Please write your solutions carefully and neatly, preferably in complete and grammatically correct sentences. Justify all of your arguments.

1. There are 26 letters in the English alphabet.
   (a) Assuming that a person’s initials consist of exactly three letters, how many distinct sets of initials are possible?
   (b) According to Wikipedia, UH Manoa has 13,952 undergraduate students. Estimate the probability that no two students have the same initials. (Assume that you have no other information about the names of the students.)

2. A 5-card hand is dealt from a standard 52-card poker deck. Find the probability of getting a full house (two cards of one rank and three of another).

3. Thirteen cards are dealt from an ordinary 52-card deck. Find the probability that the last card dealt is an ace. (The answer is amazing . . . at least I think so.)

4. A computer has 3 processors that receive \( n \) jobs, with the jobs assigned to the processors purely at random so that all of the \( 3^n \) possible assignments are equally likely. Find the probability that exactly one processor receives no jobs.

5. Use Stirling’s formula to prove that the probability of obtaining exactly \( n \) heads on \( 2n \) tosses of a fair coin is approximately \( 1/\sqrt{\pi n} \) for large values of \( n \).

6. Bowl \( A \) contains 6 red chips and 4 blue chips. Five of these 10 chips are selected at random and without replacement and put in bowl \( B \), which was originally empty. One chip is then drawn at random from bowl \( B \). Given that this chip is blue, find the conditional probability that 2 red chips and 3 blue chips were transferred from bowl \( A \) to bowl \( B \).

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1You can check your answer on Wikipedia.
2If you get stuck, there’s a hint in the TeX file.
7 (Bonus). In this exercise, you will prove Stirling’s formula:
\[
\lim_{n \to \infty} \frac{\sqrt{2\pi n} (n/e)^n}{n!} = 1.
\]
Define the greatest integer function \([ \cdot ] : \mathbb{R} \to \mathbb{Z}\) and the fractional part function \(\{ \cdot \} : \mathbb{R} \to \mathbb{Z}\) by
\[
[x] := \max\{n \in \mathbb{Z} : n \leq x\} \quad \{x\} := x - [x].
\]
So \([x]\) is the largest integer not exceeding \(x\).

(a) Show that for any integrable function \(f : [1, \infty) \to \mathbb{R}\),
\[
\sum_{k=1}^{n} \int_{k}^{k+1} f(x) \, dx = \int_{1}^{n} (n - [x])f(x) \, dx.
\]

(b) Prove that the improper integral
\[
\int_{1}^{\infty} \frac{\{x\} - \frac{1}{2}}{x} \, dx := \lim_{n \to \infty} \int_{1}^{n} \frac{\{x\} - \frac{1}{2}}{x} \, dx
\]
converges. [\textbf{Hint:} First use the mean value theorem to show that
\[
\left| \frac{1}{x} - \frac{2}{2n + 1} \right| \leq \frac{1}{2n^2}
\]
for all \(x\) in the interval \([n, n + 1]\).]

(c) Prove that the limit \(A = \lim_{n \to \infty} \frac{\log(n!)}{n \sqrt{n}}\) exists. [\textbf{Hint:} Start by writing \(\log(n!) = \sum_{k=1}^{n} \log(k)\), and then use parts (a) and (b).] Therefore \(n! \approx A\sqrt{n} (n/e)^n\) for large \(n\).

(d) Let \(c_n = \int_{0}^{\pi} \sin^n x \, dx\). Show that \(c_n < c_{n-1}\) for all \(n \geq 1\).

(e) Show that \(c_n = \frac{n-1}{n} c_{n-2}\). Use this to prove that
\[
c_{2n} = \frac{\pi (2n)!}{2^{2n}(n!)^2} \quad c_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.
\]
[\textbf{Hint:} Write \(\sin^n x = \sin(x) \sin^{n-1}(x)\) and integrate by parts.]

(f) Use parts (c) and (e) to show that \(\lim_{n \to \infty} c_n \sqrt{2n} = \frac{2\pi}{A}\) and \(\lim_{n \to \infty} c_{2n+1} \sqrt{2n+1} = A\). Conclude using part (d) that \(A = \sqrt{2\pi}\). [\textbf{Hint:} For the last part, show that \(A^2 \leq 2\pi\) and \(A^2 \geq 2\pi\).]
Solution 1 (Problem 1). (a) The multiplication rule tells us that there are
\[ N = (26)(26)(26) = (26)^3 = 17,576 \]
possible sets of initials.

(b) This is the birthday paradox again. Write \( n = 13,952 \) for ease of notation. Let us order the students of UH Manoa as \( s_1, \ldots, s_n \). The number of possible initial sequences is \( N^n = [(26)^3]^{13,952} \), and the number of sequences with no repetition is \( N P_n = N!/(N-n)! \).

If we assume that all sets of initials are equally likely\(^3\), then the probability of having no repeated set of initials among the UH student body is
\[
\frac{N P_n}{N^n} = \frac{[17,576]!}{[17,576]^{13,952}[17,576-13,952]!} \approx 1.44 \times 10^{-3574}
\]

The final answer can be obtained with Stirling’s formula.

Solution 2 (Problem 2). The number of 5-card hands is \( \binom{52}{5} \). In a 5-card hand, the number of ways to select a rank for the pair is \( \binom{13}{1} \), and there are \( \binom{4}{2} \) ways to select two out of four suits for the pair. Similarly, there are \( \binom{12}{1} \) ranks for the three-of-a-kind, and \( \binom{4}{3} \) choices of three suits for it. The multiplication rule tells us that
\[
P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}}{\binom{52}{5}} \approx 0.00144 = 0.144\%.
\]

Solution 3 (Problem 3). Let \( A_k \) be the event that exactly \( k \) aces appear in the first 12 cards dealt. Let \( B \) be the event that an ace is dealt as the 13th card. Then
\[
P(B) = \sum_{k=0}^{4} P(B|A_k)P(A_k).
\]

Now there are \( \binom{52}{12} \) possible 12-card deals. There are \( \binom{4}{k} \) ways to choose \( k \) aces, and \( \binom{48}{12-k} \) ways to choose the remaining \( 12 - k \) cards so that none are aces. The multiplication rule gives
\[
P(A_k) = \frac{\binom{4}{k}\binom{48}{12-k}}{\binom{52}{12}}.
\]

If exactly \( k \) aces appear in the first 12 cards dealt, then the probability of an ace appearing as the 13th card is \( P(B|A_k) = (4-k)/40 \). Thus
\[
P(B) = \sum_{k=0}^{4} \frac{4-k}{40} \frac{\binom{4}{k}\binom{48}{12-k}}{\binom{52}{12}} = \frac{1}{13}.
\]

Solution 4 (Problem 4). Despite the hint, it is perhaps easier to solve this problem as follows. There are \( \binom{3}{1} \) processors to which we could assign zero jobs. After choosing one, we must count the number of ways to assign \( n \) jobs to the remaining two. If the remaining two processors are labeled \( a \) and \( b \), then an assignment of jobs may be viewed as an ordered sequence of \( a \)'s and \( b \)'s of length \( n \), where an \( a \) in position \( i \) corresponds to assigning job \( i \)

\(^3\)This is evidently false since there are more common last names than others. However, as with the birthday paradox, this is the worst case one has to treat.
to processor $a$ (and similarly for $b$). There are $2^n$ such sequences, and hence $2^n$ assignments of jobs. However, we must exclude the sequence $a \cdots a$ and $b \cdots b$ because these assign all jobs to a single processor, and hence two of our three processors would receive no job at all. Hence there are $2^n - 2$ valid assignments of jobs.

By the multiplication rule, the probability of assigning no job to exactly one processor is
\[
\frac{\binom{3}{1} (2^n - 2)}{3^n}.
\]

Solution 5 (Problem 5). There are $2^n$ possible sequences of heads and tails that could occur, and we are interested in the ones where exactly $n$ heads occur. There are $\binom{2n}{n}$ possible ways to position exactly $n$ heads in a sequence of $2n$ tosses of a coin, so the probability we seek is
\[
\frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{(n!)^2 2^{2n}} \approx \frac{\sqrt{2\pi} \cdot 2n(2n/e)^{2n}}{(\sqrt{2\pi n})^2 (n/e)^{2n} 2^{2n}} = \frac{1}{\sqrt{n}}.
\]

Solution 6 (Problem 6). For integers $m$ and $n$, let $mRnB$ denote the event that $m$ red chips and $n$ blue chips are selected from the first bowl and placed into the second. Let $C$ be the event that a blue chip is selected from the second bowl. We wish to compute $P(2R3B|C)$. This can be done using Bayes’ Theorem:
\[
P(2R3B|C) = \frac{P(C|2R3B) \cdot P(2R3B)}{\sum_{k=1}^{5} P(C|kR(5-k)B) \cdot P(kR(5-k)B)}.
\]
(Note that $1 \leq k \leq 5$ because there are only 4 blue chips in bowl $A$ at the beginning.)

For a fixed $k$, we may use the hypergeometric distribution to compute that
\[
P(kR(5-k)B) = \frac{\binom{6}{k} \binom{4}{5-k}}{\binom{10}{5}}.
\]

Also, since there are only 5 chips in bowl $B$ after the initial selection, the conditional probability of $C$ is $P(C|kR(5-k)B) = (5-k)/5 = 1 - \frac{k}{5}$. Substituting these values into (1) yields
\[
P(2R3B|C) = \frac{5}{14}.
\]