $\|\Phi(e_{xx})\delta_{f(x)}\| > \delta$ for all $x \in X$.

By Lemma 5.1 and Corollary 5.2, f is expanding. So, we are left to show that f is coarse.

Let $Z \coloneqq f(X)$ and pick $g \colon Z \to X$ such that $f(g(y)) = y$ for all $y \in Z$. So, by our choice of f, it follows that

$$\|\Psi(e_{yy})\delta_{g(y)}\| = \|\Phi(e_{g(y)g(y)})e_{yy}\| = \|\Phi(e_{g(y)g(y)})\delta_{f(g(y))}\| > \delta,$$

for all $y \in Z$.

Claim 5.4. For all $\epsilon > 0$ there is $r > 0$ such that for all $y \in Z$, there is $C \subseteq X$ with $\mathrm{diam}(C) \leqslant r$, and such that $\|\Psi(e_{yy})\chi_C\| \geqslant 1 - \epsilon$.

Proof. Fix $\epsilon > 0$ and let $\gamma > 0$ be given by Lemma 5.3 for ϵ and δ. As X is metrizable, Lemma 1.8 gives $s > 0$ such that each $\Psi(e_{yy})$ is γ-s-approximable. Let $r > 0$ be given by Lemma 5.3 for s and $t = 0$. For each $y \in Z$, pick $a_z \in \mathrm{C}^*_u(X)$ with $\mathrm{prop}(a_z) \leqslant s$ such that $\|\Psi(e_{yy}) - a_z\| \leqslant$

γ. Since $\|\Psi(e_{yy})e_{g(y)g(y)}\| > \delta$ for all $y \in Z$, the result now follows from Lemma 5.3. $\square$

Claim 5.5. For all $\epsilon > 0$, there is $\eta > 0$ such that $\|\Psi(e_{yy})\chi_{X_{y,\eta}}\| \geqslant 1-\varepsilon$, for all $y \in Z$.

Proof. This follows from the proof of [42, Lemma 6.7] or, equivalently, and with a more similar terminology, from [3, Lemma 7.4]. Indeed, in [3, Lemma 7.4] the metric spaces are assumed to have the operator norm localization property. However, an inspection of the proof reveals that the argument holds under the assumption that, for all $\epsilon > 0$, there is $r > 0$ such that, for each $z \in Z$, there is $C \subseteq X$ with $\mathrm{diam}(C) \leqslant r$ satisfying $\|\Psi(e_{yy})\chi_C\| \geqslant 1-\epsilon$. This statement is nothing else but Claim 5.4. $\square$

As each $\Psi(e_{yy})$ has rank 1, Claim 5.5 implies that, for all $\epsilon > 0$, there is $\eta > 0$ such that $\|\Psi(e_{yy})(1_{\ell_2(X)} - \chi_{X_{y,\eta}})\| \leqslant \varepsilon$, for all $y \in Z$ (cf. the proof of Corollary 5.2). By Lemma 5.1, we conclude that f is coarse. $\square$

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(F. P. Baudier) Texas A&M University, Department of Mathematics, College Station, TX 77843-3368, USA
E-mail address: florent@math.tamu.edu
URL: https://www.math.tamu.edu/~florent/

(B. M. Braga) University of Virginia, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, USA
E-mail address: demendoncabraga@gmail.com
URL: https://sites.google.com/site/demendoncabraga

(I. Farah) Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematički Institut SANU, Kneza Mihaila 36, 11 000 Beograd, p.p. 367, Serbia
E-mail address: ifarah@yorku.ca
URL: https://ifarah.mathstats.yorku.ca

(A. Khukhro) Murray Edwards College, Huntingdon Road, University of Cambridge, Cambridge, CB3 0DF, United Kingdom
E-mail address: ak467@cam.ac.uk
URL: https://sites.google.com/site/anakhukhromath/

(A. Vignati) Institut de Mathématiques de Jussieu (IMJ-PRG), Université de Paris, Bâtiment Sophie Germain, 8 Place Aurélie Nemours, 75013 Paris, France
E-mail address: alessandro.vignati@imj-prg.fr
URL: http://www.automorph.net/avignati

(R. Willett) University of Hawaiʻi at Mānoa, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96816, USA
E-mail address: rufus@math.hawaii.edu
URL: https://math.hawaii.edu/~rufus/