MATH 321 / FALL 2011
ASSIGNMENT 3: DUE MONDAY, OCTOBER 3

Unless otherwise specified, your solutions should be written in complete, grammatically correct English sentences. Don’t forget to disclose with whom you worked on the assignment.

1. Solve the following (systems of) congruences. You do not need to write in complete sentences while performing the relevant computations, although you should mention if you invoke a theorem.
   (a) $5x \equiv -1 \pmod{8}$
   (b) $39x \equiv 15 \pmod{81}$
   (c) \[
   \begin{cases}
   3x & \equiv 1 \pmod{5} \\
   2x & \equiv 8 \pmod{16}.
   \end{cases}
   \]

2. Identify the pairs $\{a, a^{-1}\}$ of units of each of the following rings. You do not need to write in complete sentences.
   (a) $\mathbb{Z}/12\mathbb{Z}$
   (b) $\mathbb{Z}/8\mathbb{Z}$
   (c) $\mathbb{Z}/11\mathbb{Z}$
   (d) $\mathbb{Z}/22\mathbb{Z}$

3. (a) Let $m \geq 2$ be a natural number, and let $a \in \mathbb{Z}$. Prove that there exists an integer $e \geq 1$ such that $a^e \equiv 1 \pmod{m}$ if and only if $(a, m) = 1$. The least positive integer $e$ with this property is called the order of $a$ modulo $m$.
   (b) Now let $m = p$ be a prime number, and suppose $a \in \mathbb{Z}$ is not divisible by $p$. Let $e$ be the order of $a$ modulo $p$ (which exists by the first part of this exercise). Prove that $e \mid p - 1$. [Hint: Write $p - 1 = eq + r$ with $0 \leq r < e$ and apply Fermat’s Little Theorem.]

4. Find a formula for the first $n$ odd positive integers and prove it by induction.

5. The Fibonacci sequence $(F_n)_{n \geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ is defined by a linear recursion as follows:
   \[ F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2. \]
   In words, each term is the sum of the previous two (aside from the first and second terms in the sequence). Prove that each pair of successive terms in the Fibonacci sequence are relatively prime.