Tessellation Worksheet

In this module, we will be exploring tessellations and try to explain why Honeybees always build combs using hexagons.

What is a Tessellation?
A tessellation or tiling of the plane takes some area and covers it with shapes so that there is no overlap of the shapes and no space between the shapes. Here are some examples:

![Figure 1: Tessellations with "plus" shaped tiles and a Penrose tiling using two shapes (can you identify the two different shapes?)](image1.png)

Tessellations often also occur in art, architecture and fabrics to name a few. To the right is an image of batik fabric from Java Indonesia¹, which can be seen as a tiling. Often mosaics will also be in regular patterns such as the one to the right of a tile design in Marrakech². Traditional Polynesian tattoos often use tessellations in repeated geometric designs, including triangles and checkerboard designs such as the ones shown in figure 4.³ Other more modern tessellations include soccer balls or Tetris.

Tessellations also occur in nature, for example in the honeycomb or in the pattern on the outside of a pineapple.

![Figure 2: Batik from Java Indonesia](image2.png)  

![Figure 3: Ceramic Tiling from Marrakech](image3.png)

Can you think of other examples of tessellations?
Make Your Own Tessellation
Attached is a lesson plan from Exploratorium.edu about creating your own tessellation. It guides the student through creating their own tessellations and also introduces the concepts of Euclidean motions (translations, rotations and reflections).

Back to Honeybees...
Let’s brainstorm ideas about why Honeybees would want to choose hexagonal shapes for their combs. Here is a good link that discusses some of the factors [http://www.npr.org/blogs/krulwich/2013/05/13/183704091/what-is-it-about-bees-and-hexagons](http://www.npr.org/blogs/krulwich/2013/05/13/183704091/what-is-it-about-bees-and-hexagons)

Some of the reasons might be:
- Uniformity (all the same shape and regular)
- Convexity (both optimizes area/perimeter and for holding larva)
- Optimization of area/perimeter ratio (why?)

Regular Tessellations
Taking the first two points. Suppose that Honeybees want to build a **regular, polyhedral** tessellation. This means that only one type of polygon can be used and it must be regular (i.e. all equal sides). What are the possibilities.... Try it!

It turns out there are only 3 possible regular tessellations: triangles, squares and hexagons. Let’s use math and some algebra to explain why. Suppose we have p regular n-gons the meet up at a single point (vertex). The sum of there interior angles must be equal to 360 degrees:

\[
\text{rev}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{8},
\]

\[
deg, \quad 120^\circ, \quad 90^\circ, \quad 72^\circ, \quad 60^\circ, \quad 45^\circ,
\]

\[
\text{rad}, \quad \frac{2\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{5}, \quad \frac{\pi}{3}, \quad \frac{\pi}{4},
\]
But, not all polygons have interior angles that match these numbers. For a regular \( n \)-gon, the sum of the \( n \) exterior angles must be 360 degrees so and interior angle is \( 180 - \frac{360}{n} \). Therefore for \( p \) \( n \)-gons to meet we need

\[
180 - \frac{360}{n} = \frac{360}{p}
\]

or

\[
(n - 2)(p - 2) = 4
\]

The only integer solutions to this equation are \( n = 3, p = 6 \), \( n = 4, p = 4 \), and \( n = 6, p = 3 \) or 6 triangles, 4 squares and 3 hexagons, respectively.

**Ok, why hexagons then?**

This math problem goes back at least to 36 B when Marcus Terentius Varro gave two possible theories for why honeycomb was hexagonal, one was that it was related to the fact that bees have 6 feet and the second was the honeycomb conjecture. Namely, that no matter how you divide up the plane, if each cell has the same amount of area, then a regular hexagonal tiling has the minimal amount of perimeter, hence, beeswax required to build the hive.

This math conjecture was checked for polygons (only three possible cases as we saw above) by Pappus of Alexandria in 300 AD and only proved for convex tilings in 1943 by L. Fejes Toth. The final math proof was given by Thomas Hales in 1999.

While the math theorem about hexagons and its minimizing perimeter has been proven, the science about why honeybees chose hexagons is still open.

A recent study\(^vi\) shows how a hexagonal structure can develop naturally from circular comb being warmed, meaning that the bees wouldn’t have to do the math themselves. However, even wasp nests, which are made of paper instead of wax, are hexagons.

**What do you think? Why the hexagons in honeycomb?**

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\(^i\) http://www.winotosastro.com/batik/batikyogya.html  
\(^ii\) http://commons.wikimedia.org/wiki/File:Ceramic_Tile_Tessellations_in_Marrakech.jpg  
\(^iii\) Colored engraving of a Hawaiian chief from Beechey's *Voyage Autour du Monde* http://anthropology.si.edu/outreach/anthnote/Winter01/anthnote.html  
\(^iv\) http://www.exploratorium.edu/geometryplayground/Activities/GP_Activities_6-8/ExploringTessellations_%206-8_v4.pdf  
\(^v\) Image from http://thinkzone.wlonk.com/Units/AngleRev.htm  