1. In a ten-question true-false exam, find the probability that a student gets a grade of 70 percent or better by guessing. What if the test has 25 questions?


3. Question 3.2.28, page 117.

4. Question 3.2.32, page 118.

5. Question 3.3.7, page 129.

6. Question 3.3.9, page 129.

7. Question 3.3.14, page 129.

8. Question 3.3.15, page 129.

9 (Bonus). Let $X$ equal the number of flips of a fair coin that are required to observe heads on consecutive flips. Let $a_n$ equal the $n$th Fibonacci number, where $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

(a) Show that the pdf of $X$ satisfies
$$f_X(k) = \frac{a_{k-1}}{2^k}, \quad k = 2, 3, 4, \ldots$$

(b) In a linear algebra course, you might learn that the Fibonacci numbers are given by the formula
$$a_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].$$

Use this to prove that
$$\sum_{k \geq 2} f_X(k) = 1.$$

Conclude that there is zero probability of flipping a fair coin and never seeing a pair of consecutive heads.
Solution 1 (Problem 1). We can view the $n$ questions on the exam as independent trials, where success occurs with probability $p = 1/2$. If $X$ is a binomial random variable for $n$ trials and probability of success $p$, then we let $F_X$ be its cdf. In the question, we are asked to find the probability of getting a 70% or better, which is to say

$$P(\text{get a 70\% or better}) = P(X \geq .7n) = 1 - P(X < .7n) = \begin{cases} 1 - F_X(6) & \text{if } n = 10 \\ 1 - F_X(17) & \text{if } n = 25. \end{cases}$$

Using the table of binomial probabilities on the web site, we see that when $n = 10$, $F_X(6) \approx 0.828$, and when $n = 25$, we have $F_X(17) \approx 0.978$. Plugging this in above, we see that

$$P(\text{get a 70\% or better}) = \begin{cases} 0.172 & \text{if } n = 10 \\ 0.022 & \text{if } n = 25. \end{cases}$$

Solution 2 (Problem 3). Let $A_{\text{Black}}, A_{\text{Brown}}, A_{\text{Blue}}$ the events that he pulls out a matched pair of black, brown, or blue socks, respectively. Then these three events are mutually exclusive, so that

$$P(\text{matched pair}) = P(A_{\text{Black}}) + P(A_{\text{Brown}}) + P(A_{\text{Blue}}).$$

Suppose that we view the ten socks in his drawer as being of two types: black and not-black. Then the distribution associated to pulling two socks out is hypergeometric with $r = 2$ and $w = 8$ (here $r$ means black and $w$ means not-black). So the probability of a black pair is

$$P(A_{\text{Black}}) = \frac{\binom{2}{2} \binom{8}{0}}{\binom{10}{2}} = \frac{1}{45}.$$ 

The same calculation shows that $P(A_{\text{Blue}}) = 1/45$ since there are 2 blue socks in his drawer. For brown socks, we have $r = 6$ and $w = 4$ (where $r$ means brown and $w$ means not-brown). Then

$$P(A_{\text{Brown}}) = \frac{\binom{6}{2} \binom{4}{0}}{\binom{10}{2}} = \frac{15}{45}.$$ 

So the total probability of getting a matched pair is

$$P(\text{matched pair}) = \frac{1}{45} + \frac{15}{45} + \frac{1}{45} = \frac{17}{45} \approx 0.378.$$

Solution 3 (Problem 4). We are going to sample 10 machines and accept them if at most one is defective. Suppose that $n$ out of 100 machines is defective. Then our sampling process obeys a hypergeometric distribution, so that

$$P(\text{sample passes inspection}) = \frac{\binom{n}{0} \binom{100-n}{10} + \binom{n}{1} \binom{100-n}{9}}{\binom{100}{10}}.$$

Viewing this last expression as a function of $n$, we can plug this into Sage (or your favorite math software) and plot its value for $0 \leq n \leq 100$. Here is my Sage code:

```sage
f(n) = (binomial(n,0)*binomial(100-n,10) + binomial(n,1)*binomial(100-n,9)) / binomial(100,10)
L = [(n,f(n).numerical_approx()) for n in range(101)]
list_plot(L, axes_labels=['$n$','$P(\text{Sample \ passes})$'])
```

This gives the operating characteristic curve displayed below. It appears from the figure,
Figure 1. Here $n$ is the number of machines presumed defective.

and it can be checked using the above formula, that we will accept an incoming shipment with $\n = 15$ defective machines about 53% of the time. We will accept an incoming shipment with $\n = 16$ defective machines about 49.977% $\approx 50\%$ of the time.

**Solution 4** (Problem 7). If $p_X(K)$ is the pdf for $X$, then the cdf for $X$ satisfies

$$F_X(k) = \sum_{j \leq k} p_X(j) = p_X(k) + \sum_{j \leq k-1} p_X(j) = p_X(k) + F_X(k-1).$$

Solving for $p_X(k)$ gives

$$p_X(k) = F_X(k) - F_X(k-1) = \frac{k^2 + k}{42} - \frac{k^2 - k}{42} = \frac{x}{21}.$$
Note that we cannot apply this calculation when $k = 0$; however, the resulting formula is still valid in this case. So the relevant values of $F_X(k)$ and $p_X(k)$ are

<table>
<thead>
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<th>$k$</th>
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<th>$p_X(k)$</th>
</tr>
</thead>
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<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<tr>
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<tr>
<td>5</td>
<td>$5/7$</td>
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</tr>
<tr>
<td>6</td>
<td>$1$</td>
<td>$2/7$</td>
</tr>
</tbody>
</table>

**Solution 5** (Problem 9). (a) Consider sequences of length $k$ consisting of the symbols $H$ (heads) and $T$ (tails). Write $A_k$ for the set of sequences of length $k$ that end in $H$ and that have no consecutive $H$’s. Similarly, write $B_k$ for the set of sequences of length $k$ that end in $T$ and that have no consecutive $H$’s. Since a sequence in $A_k$ cannot have an $H$ in the $(k - 1)$st position, it follows that the first $k - 1$ symbols of a sequence in $A_k$ form a sequence in $B_{k-1}$. In particular,

$$|A_k| = |B_{k-1}|. \quad (1)$$

A sequence in $B_k$ looks like a sequence in either $A_{k-1}$ or $B_{k-1}$ with a $T$ appended to it. So we have

$$|B_k| = |A_{k-1}| + |B_{k-1}| = |B_{k-2}| + |B_{k-1}|, \quad (2)$$

where the second equality follows from (1). Notice that this last relation is the same as the relation for the Fibonacci sequence; i.e., the $k$th term is the sum of the preceding two terms. We note also that

$$B_1 = \{T\}, \quad B_2 = \{HT, TT\}.$$  
Hence $|B_1| = 1$ and $|B_2| = 2$. Since these are two consecutive terms of the Fibonacci sequence, we find (by induction) that $|B_k| = a_{k+1}$.

Now we compute the probability that a pair of heads is observed on the $k$th flip (and not before). This happens precisely when a sequence in $A_{k-1}$ has an $H$ appended to it. Since there are $2^k$ sequences of length $k$, we have

$$P(HH \text{ observed on exactly the } k\text{th flip}) = \frac{|A_{k-1}|}{2^k} = \frac{|B_{k-2}|}{2^k} \quad \text{by } (1) = \frac{a_{k-1}}{2^k}.$$  

(b) For ease of notation, set $\varphi = \frac{1 + \sqrt{5}}{2}$ and $\psi = \frac{1 - \sqrt{5}}{2}$. Now the sum in question breaks into a difference of geometric series. Sum the series and use the fact that $\varphi \psi = -1$. 