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Journal Title: Mathematics of the USSR. Sbornik.
Volume: 23(4)
Issue: 1974
Pages: 561-592
Article Author:

Article Title: V. Drinfeld; Elliptic Modules

Imprint: [Providence, R.I.; American Mathematica

ILL Number: 86466122

Call #: QA1 .M35 v.23 (1974)

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Abstract. The notion of elliptic module is introduced, generalizing the concept of an elliptic curve, and an analog of the theory of elliptic and modular curves is constructed. Here the role of the group $GL(2, \mathbb{Q})$ is played by $GL(2, k)$, where $k$ is a function field. A theorem on the coincidence of $L$-functions of modular curves and Jacquet-Langlands $L$-functions corresponding to $k$ is proved.

Bibliography: 14 items.

Introduction

A) Statement of the problem and main result. Let $k$ be a global field, $\infty$ a place of $k$, $d$ a natural number, and $k_\infty$ the completion of $k$ at $\infty$.

Definition. The triple $(k, \infty, d)$ is called admissible if a) all places of $k$, except perhaps $\infty$, are nonarchimedean, and b) $d \leq [k_\infty : k_\infty]$.

Admissible triples are of the following types:

1) $k = \mathbb{Q}$, $\infty$ is the archimedean norm, and $d = 1$;

2) $k = \mathbb{Q}$, $\infty$ is the archimedean norm, and $d = 2$;

3) $k$ is an imaginary quadratic extension of $\mathbb{Q}$, $\infty$ is the archimedean norm, and $d = 1$;

4) $k$ is a function field, and $\infty$ and $d$ are arbitrary.

The goal of this paper is the generalization of three classical theorems (connected with the first three types of admissible triples): 1) the Kronecker-Weber theorem, 2) the Eichler-Shimura theorem on $\zeta$-functions of modular curves, and 3) the fundamental theorem on complex multiplication. This generalization is connected with the fourth type of admissible triple.

Let $(k, \infty, d)$ be an admissible triple. We introduce the following notation: $A$ is the ring of elements of $k$ which are integral at all places except $\infty$; $\mathfrak{U}$ is the ring of adèles of $k$; and $\mathfrak{U}_j$ is the ring of adèles without the $\infty$ component. We formulate the classical theorems 1)–3) in a form suitable for generalization. First we introduce some necessary definitions.

Let $(k, \infty, d)$ be an admissible triple of type 1)–3), and let $K$ be a field over $A$ (i.e. there is given a homomorphism $i: A \to K$).


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Definition. An elliptic $A$-module of rank $d$ over $K$ is the following: in case 1), a homogeneous torus over $K$; in case 2), an elliptic curve over $K$; and in case 3), an elliptic curve $X$ over $K$ together with a homomorphism $\phi: A \to \text{End} X$ such that $i = D \circ \phi$, where $D: \text{End} X \to K$ is a differential.

In the same way we introduce the concept of an elliptic $A$-module of rank $d$ over $S$, where $S$ is a scheme over $A$.

Definition. Let $S$ be a scheme over $k$, $I \subset A$ a nonzero ideal, and $X$ an elliptic $A$-module of rank $d$ over $S$. A structure of level $I$ on $X$ is an isomorphism of $A$-modules over $S$, $(I^{-1}/A)^d \times S \cong X$, where $X \subset X$ is the annihilator of $I$.

If $I$ is sufficiently small, then the functor that associates to the scheme $S$ over $k$ the set of isomorphism classes of elliptic $A$-modules of rank $d$ over $S$ with structure of level $I$ is representable by a smooth $(d-1)$-dimensional manifold $N_I$. We set $N = \lim_{\to} N_I$. We can define in a natural way an action of the group $\text{GL}(d, \overline{\mathbb{Q}})$ on $N$.

Theorem 1. Let $(k, \infty, d)$ be a triple of type 1) or 3). Then $N$ is the spectrum of the maximal abelian extension of $k$, which is completely split over $\infty$. The action of $\text{GL}(d, \overline{\mathbb{Q}})$ coincides with the action from class field theory.

Theorem 2. Let $(k, \infty, d)$ be a triple of type 2), let $\overline{N}$ be a smooth compactification of $N$, and let $I$ be a prime number. Then $H^1(\overline{N}, \overline{Q}) \simeq \bigoplus_i U_i \otimes W_i$, where $U_i$ is an irreducible representation of $\text{GL}(2, \overline{Q})$ over $\overline{Q}$, and $W_i$ is a representation of $\text{Gal}(\overline{Q}/Q)$ in the space $Q^1$. Let $U_i = \bigotimes_i U_i^p$, where $U_i^p$ is an irreducible representation of $\text{GL}(2, \overline{Q})$ over $\overline{Q}$, and let $W_i^p$ be the restriction of $W_i$ on $\text{Gal}(\overline{Q}/Q)$. $W_i^p$ is an irreducible representation of $GL(2, \overline{Q})$, and let $\omega_i^p$ be the representation contragredient to $U_i^p$, and the $L$-functions are taken in the sense of Serre and Jacquet-Langlands.

1) If $p \neq l$ and $U_i^p$ is a representation of class 1 (cf. [1]), then $W_i^p$ is unramified and $L(s, W_i^p) = L(s, U_i^p)$. Here $\omega_i^p$ is the representation contragredient to $U_i^p$, and the $L$-functions are taken in the sense of Serre and Jacquet-Langlands.

2) $\bigoplus_i U_i \otimes C_0 \cong \text{Hom}_{\text{GL}(2, \mathbb{R})}(\Pi, A_0)$, where $\Pi$ is a representation of $\text{GL}(2, \mathbb{R})$ in the space of functions on $P_1(\mathbb{R})$, factored out by the constants, and $A_0$ is the space of parabolic forms* on $\text{GL}(2, \mathbb{R})$ in the sense of [10].

In this paper the concept of elliptic module for triples of type 4) is introduced, modular varieties are constructed, and analogs of Theorems 1 and 2 are proved for $d = 1, 2$.

B) Outline of the paper. The concept of elliptic module is introduced in $\S 2$. In $\S 3$ an analytic theorem is proved on the uniformization of an elliptic module over $\mathbb{R}$ with the help of a lattice in $\mathbb{R}$. In $\S 5$ a universal family of elliptic $A$-modules of rank $d$ is constructed. The proof of smoothness of the modular varieties uses properties of formal modules (cf. $\S 4$ and 4) (a formal module is the analog of a formal group). Also in $\S 5$, a congruence relation of the form $[\Delta]$ is proved. In $\S 6$ the modular varieties are constructed analytically (as factors of some domain $\Omega^d$ by a discrete group). The domain $\Omega^d$ is closely connected with the Bruhat-Tits complex of the group $\text{GL}(d, \mathbb{R})$. In $\S 7$ elliptic modules over complete discrete normed fields are studied. In $\S 8$ an analog of Theorem 1 is studied. In $\S 9$ the compactifications of modular surfaces (for $d = 2$) are considered. The analog of Theorem 2 is proved.

This article was written with the help of the Department of Mathematics and Mechanics of the Moscow State University, the Institute of Mathematics of the Academy of Sciences of the USSR, and the Institute of Mathematics of the Academy of Sciences of the USSR. The author expresses his gratitude to the above institutions for their valuable remarks and assistance.

A) Definitions and Notation. A formal series $F \in B[[x(y)], y]$ such that $F(F(x, y), z)$, a homomorphism of formal series $\beta \in B[[x]]$ such that $\beta(x) = x$), there is a canonical homomorphism of the number $d(\beta)(0)$.

Example. $F(x, y) = x + y$.

Let $B$ have characteristic $p$ and $\beta$ is defined by a series $\sum_{i=0}^{\infty} b_i x^i$, where $b_i$ is the endomorphism of multiplication modulo $p$ to the series $x^p$ will be denoted by $\beta$. The group consists of "series" $\sum_{i=0}^{\infty} b_i x^i$ in this ring by $B[[x]]$.

Let $O$ be a ring, $B$ an algebra.

Definition. A formal $O$-module of rank $d$ over $B$, and $f$ is a homomorphism of formal modules.

Example. $F(x, y) = x + y$.

This module is called additive.

The germ of a formal $O$-module is $(x, y)_*/(x, y)_p$ and $f_a \in B[[x]]$ from the definition of a formal module.

We consider the following sets of sets: $B \mapsto$ set of formal $O$-modules by some algebra $A$. (The germ of the formal $O$-modules $F$ and $F_a$, and the germ for $(F, f)$ to be a formal $O$-module $A$.

$A$ has a natural gradation $O$-modules over $B$ mod $\text{deg}(a)$ by homomorphisms $\bigoplus_{k=0}^{\infty} A_k \cong B_{\text{ker}(a)}$ of the form $ab$, where $a \in A_k$, $b \in A_{k+1}$ ideal. We denote by $A_0$ the factor.

Proposition 1.1. Let $n \geq 2$. Let $a$ and $b(a)$ (for all $a \in A$)
modular surfaces (for \( d = 2 \)) are constructed; this uses results of §7. In §§10 and 11 the analog of Theorem 2 is proved.

This article was written under the influence of the paper [14] and also of conversations with I. I. Pjateckiĭ-Šapiro. However, modular varieties have been constructed algebraically (as in [6]). The analytic description of modular varieties is based on ideas from [3] (although the congruence subgroups of \( GL(2, \mathbb{A}) \) are not Schottky groups).

The author expresses his deep gratitude to Ju. I. Manin and I. I. Pjateckiĭ-Šapiro for their valuable remarks and attention to this paper.

§1. Formal modules

A) Definitions and notation. A formal group over a ring \( B \) is, by definition, a formal series \( F \in B[[x, y]] \) such that \( F(x, y) = F(y, x) \), \( F(x, x) = 0 \), and \( F(x, F(y, z)) = F(F(x, y), z) \). A homomorphism from a formal group \( F \) into a formal group \( G \) is a series \( \beta \in B[[x]] \) such that \( \beta F(x, y) = G(\beta(x), \beta(y)) \). For any formal group \( F \) over \( B \) there is a canonical homomorphism \( D: \text{End} F \rightarrow B \) that sends the endomorphism \( \phi \) to the number \( \phi(0) \).

Example. \( F(x, y) = x + y \); this group is called additive.

Let \( B \) have characteristic \( p \). Every endomorphism of the additive group over \( B \) is defined by a series \( \sum_{i=0}^{\infty} b_i x^p^i \). Henceforth we shall identify the element \( b \in B \) with the endomorphism of multiplication by \( b \). The Frobenius endomorphism (corresponding to the series \( x^p \)) will be denoted by \( r \). Thus the ring of endomorphisms of the additive group consists of "series" \( \sum_{i=0}^{\infty} b_i x^p^i \) with the commutation rule \( rb = b^p r \). We denote this ring by \( B[r] \).

Let \( O \) be a ring, \( B \) an algebra over \( O \), and \( y: O \rightarrow B \) the natural homomorphism.

Definition. A formal \( O \)-module over \( B \) is a pair \((F, f)\), where \( F \) is a formal group over \( B \), and \( f \) is a homomorphism from \( O \) to \( \text{End} F \) such that \( D \circ f = y \).

Example. \( F(x, y) = x + y \) and \( f_a(x) = ax \) for \( a \in O \) (we write \( f_a \) instead of \( f(a) \)).

This module is called additive.

The germ of a formal \( O \)-module over \( B \) mod deg \( n \) is a pair \((F, f)\), where \( F \in B[[x, y]]/(x, y)^n \) and \( f_a \in B[[x]]/(x^n) \), and all relations between \( F \) and \( f \) which arise from the definition of a formal module are satisfied mod deg \( n \).

We consider the following functor from the category of \( O \)-algebras to the category of sets: \( B \mapsto \text{set of formal } O \text{-modules over } B \). We can clearly represent this functor by some algebra \( \Lambda_O \). (The generators of \( \Lambda_O \) are the "indeterminate coefficients" of the series \( F \) and \( f_a \), and the relations between them are those which are required for \((F, f)\) to be a formal \( O \)-module.)

\( \Lambda_O \) has a natural gradation. It is easy to check that the set of germs of formal \( O \)-modules over \( B \) mod deg \((n+1)\) is canonically isomorphic to the set of \( O \)-module homomorphisms \( \bigoplus_{k=0}^{n+1} \Lambda_Q^k \rightarrow B \) such that \( \psi(ab) = \psi(a)\psi(b) \) and \( \psi(1) = 1 \). Elements of the form \( ab \), where \( a \in \Lambda_O \), \( b \in \Lambda_O \), deg \( a > 1 \), and deg \( b > 1 \), generate a homogeneous ideal. We denote by \( \tilde{\Lambda}_O \) the factor of \( \Lambda_O \) by this ideal.

Proposition 1.1. Let \( n \geq 2 \). Then \( \tilde{\Lambda}_O^{-1} \) (as an \( O \)-module) can be defined by generators \( a \) and \( h(a) \) (for all \( a \in O \)) and the relations

\[ a_{n-1} = h(a) \text{ (for all } n \geq 2 \text{).} \]
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\[ a(a^n - a) = \begin{cases} h(a), & \text{if } n \text{ is not a power of a prime number} \\ h(a) \circ p, & \text{if } n = p^k. \end{cases} \]  

(1)

\[ h(a + b) - h(a) - h(b) = aC_n(a, b), \]

\[ ah(b) + bh(a) = h(a, b). \]  

(2)

(3)

(Here \( C_n(x, y) = (x + y)^n - x^n - y^n \) if \( n \) is not a power of a prime number, and \( C_n(x, y) = ((x + y)^n - x^n - y^n)/p \) if \( n = p^k \).)

\[ \text{Proof.} \] Clearly, \( \tilde{\Lambda}_n^{n-1} \) can be defined by the generators \( h(a) \) and \( c_i \) \((0 < i < n)\) with the relations

\[ a(x + y + \Phi(x, y)) + h(a)(x + y)^n = ax + ay + h(a)x^n + h(a)y^n + \Phi(ax, ay), \]

\[ ax + h(a)x^n + bx + h(b)x^n + \Phi(ax, bx) = (a + b)x + h(a + b)x^n, \]

\[ abx + h(ab)x^n = a(bx + h(b)x^n) + h(a)(bx)^n, \]

\[ \Phi(x, y) = \Phi(y, x), \quad \Phi(x, z) + \Phi(y, z) = \Phi(x + y, z) \]

(where \( \Phi(x, y) \) is defined by \( \sum_{1 \leq i \leq n} c_i x^i y^{n-i} \)). It is well known that from the last two relations follows \( \Phi(x, y) = aC_n(x, y) \) (cf. [7]). □

B) The case when \( O \) is a field.

**Proposition 1.2.** 1) If \( O \) is a field, then every formal \( O \)-module is isomorphic to an additive module.

2) If \( O \) is infinite, then there exists a unique isomorphism with an additive module, whose derivative at zero equals 1. In this case \( \Lambda_n \cong O[c_1, c_2, \ldots] \), where \( \deg c_i = i \).

\[ \text{Proof.} \] The case of characteristic zero is well known (cf. [7]). Let \( \text{char } O = p \), let \( B \) be an algebra over \( O \), and let \( (F, f) \) be a formal \( O \)-module over \( B \). Then multiplication by \( p \) in \( F \) is zero; therefore (cf. [7]) \( F \) is isomorphic to an additive group. We can assume that \( F \) is additive. Let \( f : O \to B[[x]] \) have the form \( f(a) = a + \phi(a)x \) \( (mod \, x^{k+1}) \). We shall prove that there exists a \( s \in B \) such that \( (1 + sx)/(1 + x)^{k-1} = a + \phi(a) + \phi(b) + \phi(a \circ b) = a\phi(b) + b\phi(a) \). If \( O \subset F \) \( (mod \, p^k) \), then the existence and uniqueness of \( s \) is clear \( (s = \phi(a)/(a^{p^k} - a), \) where \( a \in O, \phi \in F \) \( (mod \, p^k) \).

If \( O \subset F \) \( (mod \, p^k) \), then \( \phi \) is differentiation; therefore \( \phi = 0 \). □

C) Computation of \( \Lambda_n \), when \( O \) is the ring of integers of a local nonarchimedean field. From now on, \( O \) will be the ring of integers of a local nonarchimedean field \( K \), \( \pi \) will be a prime element of \( O \), \( p \) the characteristic of \( O/(\pi) \) and \( q \) the order of \( O/(\pi) \).

**Proposition 1.3.** \( \tilde{\Lambda}_n^{n-1} \cong O \). More precisely:

1) If \( n \) is not a power of \( q \), then \( h(a) = (a^n - a)u \) and \( aC_n(x, y) = u[(x + y)^n - x^n - y^n] \), where \( u \) is a generator of \( \tilde{\Lambda}_n^{n-1} \).

2) If \( n = q^r \), then \( h(a) = (a^n - a)u/p \) and \( a = pu/\pi \), where \( u \) is a generator of \( \tilde{\Lambda}_n^{n-1} \).

**Proof.** 1) If \( n \) is not a power of \( p \), then \( h(a) \) can be expressed through \( a \) by using formula (1). If \( n \) is a power of \( p \), then \( h(a) \) is determined from (2).

2) Let \( n \) be a power of \( q \), so \( h(a^n - a)/n \), and \( a \mapsto f/n \). It means \( M = \Lambda_n^{n-1}/h(n) \). If \( x \in \Lambda_n^{n-1} \), then \( h(a^n - a)/n \) \( (mod \, n^2) \) \( (mod \, p) \), then \( \tilde{h}(a) \) \( (mod \, (p^n - 1)x) \), and so \( \tilde{h} = 0 \). Then \( \tilde{h} \) is zero. □

**Corollary.** 1) Every germ of \( \Lambda_n \) is \( O \)-consistent.

2) If \( B \to C \) is an epimorphism, then \( C \) arises from an \( O \)-module over \( B \).

**Proposition 1.5.** 1) Let \( (F, f) = (G, g) \) \( (mod \, n) \).

a) If \( n \) is not a power of \( p \), then \( h(a) \) is the same for \( F \) and \( G \).

b) If \( n \) is a power of \( q \), then \( h(a) \) is the same for \( F \) and \( G \).

2) Let \( q \in B[[x]] \) \( (mod \, p^k) \).

Then

\[ \tilde{h}(a) = g_0(a) \]

**Proof.** Statement 1 is a restatement of the previous proposition. □

**Corollary.** The formal \( O \)-modules are isomorphic if and only if the coefficients of \( f_p \) are equal.

**D) Classification of formal o-modules.**
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using formula (1). If $n$ is a power of $p$, but not a power of $q$, then there exists $a \in O$ such that $a^n - a \not\equiv 0 \pmod{q}$; from (3) it follows that $(a^n - a)h(b) = (b^n - b)h(a); \ a$ can be determined from (2).

2) Let $n$ be a power of $q$. There exists an epimorphism $\varLambda^n_{\mathcal{O}} \rightarrow O$ sending $h(a)$ to $(a^n - a)/\pi$, and $a \mapsto a/\pi$. It remains to prove that $\varLambda^n_{\mathcal{O}}$ is generated by $h(a)$. Let $M = \varLambda^n_{\mathcal{O}}/\langle h(n) \rangle$. If $x \in \varLambda^n_{\mathcal{O}}$, then we denote by $\overline{x}$ the image of $x$ in $M$. Since $h(a^n) = \pi h(b) = \pi^n h(b)$, then $\overline{h(n)}h = 0$ for any $b \in O$. In particular, $\overline{h}(p) = 0$. But $h(p) = (p^n - 1)a$, and so $\overline{a} = 0$. Then $M$ is an $O/(\pi)$-module, and $\overline{f}: O/(\pi) \rightarrow M$ is differentiable. Therefore $\overline{h} = 0$, i.e. $M = 0$. □

Proposition 1.4. $\varLambda^n_{\mathcal{O}} \cong O[g_1, g_2, \ldots], \deg g_i = i$.

Proof. It follows from Proposition 1.3 that there exists an epimorphism $O[g_1, g_2, \ldots] \rightarrow \varLambda^n_{\mathcal{O}}$ consistent with the gradation. Proposition 1.2 implies that $\varLambda^n_{\mathcal{O}} \otimes K \cong K[c_1, c_2, \ldots]$ and $\deg c_i = i$. Therefore the epimorphism we have constructed is an isomorphism. □

Corollary. 1) Every germ of a formal $O$-module arises from a formal $O$-module.

2) If $B \rightarrow C$ is an epimorphism of $O$-algebras, then every formal $O$-module over $B$ arises from an $O$-module.

Proposition 1.5. 1) Let $(F, f)$ and $(G, g)$ be formal $O$-modules over $B$, and let $(F, f) = (G, g) \mod deg n$.

a) If $n$ is not a power of $g$, then

$$F(x, y) = G(x, y) + \left[\left(x+y\right)^n - x^n - y^n\right] \mod deg (n+1),$$

$$f_a(x) = g_a(x) + \left(a^n - a\right)x^n \mod deg (n+1), \ a \in B;$$

b) If $n$ is a power of $g$, then

$$F(x, y) = G(x, y) + \frac{a}{a^n}C_n(x, y) \mod deg (n+1),$$

$$f_a(x) = g_a(x) + \frac{a^n - a}{\pi}x^n \mod deg (n+1), \ a \in B.$$

2) Let

$$\varphi(B[[x]], \varphi(x) = x - ux^n \mod deg (n+1),$$

Then

$$F(x, y) = G(x, y) + \left[\left(x+y\right)^n - x^n - y^n\right] \mod deg (n+1),$$

$$f_a(x) = g_a(x) + \left(a^n - a\right)x^n \mod deg (n+1).$$

Proof. Statement 1 is a reformulation of Proposition 1.3, and statement 2 can be checked directly. □

Corollary. The formal $O$-module $(F, f)$ is isomorphic to an additive module if and only if the coefficients of $f_a$ are divisible by $n$.

D. Classification of formal $O$-modules over fields of "finite characteristic."
A homomorphism of formal $O$-modules is a homomorphism of formal groups that commutes with the action of $O$.

Let $E$ be a field over $O/(n)$ and let $\phi$ be a homomorphism of formal $O$-modules over $E$. It is known [7] that if $\phi \neq 0$, then $\phi(x) = \psi(x^{p^k})$ and $\psi'(0) \neq 0$. Since $\phi$ commutes with the action of $O$, it follows that $\log_p q|k$. The number $k/\log_p q$ is called the height of $\phi$; the height of the zero homomorphism is $\infty$. The height of a formal $O$-module is the height of the endomorphism of multiplication by $n$.

**Remark.** If $O'$ is the integral closure of $O$ in a finite extension of $K$ and $n = [O':O]$, then every formal $O'$-module is also an $O$-module, and its $O$-height is $n$ times its $O'$-height.

**Proposition 1.6.**
1) There exist modules of arbitrary height.
2) There exist nonzero homomorphisms only between modules of the same height.
3) A formal $O$-module of height $h$ is isomorphic to the additive module $\mod deg q^h$.

**Proof.** 1) We consider a homomorphism $\lambda: \Lambda \rightarrow [g_1, g_2, \ldots] \rightarrow E$ such that $\lambda(g_{i,q^h}^{-1}) \neq 0$ and $\lambda(g_1) = 0$ for $i < q^h - 1$. To this corresponds a formal $O$-module over $E$ of height $h$.
2) This follows from the fact that the height of the composition of homomorphisms equals the sum of their heights.
3) This follows from Proposition 1.5.

**Proposition 1.7.**
1) All formal $O$-modules of height $h < \infty$ over a separably closed field $E$ are isomorphic.
2) The ring of endomorphisms of such a module is isomorphic to the ring of integers of a central division algebra over $K$ with invariant $1/h$.

**Proof.** A formal $O$-module $(F, f)$ will be called normal if the following conditions are satisfied:
1) $f_i(x) = x^{q^h}$,
2) $F \in F_{q^h}[x, y]$ for $a \in F_{q^h}[x]$ for $a \in O$,
3) $F(x, y) = x + y \mod q^{h}$; $f_i(x) = ax \mod q^{h}$.

a) Every formal $O$-module over $E$ of height $h$ is isomorphic to a normal one. Indeed, as in [7], by means of a change of variables we will have $f_i(x) = x^{q^h}$. Then conditions 1) and 2) will be satisfied. By Proposition 1.6 we perform a change of variables with coefficients from $F_{q^h}$, after which condition 3) is satisfied (and, as before, conditions 1) and 2)).

b) With the help of Proposition 1.5 it can be shown that between any two normal formal $O$-modules of height $h$ there exists an isomorphism which is the identity mod $deg(q^h + 1)$.

c) Let $L$ be the ring of endomorphisms of a normal formal $O$-module of height $h$. Clearly, $O \subset L$, $L$ contains no divisors of zero, and $L$ is complete in the $\pi$-adic topology. It follows from b) that every germ of an endomorphism of our module mod $q^h$ with coefficients from $F_{q^h}$ arises from an endomorphism. Therefore it follows that $\dim L/nL = q^h$.

Therefore $L \otimes K$ is a central division algebra on $L \otimes K$, and so $L$ is a maximal subring of $\sum_{i=1}^d a_i q_i^{h_i}$ that the invariant of $L$.

**§ 2. Elliptic $A$-modules.**

A) **Definitions and notation.**

Let $A$ be an elliptic $A$-module of height $h$. In the algebra $A$ (as in §1). We shall identify the elements of the $\mathbb{Q}$-algebra $A$ by $b$. Every endomorphism of the algebra $A$ is $\sum_i b_i t^i$, and, moreover, $b = b_0$.

We have two homomorphisms:

The following notation will be used:

The field of characteristic $p$; $\infty$ corresponding to the place $v$ of $K$ to $v$ (or its continuation to $A$). We now $a_i \in K$ for $i \leq 1$ for $v \neq \infty$ if $v = \infty$.

Let $K$ be a field over $A$ (or $K$).

**Definition.** An elliptic $A$-module is one for which $i = D \circ \phi$ and $\phi \neq 0$.

**B) Rank and places of $i$**

Let $i = i^*(\Spec K) \in \Spec A$ (i.e. the ideal of $A$).

Thus $i$ is an imbedding if and only if $i = 0$.

**Definition.** An elliptic $A$-module is one that $i = D \circ \phi$ and $\phi \neq 0$.

**Proposition 2.1.**

a) $\phi$ is a homomorphism.

b) There exists $a > 0$ such that $\deg a_i = p^h$.

**Proof.**

a) If $\ker \phi \neq 0$, then $\ker L$ consists of divisors of zero, and so $\ker L$ is the identity mod $\ker (\phi)$.

b) Clearly $\deg \phi(ab) = \deg \phi(a) \cdot \deg \phi(b)$. Hence $\deg \phi(a) = 0$ if and only if $a = a_i$.

Therefore $\deg \phi = \deg \phi$ is a homomorphism.

**Definition.** $d$ is called the **rank** of $\phi$.

**Example.** Let $A = F_{q^h}$.

The representation of the form $\sum_{i=1}^d a_i q_i^{h_i}$ is equivalent to the representation of the form $\sum_{i=1}^d a_i q_i^{h_i}$.

**Proposition 2.2.** Let $K$ be an elliptic $A$-module. Let $K$ be the algebra $A$.
it follows that \( \dim L/\mathfrak{n}L = h^2 \), and that the center of \( L/\mathfrak{n}L \) coincides with \( O/(\mathfrak{n}) \). Therefore \( L \otimes K \) is a central division algebra of dimension \( h^2 \). The height is a norm on \( L \otimes K \), and so \( L \) is a maximal order in \( L \otimes K \). It follows from the relation 
\[ r_{L \otimes K} = \operatorname{deg} \Phi \] 
that the invariant of \( L \otimes K \) equals \( 1/h \). \( \square \)

\section{Elliptic modules (algebraic approach)}

A) Definitions and notation. Let \( B \) be a ring of characteristic \( p \). We denote by \( \tau \) the endomorphism of the (algebraic) additive group over \( B \) that sends \( t \) to \( t^p \) (just as in \( \S 1 \)). We shall identify the element \( b \in B \) with the endomorphism of multiplication by \( b \). Every endomorphism of the (algebraic) additive group over \( B \) has the form \( \sum b_i t_i \), and, moreover, \( \tau b = b^p \). We denote the ring of such “polynomials” by \( B[\tau] \).

We have two homomorphisms: \( \ell : B \rightarrow B[\tau], \ell(b) = b, \) and \( D : B[\tau] \rightarrow B, D(\sum b_i t_i) = b_0 \).

The following notation will be used throughout this paper. \( k \) is a global field of characteristic \( p \); \( \infty \) is a fixed place of \( k \); \( k_v \) is the completion of \( k \) corresponding to the place \( v; \left| \cdot \right|_v \) is the normed absolute value corresponding to \( v \) (or its continuation to a finite extension \( k_v \)); we write \( \left| \cdot \right| \) for \( \left| \cdot \right|_\infty \); \( A = \{ x \in k \mid |x|_v \leq 1 \text{ for } v \neq \infty \} \); if \( v \in \text{Spec } A \), then \( A_v \) is the completion of \( A \) at \( \left| \cdot \right|_v \).

Let \( K \) be a field over \( A \) (i.e. there is defined \( i : A \rightarrow K). \) We call the place \( i^* \text{Spec } A \) the “ideal Ker \( i \)” the “characteristic” (notation: “char”). Thus \( i \) is an imbedding if and only if \( K \) has general “characteristic.”

Definition. An elliptic \( A \)-module over \( K \) is a homomorphism \( \psi : A \rightarrow K[\tau] \) such that \( i = D \circ \psi \) and \( \psi \neq e \circ i \).

B) Rank and places of finite order. We define a mapping \( \deg : K[\tau] \rightarrow \mathbb{Z} \) in the following way: \( \deg \sum a_i t^i = p^n \) for \( a_i \neq 0; \deg 0 = 0 \).

Proposition 2.1. a) \( \psi \) is an imbedding.

b) There exists \( d > 0 \) such that \( \deg \psi(a) = |a|^d \) for \( a \in A \).

Proof. a) If Ker \( \psi \neq 0 \), then Ker \( \psi \) would be a maximal ideal (since \( K[\tau] \) has no divisors of zero), and so \( \text{Im } \psi \) would be a field, i.e. \( \text{Im } \psi \subset \mathbb{K}(\tau) \) and so \( \psi = e \circ i \).

b) Clearly

\[ \deg \psi(ab) = \deg \psi(a) \cdot \deg \psi(b), \quad \deg \psi(a + b) \leq \max(\deg \psi(a), \deg \psi(b)), \]

\[ \deg \psi(a) = 0 \text{ if and only if } a = 0, \quad \deg \psi(a) \geq 1 \text{ for } a \neq 0, \text{ and } \deg \psi(a) > 1 \text{ for some } a \in A. \]

Therefore \( \deg \circ \psi \) extends to a nontrivial absolute value on \( k \), which cannot correspond to a finite place. \( \square \)

Definition. \( d \) is called the rank of the elliptic \( A \)-module \( \psi \).

Example. Let \( A = \mathbb{F}_q[x] \). Then \( \psi|_{\mathbb{F}_q} = e \circ i|_{\mathbb{F}_q} \). The representation of \( \psi \) is equivalent to the representation of \( \phi(x) \in \mathbb{K}[\tau] \). In place of \( \phi(x) \) one can take any element of the form \( (x) + \sum a_j x^j \tau_{L \otimes K}, \) where \( d \geq 1, a_d \neq 0, a_j \in K \). The rank of such a module equals \( d \).

The representation of an elliptic \( A \)-module over \( K \) changes any \( K \)-algebra into an \( A \)-module. Let \( K \) be the algebraic closure of \( K \).

Proposition 2.2. Let \( K \) be a divisible \( A \)-module. If \( a \in A, a \neq 0, \) then the number
of points of order $a$ in $K$ does not exceed $|a|^d$, where $d$ is the rank of the elliptic $A$-module. Equality holds if and only if $i(a) \neq 0$. The torsion submodule in $\mathcal{K}$ is isomorphic to $\bigoplus_{\nu \in \text{Spec } A} (K_{\nu}/A_{\nu})^{i_{\nu}}$, where $i_{\nu} < d$ for $\nu \neq \text{"char" } K$.

Proof. Every divisible torsion $A$-module is isomorphic to $\bigoplus_{\nu \in \text{Spec } A} (K_{\nu}/A_{\nu})^{i_{\nu}}$, where the $i_{\nu}$ are found by counting the number of places of order $a$. □

Corollary. The rank of an elliptic $A$-module is a natural number.

Remark. Since $K[r] \subset K[r]$, every elliptic $A$-module over $K$ defines a formal $A$-module over $K$. If $K$ has general characteristic, then every formal $A$-module over $K$ uniquely extends to a formal $k$-module. If $\nu = \text{"char" } K = \nu \in \text{Spec } A$, then every formal $A$-module over $K$ is uniquely extended from an $A_{\nu}$-module. The height of the formal $A_{\nu}$-module corresponding to the elliptic $A$-module is equal to $d - i_{\nu}$.

C) Isogenies. Let $\phi: A \to K[r]$ and $\psi: A \to K[r]$ be elliptic $A$-modules over $K$. A homomorphism from $\phi$ to $\psi$ is an element $\alpha \in K[r]$ such that $\alpha \phi(a) = \psi(a) \alpha$ for $a \in A$. A nonzero homomorphism is called an isogeny.

Remark. By comparing powers, it follows that isogenies exist only between modules of the same rank.

Every homomorphism of elliptic $A$-modules is also a homomorphism of their additive groups, and so one can consider the kernel of a homomorphism of elliptic modules.

The kernel of an isogeny is a finite $A$-invariant group subscheme in the additive group.

Proposition 2.3. Let $\phi$ be an elliptic $A$-module. For a finite group subscheme $H$ of the additive group, invariant with respect to $A$, to be the kernel of an isogeny from $\phi$ to some other module, it is necessary and sufficient that

a) if $K$ has general "characteristic", then $H$ must be reduced;

b) if $\nu = \text{"char" } K \in \text{Spec } A$ and $q$ is the order of the residue field of $\nu$, then $H_{\nu} = \text{Spec } k[1]/(q^{i_{\nu}})$, where $H_{\nu}$ is the connected component of $H$.

Proof. An additive group is obtained by factoring the additive group by $H$. Let $\psi: A \to K[r]$ be a homomorphism whose kernel is $H$. Since $H$ is invariant with respect to $A$, there exists a unique homomorphism $\psi: A \to K[r]$ such that $\psi(a) = \phi(a) \psi(a)$ for $a \in A$. Clearly $\psi \phi = \psi$ and $D(\psi(a)) = [i(a)]^n$, where $n$ is the order of $H_{\nu}$. Therefore, for $\psi$ to be an elliptic module, it is necessary and sufficient that $i(a^n) = i(a)$ for $a \in A$. □

Corollary. Any isogeny can be multiplied by another isogeny to an endomorphism which is multiplication by $a \in A$, $a \neq 0$.

Proposition 2.4. Let $X$ and $Y$ be elliptic $A$-modules of rank $d$ over $K$. Then $	ext{Hom}(X, Y)$ is a projective $A$-module of finite dimension (not exceeding $d^2$). If $\nu \in \text{Spec } A$, then the homomorphism $\text{Hom}(X, Y) \otimes_A A_{\nu} \to \text{Hom}_{A_{\nu}}(T_{\nu}X, T_{\nu}Y)$ is injective (here $T_{\nu}X$ is the $\nu$-component of the torsion submodule of $K$), and its cokernel is torsion-free.

Proof (cf. [2]). 1) $	ext{Hom}(X, Y)$ is a torsion-free $A$-module. If $u, w \in \text{Hom}(X, Y)$, then $\text{deg}(u + w) \leq \max(\text{deg } u, \text{deg } w)$.

2) Let $\phi$ be an elliptic $A$-module. If $\nu \neq \text{"char" } K$, then $\text{Hom}(X, Y) \otimes_A A_{\nu} \to \text{Hom}_{A_{\nu}}(T_{\nu}X, T_{\nu}Y)$ is a module of finite type.

Corollary. Let $X$ be an $A$-module. Then $\text{End } X$ is a projective $A$-module, dim $\text{End } A = \text{dim } \text{End } A_v = \nu$.

§3.

A lattice over $L$ is a finitely generated free discrete subgroup of $\mathbb{L}$ with respect to $\text{Gal}(L^{\text{sep}}/L)$. Let $\mathcal{E}(\Gamma)$ be the set of lattices isomorphic from $\Gamma_1$ into $\Gamma_2$. A lattice isomorphism is called a lattice isomorphism.

Proposition 3.1. The category $\mathcal{E}(\Gamma)$ is a category of lattices.

Proof. 1) Let $L$ be a lattice over $L$. Then $L$ is a lattice over $L$ if $L$ is a lattice over $L$ and $L$ is a lattice over $L$.

Lemma. Let $E$ be a field over $L$. Then $g(z) = \prod_{\alpha \in \Delta}(z - \alpha)$. Then $g(z) = \prod_{\alpha \in \Delta}(z - \alpha)$.

Proof. Clearly $g(z + w) = g(z) + g(w)$, and $g(z + w) + g(z) + g(w)$ divides $g(z + w) - g(z) - g(w)$. □

Since $\Gamma$ is the union of lattices, $f(z + w) = f(z) + f(w)$. Clearly $f(z)$ is a lattice, and $f(z)$ is a lattice for all $z \in L$. Therefore $f(z) = \prod_{\alpha \in \Delta}(z - \alpha)$.

2) Let $\phi$ be an elliptic $A$-module. Let $\mathcal{E}(k)$ be the set of formal $k$-modules over $L$ (cf. §1.2, there exists a unique $f$ such that $f(z)$ is a lattice over $L$), and shall prove that the formal homomorphism $|a| > 1$ and $\phi(a) = a + \sum_{\alpha \in \Delta} a_{\alpha}$.
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deg(u + w) ≤ max(deg u, deg w) and deg(aw) = |a|d deg u, deg u ≥ 1 for u ≠ 0. Therefore, if V ⊆ Hom(X, Y) ⊅ A k is a finite-dimensional subspace, then V ∩ Hom(X, Y) is a module of finite type.

2) Let a ∈ A such that |a|∗ < 1 and |a|w = 1 for w ≠ v, ∞. Then the homomorphism Hom(X, Y)/(a^k) → Hom,top(X, Y)/(a^k) is injective, and so lim Hom(X, Y)/(a^k) ⊆ Hom,X X, Y)/(a^k). On the other hand, Hom(X, Y) ⊅ A k → lim Hom(X, Y)/(a^k) is a monomorphism. □

Corollary. Let X be an elliptic A-module of rank d over K. Then End X is a projective module, dim End X ≤ d^2, and End X ⊅ A k∞ is a division ring. If K has general "characteristic", then End X is commutative and dim End X ≤ d.

§ 3. Elliptic modules (analytic approach)

Let L be a finite extension of k∞, and let L^s be the separable closure of L. A lattice over L is a finitely-generated discrete A-submodule in L^s, invariant with respect to Gal(L^s/L). Let Γ_1 and Γ_2 be lattices over L of dimension d. A morphism from Γ_1 into Γ_2 is a number α ∈ L such that αΓ_1 ⊆ Γ_2. Composition of morphisms is defined by multiplication of numbers.

Proposition 3.1. The category of elliptic modules of rank d over L is isomorphic to the category of lattices of dimension d over L.

Proof. 1) Let L be a lattice of dimension d over L. We set

f(z) = z \prod_{\alpha \in \Lambda} \left( 1 - \frac{z}{\alpha} \right).

(This product clearly converges uniformly in every circle, since f is an entire function.) First we prove the following lemma.

Lemma. Let E be a field, char E = p, and let Δ be a finite subgroup of E. Let g(z) = \sum_{\alpha \in \Delta} (z - \alpha). Then g(z + w) = g(z) + g(w).

Proof. Clearly g(z + w) - g(z) - g(w) = 0 for z ∈ Δ or w ∈ Δ. Therefore g(z)g(w) divides g(z + w) - g(z) - g(w). Comparing powers, we obtain g(z + w) = g(z) + g(w).

Since f is the union of an increasing sequence of finite subgroups, we see that f(z + w) = f(z) + f(w). Clearly f induces a group isomorphism \overline{L}/\Gamma \cong \overline{L}. Since Γ is an A-module, the structure of an A-module passes over from \overline{L}/\Gamma to \overline{L}. If a ∈ A and a ≠ 0 then f(az) and \prod_{\beta \in 1/\alpha \Gamma \Gamma} (f(az) - f(\beta)) are analytic functions of z with the same divisors. Therefore f(az) = P_a f(z), where P_a is a polynomial of degree |a|^d.

2) Let φ be an elliptic A-module of rank d over L. There is a corresponding formal k-module over L (cf. the remark at the end of § 2B). According to Proposition 1.2, there exists a unique f = 1 + \sum t_i b_i \in L[[t]] such that f(a) = φ(a)f for a ∈ A. We shall prove that the formal homomorphism f is an analytic homomorphism. Let a ∈ A, |a| > 1 and φ(a) = a + \sum t_i a_i t_i. Let i > s. From the identity f(a) = φ(a)f we obtain

(a^d - a) b_i = \sum_{j=1}^{s} a_j b_{i+j}.

Let \( c_i = |b_i|^{p^{-i}} \). Then \( |a_i| |c_i| \leq \max_{1 \leq j \leq s} (|a_i|^{1 - \theta} a_j^{-\theta} - |c_i|) \). Let \( 1/|a| < \theta < 1 \), and \( c_i \approx a_j \). \( \max_{1 \leq j \leq s} \epsilon_{i-j} \) for sufficiently large \( i \). Therefore \( c_i \rightarrow 0 \). Let \( \Gamma \subset T \) be the kernel of \( \rho \). Clearly \( \Gamma \subset L^2 \), \( \Gamma \) is invariant with respect to \( \text{Gal}(L^2/L) \), \( \Gamma \) is an \( A \)-module, and \( (1/a)\Gamma \approx (A/\langle a \rangle)^a \). Since \( \Gamma \) is discrete, \( \Gamma \) is an lattice over \( L \) of dimension \( d \).

3) Let \( \Gamma_1 \) and \( \Gamma_2 \) be lattices of dimension \( d \) over \( L \), \( \alpha \in L \), and \( \alpha \Gamma_1 \subset \Gamma_2 \).

Let \( f_1 \) and \( f_2 \) be the entire functions constructed in 1) from the lattices \( \Gamma_1 \) and \( \Gamma_2 \). The function \( f_2(\alpha z) \) is invariant with respect to \( \Gamma_1 \). Repeating the discussion from step 1), we get \( f_2(\alpha z) = P(f_1(z)) \), where \( P \) is a polynomial. \( P \) defines a homomorphism of elliptic \( A \)-modules. On the other hand, let the polynomial \( P \) define a homomorphism of elliptic \( A \)-modules. Then \( P \circ f_1 \) is a formal homomorphism from an additive \( k \)-module into a formal \( k \)-module corresponding to the second elliptic \( A \)-module. It follows from Proposition 1.2 that \( P(f_1(z)) = f_2(\alpha z) \) for a uniquely defined \( \alpha \in \mathbb{L} \).

Clearly, \( \alpha \Gamma_1 \subset \Gamma_2 \). \( \Box \)

Corollary. For any \( A \) and \( d \) there exist elliptic \( A \)-modules of rank \( d \) over \( k_\mathbb{L}^\mathbb{L} \).

\( \S 4. \) Universal deformations of formal modules

A) Deformations of zero level (cf. [13]). We shall use the same notation as in \( \S 3 \). Let \( O^{nr} \) be a maximal unramified extension of \( O \), and let \( \hat{O}^{nr} \) be the completion of \( O^{nr} \). We consider the category \( \mathcal{C} \) whose objects are complete local \( \hat{O}^{nr} \)-algebras whose residue fields are isomorphic to \( \hat{O}^{nr}/(\pi) \). The morphisms of \( \mathcal{C} \) are local homomorphisms of \( \hat{O}^{nr} \)-algebras.

Let \( (G, g) \) be a formal \( O \)-module over \( \hat{O}^{nr}/(\pi) \), and let \( R \in \mathcal{C} \). A deformation of the module \( (G, g) \) with basis \( R \) is a formal \( O \)-module \((F, /)\) over \( R \) whose reduction modulo a maximal ideal is \( (G, g) \).

Proposition 4.1. Let \((F, /)\) and \((F', /')\) be deformations of the modules \((G, g)\) and \((G', g')\) with basis \( R \). Let \( \phi: \text{ker}(F, /) \rightarrow \text{ker}(F', /') \) be a homomorphism inducing the zero homomorphism \((G, g) \approx (G', g')\). If the height of \((G, g)\) is finite, then \( \phi = 0 \).

Proof. Let \( m \in R \) be a maximal ideal. It suffices to consider the case when \( m^{r+1} = 0 \) and \( \phi \equiv 0 \) mod \( m^r \), \( r \geq 1 \). Then \( F'(\phi(x), \phi(y)) = \phi(F(x), F(y)) \), \( F'_a(\phi(x)) = a \phi(x) \), and so \( \phi(F(x, y)) = \phi(F(x), F(y)) \) and \( \phi(\partial_a(x)) = a \phi(x) \). Let \( l: m^r \rightarrow R/m \) be a linear function, \( \phi = l(\phi) \). Then \( \phi \) is a homomorphism from \((G, g)\) into an additive \( O \)-module, and so \( \phi = 0 \). \( \Box \)

Deformations \((F_1, /_1)\) and \((F_2, /_2)\) of the module \((G, g)\) are called isomorphic if there exists an isomorphism \((F_1, /_1) \approx (F_2, /_2)\) inducing the identity automorphism on \((G, g)\). (If \((G, g)\) has finite height, then such an isomorphism is unique.)

Proposition 4.2. Let \((G, g)\) be a formal \( O \)-module over \( \hat{O}^{nr} \) of finite height \( h \).

The functor that associates to \( R \in \mathcal{C} \) the set of deformations of the module \((G, g)\) up to isomorphism is represented by the algebra \( \hat{O}^{nr}/[t_1, \ldots, t_{h-1}] \).

Proof. Let \( O[\epsilon_1, \epsilon_2, \ldots] = \Lambda_\mathbb{L} \hat{O}^{nr}/(\pi) \) be the homomorphism corresponding to \((G, g)\). We can assume that under this homomorphism \( g_i \rightarrow 0 \) for \( i < q^h - 1 \). Let \((F^0, /^0)\) be a deformation of \((G, g)\) corresponding homomorphism \( h \rightarrow 0 \). Let \( \epsilon_i \rightarrow 0 \).

Let \( M \) be a vector space \((G, g)\) with coefficients in \( M \).

\[ \Delta(y, z) + \Delta(x, y, G(y, z)) \]
\[ \delta_a(x) + \delta_a(y) + \delta_a(z) + \delta_a(x + y + z) + \Delta(g_a(x, y, z)) \]

A coboundary of the series
\[ \Delta(x, y) \equiv \psi(G(x, y)) \]

Let \( R \in \mathcal{C} \), \( m \in R \) a maximal ideal, and let \((G, g)\) with basis \( R \).

Lemma 1) The existence of a deformation \((F', /')\) over \( R \) that is congruent to \((G, g)\) up to isomorphism \((F', /') \equiv (G, g)\) with coefficients in \( \hat{O}^{nr}/(\pi) \).

2) Two cocycles with corresponding deformations are equal.

Proof. If \((F', /') \equiv (F'', /'')\) and \((F', /') \equiv (F'', /'')\) is the isomorphism, then assertions can be verified directly.

Let \( \psi \equiv \hat{O}^{nr}/[t_1, \ldots, t_{h-1}] \mod m^r \). For the proof of uniqueness of a homomorphism \( \psi_0(\nu(F^0, /^0), \rho) \equiv (F', /') \).

Let \( \psi(t_i) = \phi(t_i) + \epsilon_i \in m^r \). Then \( \phi_0(\nu(F^0, /^0), \rho) \equiv \psi_0(\nu(F^0, /^0), \rho) \).

It remains to show that the change of coefficients in \( \hat{O}^{nr}/(\pi) \). This is clear, and the second was established.

a) The coboundary \( x^i \) is congruent to \((x + y)^i \) modulo \( m^r \).

The coboundary \( x^i \) is congruent to \((x + y)^i \) modulo \( m^r \).
Let $1/|a| < 0 < 1$, and $c_i = 0$.

Let $\Gamma \subset \mathbb{L}$ be the kernel of $f$, $L = \Gamma \subset \mathbb{L}, \Gamma$ is an $A$-module, and we have $L$ of dimension $d$.

Let $\alpha \in L, a \in L$, and $\alpha \Gamma \subset \mathbb{L}_\alpha$.

From the lattices $\Gamma_1 \subset \mathbb{L}$ and $\Gamma_2$, we get the discussion from the previous paragraph.

To define a homomorphism $P$ from a homomorphism from an algebraic $A$-module, we need the second elliptic $A$-module. It is uniquely defined $\alpha \in L$.

Elliptic modules of rank $d$ over $k^n$.

Elliptic modules

We will use the same notation as in II. Let $\hat{\mathcal{O}}^{nr}$ be the complete local $\hat{\mathcal{O}}^{nr}$.

The morphisms of $C$ are

Let $R \subset C$. A deformation of $(F, f)$ over $R$ whose reduction

beings the study of the modules $(G, g)$

Since $\Delta$ is a morphism inducing the formal group $G$, $g$ is finite, then $\phi = 0$.

Consider the case when $\phi(x) + \phi(y) + f_\phi(\phi(x)) = 0$. Let $m' \to R/m$ be a reduction of $(G, g)$ into an additive

morphism corresponding to $g$ are called isomorphic if $\phi$ is the identity automorphism 

and the identity automorphism is unique.

$\hat{\mathcal{O}}^{nr}$ of finite height $h$.

(1) Elliptic modules

The morphism corresponding to $0$ for $i < q^b - 1$. Let

$\Delta = (\Delta, \delta)$ denote the isomorphism of the deformations $(F', f')$ and $(F'', f'')$ is the identity mod $m'$ (by Proposition 4.1). The remaining assertions can be verified directly.

Let $\phi: \hat{\mathcal{O}}^{nr}[t_1, \ldots, t_{h-1}] \to R$ be a homomorphism such that $\phi(f^0, f^0) = (F, f)$ mod $m'$. For the proof of the proposition it suffices to show the existence and uniqueness of a homomorphism $\psi: \hat{\mathcal{O}}^{nr}[t_1, \ldots, t_{h-1}] \to R$ such that $\psi = \phi$ mod $m'$ and $\psi(f^0, f^0) \approx (F, f)$. Let $\phi: \hat{\mathcal{O}}^{nr}[t_1, \ldots, t_{h-1}] \to R$ be a homomorphism such that $\phi(t_0) = \phi(t_0) + \epsilon_i + \delta_i \in m'$. Then the difference between the cocycles corresponding to $\phi_h(f^0, f^0)$ and $\psi_h(f^0, f^0)$ has the form $\Sigma_{i=1}^{h-1} \epsilon_i(\Delta_i, \delta_i)$, where $(\Delta_i, \delta_i)$ is a cocycle with coefficients in $\hat{\mathcal{O}}^{nr}/(m)$, which depends neither on $\epsilon_i$ nor on $r$.

Then

$$\Delta_i = 0 \mod deg q^i, \Delta_i = 0 \mod deg (q^i + 1).$$

It remains to show that the classes $(\Delta_i, \delta_i)$ form a basis for the cohomology with coefficients in $\hat{\mathcal{O}}^{nr}/(m)$. This follows from the following two assertions (the first is clear, and the second was essentially proved in §1B).

a) The coboundary $x^n$ is congruent to

$$((x + y)^n - x^n - y^n, (a^n - a) x^n) \mod deg (n + 1).$$

The coboundary $x^n q_i$ is congruent to

$$\left\{\frac{1}{\pi} C_{\pi} q_i, h_i - a^{\pi} \frac{a^{d+h}}{\pi} x^{d+h}\right\} \mod deg (q^i + 1), h_i = 0.$$
b) Let \((\Lambda, \delta)\) be a cocycle, and let \((\Lambda, \delta) \equiv 0 \mod \deg n\). If \(n\) is not a power of \(q\), then
\[
(\Lambda, \delta) \equiv \{v[(x + y)^n - x^n - y^n], v(a^n - a)x^n\} \mod \deg (n + 1).
\]
If \(n\) is a power of \(q\), then
\[
(\Lambda, \delta) \equiv \left\{ h \cdot \frac{b}{a} c_n(x, y), h \cdot a^n - a x^n \right\} \mod \deg (n + 1).
\]

B) Deformations of arbitrary level. Let \(R \in C\), and let \(m \subset R\) be a maximal ideal. The assignment of a formal \(O\)-module \((F, \phi)\) over \(R\) turns \(m\) into an \(O\)-module. Let the reduction of \((F, \phi)\) modulo \(m\) have finite height \(h\). Let \(n \in \mathbb{Z}\), \(n \geq 0\).

Definition. A structure of level \(n\) on a formal \(O\)-module \((F, \phi)\) is an \(O\)-module homomorphism \(\phi: ((1/q^n)O/O)^h \to n\) such that \(f_n(x)\) is divisible by
\[
\prod_{\alpha \in (1/q^n)_{x^n}} (x - \alpha).
\]

For \(n \geq 1\) it follows that \(f_n(x)\) and \(\prod_{\alpha \in (1/q^n)_{x^n}} (x - \alpha)\) divide each other.

Indeed, \(f_n(x)\) is divisible by \(\prod_{\alpha \in (1/q^n)_{x^n}} (x - \alpha)\) if and only if \(\phi(x)\) remains to show that \(\phi(x)\) is invertible modulo \(\mathfrak{p}^n\), which is equivalent to the square of the maximal ideal of \(R\).

Setting \(r = h\), we obtain a deformation of level \(n\).

Proposition 4.3. 1) The functor that associates to \(R \in C\) the set of deformations of level \(n\) of the module \((G, g)\) up to isomorphism is represented by some ring \(D_n\).

2) \(D_n\) is a regular ring. Let \(n \geq 1\), and let \(e_i\) \((i = 1, \ldots, h)\) be a basis for \(((1/q^n)O/O)^h\) as an \(O/(q^n)\)-module. The images of \(e_i\) in \(D_n\) under the universal deformation of level \(n\) form a system of local parameters.

3) Let \(m \leq n\). The homomorphism \(D_m \to D_n\) is finite and flat.

Proof. a) Let \((F, \phi)\) be a universal deformation with basis \(D_0 \cong \mathfrak{D}_{n}[[t_1, \ldots, t_{n-1}]]\).

Let \(0 \leq r \leq h\). Consider the functor \(\Phi_{\mathfrak{p}}\) that associates to each \(\mathfrak{p}\)-algebra \(R \in C\) the set of homomorphisms \(\phi\) from \(((1/q^n)O/O)^h\) into \(R\) such that \(f_n(x)\) is divisible by
\[
\prod_{\alpha \in (1/q^n)_{x^n}} (x - \alpha).
\]

Lemma. \(\Phi_{\mathfrak{p}}\) is represented by a ring \(L_r\) having the following properties:

1) \(L_r\) is a regular ring. Let \(e_i\) \((i = 1, \ldots, r)\) be a basis for \(((1/q^n)O/O)^h\). Then the images of \(e_i\) in \(L_r\) and also \(t_1, \ldots, t_{n-1}\) form a system of local parameters.

2) The homomorphism \(L_{r-1} \to L_r\) is finite and flat.

Proof. For \(r = 0\) the lemma is true. Suppose that \(r \geq 1\) and that the lemma has been proved for \(\Phi_{r-1}\). Let \(\phi_{r-1} : ((1/q^n)O/O)^{r-1} \to L_{r-1}\) be the homomorphism mentioned in the definition of \(\Phi_{r-1}\). We set \(\theta_i = \phi_{r-1}(e_i)\) \((1 \leq i \leq r - 1)\) and

Let \(L_r = L_{r-1}[t_r]/(\phi(t_r))\).

so that the restriction of \(\phi_r\) to the second summand of \(L_{r-1}\) and then \(L_r\) is regular, and \(\theta_1, \ldots, \theta_{r-1}\) form a system of local parameters.

It remains to prove that \(L_r\) represents \(\Phi_{\mathfrak{p}}\).

Proposition 4.4. Let \((F, \phi)\) be a universal deformation of level \(n\). Let \(P \in ((1/q^n)O/O)^h\) be a maximal ideal of \(R\).

\[H_{\mathfrak{p}(\mathfrak{p})}^1(S_{\mathfrak{p}}(P)) = 0\]

is an \(O\)-invariant group subspace of \(H_{\mathfrak{p}(\mathfrak{p})}^1(S_{\mathfrak{p}}(P))\).

is an imbedding, then the corresponding maximal ideal of \(R\) is a structure of level \(n\).

Proof. It is enough to consider the case \(r = 0\).

Indeed, \(f_n(x)\) is divisible by \(\prod_{\alpha \in (1/q^n)_{x^n}} (x - \alpha)\) if and only if \(\phi(x)\) remains to show that \(\phi(x)\) is invertible modulo \(\mathfrak{p}^n\), which is equivalent to the square of the maximal ideal of \(R\).
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$g(x) = \frac{f_n(x)}{\prod_{\alpha \in (1/\pi) O^*} (x - \phi_{r-1}(\alpha))}.$

Let $L_r = L_{r-1}[[\theta_1]]/g(\theta_r)$. We define a homomorphism

$\varphi_r : (\frac{1}{\pi} O/O)^{r-1} \oplus (\frac{1}{\pi} O/O) \rightarrow L_r$

so that the restriction of $\varphi_r$ to the first summand coincides with $\phi_{r-1}$ and the restriction of $\phi_r$ to the second summand sends $1/\pi$ into $\theta_r$. Clearly, $L_r$ is finite and flat over $L_{r-1}$; furthermore,

$L_n(b_1, \ldots, b_r, t_1, \ldots, t_{h-1}) = \delta^r(x),$

and so $L_r$ is regular, and $\theta_1, \ldots, \theta_r, t_1, \ldots, t_{h-1}$ form a system of local parameters. It remains to prove that $L_r$ represents $\Phi_r$. It is enough to show that $f_n(x)$ is divisible by

$\prod_{\alpha \in (1/\pi) O^*} (x - \phi_r(\alpha)).$

Indeed, $f_n(x)$ is divisible by $x - \phi_r(\alpha)$ for $\alpha \in ((1/\pi) O/O)^{r}$. Since $L_r$ is regular, it remains to show that $\phi_r$ is injective. Indeed, if $\phi_r(\Sigma \alpha_i \theta_i) = 0$, then $\Sigma \alpha_i \theta_i$ belongs to the square of the maximal ideal of $L_r$, and so $\alpha_i \in (1/\pi) O$. □

Setting $r = h$, we obtain assertions 1)–3) for $n = 1$.

b) Suppose that $n \geq 1$ and that statements 1) and 2) about $D_n$ have been proved. Let $e_i (1 \leq i \leq h)$ be a basis for $((1/\pi^n) O/O)^h$, and let $b_i$ be the image of $e_i$ in $D_n$. Clearly

$D_{n+1} = D_n[[y_1, \ldots, y_h]]/(f_n(y_1) - b_1, \ldots, f_n(y_h) - b_h).$

Therefore $D_{n+1}$ is regular, $(y_1, \ldots, y_h)$ is a system of local parameters in $D_{n+1}$, and the homomorphism $D_n \rightarrow D_{n+1}$ is finite and flat. □

Proposition 4.4. Let $(F, \phi)$ be a formal $O$-module over $R \in C$ with structure $\phi$ of level $n$. Let $P \subset ((1/\pi^n) O/O)^h$ be a submodule. Then

$H \stackrel{\text{def}}{=} \text{Spf} \ R[[x]]/\prod_{\alpha \in P} (x - \phi(\alpha)) \subset \text{Spf} \ R[[x]]$

is an $O$-invariant group subscheme, and the factor $G = F/H$ is a formal $O$-module. If

$(\frac{1}{\pi^m} O/O)^h \rightarrow (\frac{1}{\pi^n} O/O)^h/P$

is an embedding, then the corresponding homomorphism from $((1/\pi^n) O/O)^h$ into the maximal ideal of $R$ is a structure of level $m$.

Proof. It is enough to consider the case $R = D_n$ (cf. Proposition 4.3). Clearly $H$ is the minimal closed subscheme in Spf $R[[x]]$ containing Spf $R[[x]]/(x - \phi(\alpha))$ for all $\alpha \in P$. Since $F(\phi(\alpha), \phi(\beta)) = \phi(\alpha + \beta)$ and Spf $R[[x]]/(x - \phi(\alpha + \beta))$ is contained in $H$
for $a, \beta \in P$, it follows that $H$ is invariant under addition. Similarly it is proved that $H$ is invariant under the action of $O$. Since $\phi(a) \neq 0$ for $a \neq 0$, the homomorphism from $F$ to $G$ induces a nonzero tangent mapping. Since $D_n$ has no divisors of zero, $G$ is a formal $O$-module. Since the homomorphism from $((1/n^m)O/O)^h$ into the maximal ideal $D_n$ is injective, and $D_n$ is regular, it follows that this homomorphism is a structure of level $m$. \hfill \Box

C) Deformations of divisible modules. In this section, a formal group will be a group object in the category of formal schemes. (For example, a discrete group is a formal group.) Let $R \in C$. A divisible $O$-module over $R$ is a formal group $F$ over $R$ together with a homomorphism $f: O \to F$ such that $F_{loc}$ is a formal $O$-module, and

$$F/F_{loc} \cong \text{Spf } R \times (K/O)^1$$

(j < \infty). (For $R = \tilde{O}^n/(\pi)$, the sequence $0 \to F_{loc} \to F \to F/F_{loc} \to 0$ splits.)

Let the reduction of $F_{loc}$ modulo the maximal ideal have finite height $h$. A structure of level $n$ on the divisible module $(F, f)$ is a homomorphism $((1/n^m)O/O)^{j+h} \xrightarrow{\phi} \text{Mor}(\text{Spf } R, F)$, including a structure of level $n$ on $F_{loc}$ and an epimorphism

$$\left(\frac{1}{n^m}O/O\right)^{j+h} \xrightarrow{\phi} \left(\frac{1}{n^m}O/O\right) \subset F/F_{loc}$$

The concept of a deformation of level $n$ is introduced as for formal modules. Proposition 4.1 is easily generalized to the case of divisible modules.

Proposition 4.5. Let $(G, g)$ be a divisible $O$-module over $\tilde{O}^n/(\pi)$ with structure of level $n$ such that $G_{loc}$ has height $h$, and $G/G_{loc} \cong (K/O)^1$.

Let $n \in \mathbb{Z}, n \geq 0$. The functor that associates to $R \in C$ the set of deformations of level $n$ of the module $(G, g)$ with basis $R$ up to isomorphism is represented by the ring $E_n \cong D_n[[d_{1}, \ldots, d_{j}]]$, where $D_n$ is defined as in Proposition 4.3. In particular, $E_n$ is a regular ring of dimension $j + h$, and $E_0$ is smooth over $\tilde{O}^n$. If $m \leq n$, then the homomorphism $E_m \to E_n$ is finite and flat.

Proof. Let $R \in C$, and let $(F, f)$ be a deformation of level $n$ of the formal module $G_{loc}$. Clearly its extensions by a deformation of the divisible module $G$ are classified by the elements of $\text{Exp}(\Gamma, \text{Mor}(\text{Spf } R, F))$, where $\Gamma$ is a factor of $G/G_{loc}$ by elements of order $n^m$. If $M$ is a module over $O$, complete in the $\pi$-adic topology, then $M = \text{Exp}(K/O, M)$. Therefore, fixing an isomorphism $\Gamma \cong (K/O)^1$, we can identify

$$\text{Exp}(\Gamma, \text{Mor}(\text{Spf } R, F))$$

with $\text{Mor}(\text{Spf } R, F)$.

Remark. Let $O'$ be the integral closure of $O$ in a finite extension of $K$. By $O'^n$ we mean the maximal unramified extension of $O$ in $O'^n$. Let $(G, g)$ be a divisible $O'$-module such that $G_{loc}$ has finite height, and let $E'$ be the basis of a universal deformation (of level zero) of $(G, g)$. We can consider $(G, g)$ as an $O$-module; let $E$ be the basis of a universal deformation of the $O$-module $(G, g)$. Clearly the group of automorphisms of $(G, g)$ acts on $E$, and, in particular, $O'^{\ast}$ acts on $E$. It is easy to show that if $\Gamma \subset O'^{\ast}$ is a subgroup which generates $O'$ as an $O$-module, then $\text{Spf } E' = (\text{Spf } E)^\Gamma$.

A) Endomorphisms of the form $f_{\lambda}$

Proposition 5.1. Let $B = \mathbb{Z}/n\mathbb{Z}$, $f_1, f_2 \in \text{End}(A)$, $f_i = \sum_{a_{i1} \in \mathbb{Z}} a_{i1} f_i^a_{i1}$, $d_i = \sum_{a_{i1} \in \mathbb{Z}} a_{i1}$.

1) If $d_1 \neq d_2$, then $h = 0$.

2) If $d_1 = d_2$ and $h \neq 0$, then $B_{\lambda}$

Proof. It suffices to consider the case $d < 0$. Let $h_1 < m$. Then $h/h_1 = f_{\lambda}^a_{\lambda} = a_{\lambda} f_{\lambda}^a_{\lambda}$.

2) Let $d = d_2 = d$ and let $a_{\lambda} = f_{\lambda}^a_{\lambda}$.

Proposition 5.2. Let $B = \text{End}(A)$, $d > 0$, with $a_{\lambda}$ invertible and $f_{\lambda}^a_{\lambda} = 1 + \sum_{\lambda \in \mathbb{Z}} a_{\lambda} f_{\lambda}^a_{\lambda}$.

has degree $d$.

Proof. The uniqueness for $k$ follows.

Let $m \in B$ be an ideal, $m < n$, and

$$(1 - f_{\lambda}^a_{\lambda})$$

is less than $n$. From this follows.

B) Construction of moduli

Definition. Let $S$ be a scheme and $\xi$ be a line bundle over $S$ together with a map of the form $\xi_{\lambda} = \sum a_{\lambda} \xi_{\lambda}$.

If $a \in A$ is the differential of any endomorphism of $\xi$, then $\text{End}(\xi)$ is of rank $d$ in the sense of \S 1.1.1. Any group schemes over $S$ that are of this type.

Remark. An elliptic moduli of endomorphism $\phi(a)$ has the form $\xi_{\lambda}$.

Proposition 5.2, every elliptic moduli is isomorphic to a standard moduli of rank $d$.

Let $I \subset A$ be an ideal. We define $I$.

Let $X$ be an elliptic module of rank $d$.

annihilator of $I$. Clearly $X_I$ is an ideal in Spec $A$ does not intersect...
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\section*{§5. Modular Manifolds}

A) \textbf{Endomorphisms of the additive group.}

\textbf{Proposition 5.1.} Let $B$ be a ring of characteristic $p$ with $\text{Spec } B$ connected. Let $\{ f_i \}_{i \in B/\mathfrak{m}}$, $f_i = \sum_{i=0}^{d_i} a_{ij}^i$, $d_i > 0$, $a_{ij}$ invertible for $i > d$, $h \in O/\mathfrak{m}$, and $h_1 = f_i h$.  

1. If $d_1 = d_2$, then $h = 0$.

2. If $d_1 = d_2$ and $h \neq 0$, then $h$ has the form $\sum_{i=0}^{d_1} h_i^i$, and $h_1$ is invertible.

\textbf{Proof.} It suffices to consider the case when $B$ is a local Artinian ring. We proceed by induction on the length of $B$.

1) Assume that the coefficients of $h$ are nilpotent. (This is also true if $d_1 \neq d_2$.)

We shall show that $h = 0$. Let $M \subset B$ be an ideal such that $m^2 = 0$ and $h = \sum_{i=0}^{n} b_i^i$, $b_i \in m$. Then $h_1 = f_2 b_i = a_{i0} b_i$. Since $d_1 > 0$, we have $h_n = 0$.

2) Let $d_1 = d_2$ and $h = \sum_{i=0}^{d_1} h_i^i$; let $m \subset B$ be an ideal, $m^2 = 0$, $h_i \in m$. Equating the coefficients for $h^i$, we obtain $h_i = 0$. \hfill $\square$

\textbf{Proposition 5.2.} Let $B$ be a ring, char $B = p$. Let $f = \sum_{i=0}^{n} a_i^i \in B/\mathfrak{m}$, and let $d > 0$, with $a_d$ invertible and $a_i$ nilpotent for $i > d$. Then there exists a unique element of the form $1 + \sum_{i=0}^{n} a_i^i \in B/\mathfrak{m}$ such that the $a_i$ are nilpotent and $\left(1 + \sum_{i=0}^{n} a_i^i\right) a_i \left(1 + \sum_{i=0}^{n} a_i^i\right)^{-1}$ has degree $d$.

\textbf{Proof.} The uniqueness follows from Proposition 5.1.

Let $m \subset B$ be an ideal, $m^2 = 0$, $a_i \in m$ for $i > d$, and $n > d$. Then the degree of $\left(1 - \frac{a_n}{a_{n-1}^n} \tau^{n-d} - \frac{a_n}{a_{n-2}^n} \tau^{n-d} \right)$ is less than $n$. From this follows the existence. \hfill $\square$

B) \textbf{Construction of modular schemes.}

\textbf{Definition.} Let $S$ be a scheme over $A$. An \textit{elliptic $A$-module} over $S$ of rank $d$ is a line bundle $L$ over $S$ together with a homomorphism $\psi: A \to \text{End} L$ (where $\text{End} L$ is the ring of endomorphisms of $L$ as a group scheme over $S$) such that 1) for any $a \in A$ the differential of $\psi(a)$ is multiplication by $a_i$, and 2) for any field $K$ and homomorphism $\text{Spec } K \to S$, the corresponding homomorphism $A \to K_{\mathfrak{m}}$ is an elliptic module of rank $d$ in the sense of §1. A homomorphism of elliptic modules is a homomorphism of group schemes over $S$ that agrees with the action of $A$.

\textbf{Remark.} An elliptic module over $S$ is called \textit{standard} if for every $a \in A$ the endomorphism $\psi(a)$ has the form $\sum_{i=0}^{d-1} a_i b_i^i$, where $b_i \in H^0(S, L^{1-p})$. According to Proposition 5.2, every elliptic module is isomorphic to a standard module, and every isomorphism of a standard module is linear.

Let $I \subset A$ be an ideal. We denote by $V(I)$ the set of simple ideals containing $I$. Let $X$ be an elliptic module over $S$ of rank $d$, let $I \neq 0$, and let $X_I \subset X$ be the annihilator of $I$. Clearly, $X_I$ is a finite flat group scheme over $S$. If the image of $S$ in Spec $A$ does not intersect $V(I)$, then $X_I$ is \textit{étale} over $S$. 

\textbf{Proof.} The uniqueness follows from Proposition 5.1.

Let $m \subset B$ be an ideal, $m^2 = 0$, $a_i \in m$ for $i > d$, and $n > d$. Then the degree of $\left(1 - \frac{a_n}{a_{n-1}^n} \tau^{n-d} - \frac{a_n}{a_{n-2}^n} \tau^{n-d} \right)$ is less than $n$. From this follows the existence. \hfill $\square$
Definition. A structure of level $l$ on $X$ is a homomorphism of $A$-modules $\psi: (l^{-1}/A)^d \rightarrow \operatorname{Mor}(S, X)$ such that, for any $m \in V(l)$, $X_m$ as a divisor coincides with the sum of the divisors $\psi(a)$, $a \in m^{-1}/A$.

Remark. If the image of $S$ in $\operatorname{Spec} A$ does not intersect $V(l)$, then a structure of level $l$ is an isomorphism $(l^{-1}/A)^d \times S \cong X_l$.

Proposition 5.3. Let $l \subset A$ be an ideal such that $l \neq 0$ and $V(l)$ contains more than one element. The functor that associates to the scheme $S$ over $A$ the set of elliptic $A$-modules of rank $d$ with structure of level $l$ up to isomorphism is represented by a scheme $M^{(d)}_A$ of finite type over $A$.

Proof. Let $m \in V(l)$. It is enough to prove that the restriction of our functor to the category of schemes over $Spec A - m$ is representable. Indeed, if $S$ is a scheme over $Spec A - m$ and $X$ is an elliptic $A$-module of rank $d$ over $S$ with structure of level $l$, then a choice of nonzero elements $(m^{-1}/A)^d$ defines a trivialization of the bundle $X$.

C) Deformations of elliptic modules. Let $V = \operatorname{Spec} A$ and $0 = A_0$. The definition of the category $C$ was given in §4. Let $X$ be an elliptic module of rank $d$ over $A_0$, with structure of level $v^n$. We consider the functor that associates to $R \in C$ the set of deformations of level $v^n$ of the module $X$ with basis $R$ up to isomorphism. This functor can be represented in the following way: Let $I \subset A$ be nonzero ideal, $V(I) \neq \emptyset$, $I \neq A$. We lift in any way the structure of level $v^n$ on $X$ to a structure of level $l v^n$. Let $y$ be the corresponding point in $M^{(d)}_{l v^n} \otimes A_0$, and $F_n$ its completion at a local ring. Then $F_n$ represents our functor.

Let $R \in C$, and let $Y$ be an elliptic module over $R$. Then $\hat{Y} = \lim_{\to} Y_{v^n}$ is a divisible $A_0$-module. A structure of level $v^n$ on $Y$ defines a structure of level $n$ on $\hat{Y}$. Thus there is a homomorphism $E_n \rightarrow F_n$, where $E_n$ was defined in §40.

Proposition 5.4. This homomorphism $E_n \rightarrow F_n$ is an isomorphism.

Proof. a) If $R \in C$ and $Y$ is an elliptic module over $R$, $\hat{Y} = \lim_{\to} Y_{v^n}$, then a structure of level $n$ on $\hat{Y}$ defines a structure of level $v^n$ on $Y$. Thus it suffices to prove the proposition for $n = 0$.

b) We reduce the proof to the case $A = F_p[x]$, $v = (x)$. Let $x \in v$. We may consider an elliptic $A$-module as an $F_p[x]$-module, and a divisible $A_0$-module as an $F_p[[x]]$-module. Let $E_0$ be a basis of the universal deformation of the $A$-module $X$, and $F_0$ a basis of the universal deformation of the $F_p[x]$-module $X$. The diagram

$$
\begin{array}{ccc}
E_0 & \rightarrow & F_0 \\
\uparrow & & \uparrow \\
E_0' & \rightarrow & F_0'
\end{array}
$$

is commutative.

Let $A_x$ be the localization of $A$ by the set of elements of $A$ which are relatively prime to $x$. We define an action of $A_x$ on $F_0'$. Let $a \in A \cap A_x$, and let $b$ be a deformation of the elliptic $F_p[x]$-module $Y$ with structure of level $l$ up to isomorphism, deformation $Y'$ with structure of level $l$. (Since $b(0) \neq 0$, $a \neq b$. Since $Y_{(b)}$ is finite over $F_p$, it extends to a subscheme $l \subset Y$. The subgroup $A_x \cap A$ acts on $F_0'$. We define an action of $A_x$ on $F_0'$. It is clear that $F_0'$ is a $A_x$-module.)

On the other hand (cf. §4C), the following diagram is commutative:

$$
\begin{array}{ccc}
F_0' & \rightarrow & F_0' \\
\uparrow & & \uparrow \\
F_0' & \rightarrow & F_0'
\end{array}
$$

Thus the homomorphism $E_n \rightarrow F_n$ is an isomorphism.

c) It remains to clear up the case $A = F_p[[x]]/(x^d)$, where $a_x \neq 0$. Clearly, $F_0 = \hat{F}_0$, and the formal module $\hat{X}_l$ is an elliptic module $X$. Then it follows from Proposition 4.4 that $\beta_i = 0$ for $i \leq d - 1$, and so $\sum_{i=0}^{d-1} \beta_i p^i = 0$. Since $A_x = F_p[[x]]/(x^d)$, $\beta_i = 0$ for $h \leq i \leq d - 1$. □

Corollary. Suppose the condition is satisfied, and let $X$ be an $A$-module. Let $Y$ be a smooth $d$-dimensional manifold over $V(l)$. If $l \subset I$, then the morphism $\hat{Y}$ is an isomorphism.

D) Group actions. We set $I$ be the ring of adeles without $0$ addeles and $\lim A/l$.

We define an action of the group $A$ on $X$ as follows: let $X$ be an $A$-module, and let $E_0$ be a basis of the universal deformation of the $A$-module $X$. We regard $g$ as an endomorphism of $X$ from Proposition 4.4 that the deformation $\hat{X}_l$ is an $A$-module. We define $\psi_1: (k/A)^d \rightarrow \operatorname{Mor}(S, X)$ such that $\psi(a)$ is a structure of level $l$. □
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isomorphism of \( A \)-modules \( \psi : X \to Y \), where \( \psi \) is a divisor coincides with the restriction of our functor to \( A \). Indeed, if \( S \) is a scheme over \( \mathbb{C} \) with structure sheaf \( \mathcal{O} \) of rank \( d \) over \( A \), then the homomorphism is represented by \( \mathcal{O} \). The definition of deformation of level \( \nu \) on \( \mathcal{O} \) can be represented by \( \nu \downarrow \mathcal{O} \), \( \nu \sqcup = \mathcal{O} \). We lift in any way to \( \nu \sqcup = \mathcal{O} \). Then \( F_{\nu} \) represents our homomorphism.

where \( \nu \) is a structure of level \( \nu \) on \( \mathcal{O} \).

Thus it suffices to prove that \( \psi \mid \nu \sqcup = \mathcal{O} \).

Let \( x \in \nu \). We may consider \( \mathcal{O} \)-module as an \( \mathbb{F}_{p}[\kappa][x] \)-module. Then \( \mathbb{F}_{p}[\kappa][x] \)-module \( \hat{X} \), and \( \hat{X} \) is the \( \mathcal{O} \)-module. The diagram

of the elliptic \( \mathbb{F}_{p}[\kappa]\)-module \( X \) with \( R \in C \). Then there exists a unique, up to isomorphism, deformation \( Y' \) with basis \( R \) for which the endomorphism \( X \to Y' \) extends to a homomorphism \( Y \to Y' \). (\( Y' \) is obtained in the following way: Let \( b \in \mathbb{F}_{p}[\kappa], \) \( b(0) \neq 0 \), \( d \). Since \( Y' \mid \nu \sqcup = \mathcal{O} \) is étale over \( \text{Spec} \ R \), the subscheme \( X_{\nu} \subset \hat{X} \) uniquely extends to a subscheme \( \hat{Y} \subset \hat{Y} \), étale and finite over \( \text{Spec} \ R \). Let \( Y' = Y/\text{H.} \) Thus the subgroup \( X_{\nu} \cap X \text{ acts on } F_0 \), and \( X_{\nu} \cap F_0[x] \) acts trivially. This allows us to define an action of \( A_{\nu} \) on \( F_0[x] \). It follows from Proposition 5.1 that \( \text{Spf} \ F_0 = (\text{Spf} \ F_0)^{X_{\nu}} \). On the other hand (cf. \S 4(C)), the group \( A_{\nu} \subset A \text{ acts on } F_0 \), and \( \text{Spf} \ F_0 = (\text{Spf} \ F_0)^{X_{\nu}} \). Thus the homomorphism \( F_0 \to F_0 \) is consistent with the action of \( A_{\nu} \). Since the bottom row of the commutative diagram is an isomorphism, so is the upper row.

c) It remains to clear up the case \( n = 0 \), \( A = \mathbb{F}_{p}[\kappa], \nu = (x) \). Suppose the elliptic \( \mathbb{F}_{p}[\kappa]\)-module \( X \) is defined by a homomorphism \( \mathbb{F}_{p}[\kappa] \to \mathbb{F}_{p}[\kappa], \) where \( x \to \sum_{i=0}^{d-1} a_i r^i \). Since \( a_i \neq 0 \). Clearly, \( F_0 = \mathbb{F}_{p}[\kappa], \) and the universal deformation of the elliptic module \( X \) has the form \( x + \sum_{i=0}^{d-1} a_i r^i + r^d \). Since \( E_0 \approx F_0 \), it suffices to show that a morphism \( \text{Spf} \ F_0 \to \text{Spf} \ E_0 \) induces a monomorphism of tangent spaces. In other words, we must prove that if \( \beta_1, \ldots, \beta_{d-1} \in \mathbb{F}_{p}[\kappa] \) and the deformation

with basis \( \mathbb{F}_{p}[\kappa][x]/(x) \) induces the trivial deformation of the divisible module \( \mathbb{F}_{p}[\kappa][x]/(x) \), then \( \beta_i = 0 \) for \( 1 \leq i \leq h - 1 \). Indeed, it follows from the triviality of the deformation of the formal module \( \mathbb{F}_{p}[\kappa][x]/(x) \) that \( \beta_i = 0 \) for \( 1 \leq i \leq h - 1 \). Let \( r \in \mathbb{F}_{p}[\kappa] \) be a point of order \( x \) of the module \( X \). Then it follows from the triviality of the deformation \( \hat{X} \) that

and so \( \sum_{i=h}^{d-1} \beta_i r^i = 0 \). Since \( r \) can assume \( d-1 \) different values, it follows that \( \beta_i = 0 \) for \( h \leq i \leq d - 1 \).

Corollary. Suppose the conditions of Proposition 5.3 are satisfied. Then \( M_d \) is a smooth \( d \)-dimensional manifold. The morphism \( M_d \to \text{Spec} \ A \to V(l), l \not\in I \), is smooth over \( \text{Spec} \ A \to V(l), l \not\in I \), and \( M_d \) is finite and flat.

D) Group actions. We set \( M_d = \lim_{\to} M_d \). Let \( \mathbb{A} \) be the ring of adeles of \( k \), and let \( \mathbb{A}_l \) be the ring of adeles without the component at \( \infty \) (Thus \( \mathbb{A} = \mathbb{A}_l \times k_\infty \) and \( \mathbb{A} = \lim_{\to} A / l \)). We define an action of the group \( GL(d, \mathbb{A}_l)/k^* \) on \( M_d \). Let \( S \) be a scheme over \( A \), and let \( X \) be an elliptic module over \( S \) of rank \( d \) together with a homomorphism \( \psi : (k/A)^d \to \text{Mor}(S, X) \) such that for any nonzero ideal \( I \subset A \) the restriction of \( \psi \) to \( (I^{-1}/A)^d \) is a structure of level \( I \). Let \( g \in GL(d, \mathbb{A}_l) \) be a matrix with coefficients in \( \mathbb{A} \). We can regard \( g \) as an endomorphism of \( (k/A)^d \). Its kernel \( P \) is finite. It follows from Proposition 4.4 that the divisor \( H \subset P \), which is equal to the sum of the divisors \( \psi(a), a \in P \), is an \( A \)-invariant group subscheme, and \( X/H \) is an elliptic \( A \)-module.

We define \( \psi : (k/A)^d \to \text{Mor}(S, X/H) \) so the diagram
\[
\begin{align*}
(k/A)^d & \cong \text{Mor}(S, X) \\
\times & \downarrow \\
(k/A)^d & \gamma \rightarrow \text{Mor}(S, X/H)
\end{align*}
\]

is commutative.

It follows from Proposition 4.4 that for any \( l \) the restriction of \( \psi_1 \) to \((I^{-1}/A)^d\) is a structure of level \( l \). We obtain a left action on \( M^d \) of the subgroup of matrices in \( \text{GL}(d, \mathbb{A}_f) \) with coefficients in \( \mathbb{A} \). Since the subgroup of nonzero elements of \( A \) acts trivially on \( M^d \), we obtain an action of \( \text{GL}(d, \mathbb{A}_f)/k^* \).

Let \( I \subset A \) be an ideal satisfying the conditions of Proposition 5.3, and let \( U_I \) be the kernel of the homomorphism \( \text{GL}(d, \mathbb{A}) \rightarrow \text{GL}(d, A/I) \). Then \( M_I^d = U_I \backslash M^d \). Indeed, if \( f, g \in I \), then the morphism

\[
M_I^d \times_{\text{Spec } A} (\text{Spec } A - V(f)) \rightarrow M_I^d \times_{\text{Spec } A} (\text{Spec } A - V(g))
\]

is a bundle with structure group \( U_I/U_f \). It follows from the normality of \( M_I^d \) that \( M^d_I = U_I \setminus M^d \).

E) Congruence relations. Recall the construction of the induced ring \([4] \). Let \( G \) be a topological group, \( H \subset G \) a closed subgroup, and \( B \) a discrete ring on which the group \( H \) acts continuously on the right. Consider the ring \( C \) of continuous functions \( f : B \rightarrow G \) such that \( f(gh) = f(g) \cdot h \) for \( g \in G \) and \( h \in H \). This is called the induced ring; \( G \) acts on it by the formula \( f(g) \cdot (g') = f(gg') \). We shall say that \( \text{Spec } C \) is induced by the scheme \( \text{Spec } B \).

Let \( \nu \in \text{Spec } A \), and let \( M^d_{(\nu)} \) be the fiber of \( M^d \) over \( \nu \). To each point of \( M^d_{(\nu)} \), there corresponds an elliptic \( A_\nu \)-module, to which in turn corresponds a formal \( A_\nu \)-module. Let \( W \subset M^d_{(\nu)} \) be the set of points for which the corresponding formal \( A_\nu \)-module has height 1.

Proposition 5.5. 1) \( W \) is an open, affine, everywhere dense \( \text{GL}(d, \mathbb{A}_f) \)-invariant subset.

2) Let \( B \subset \text{GL}(d, k_\nu) \) be a group of matrices \( (a_{ij}) \) such that \( a_{ii} = 0 \) for \( i > 1 \). Let \( B' \) be the preimage of \( B \) in \( \text{GL}(d, \mathbb{A}_f) \). Then the \( \text{GL}(d, \mathbb{A}_f) \)-scheme \( W \) induced by any \( B' \)-scheme \( W_0 \) has the following properties:

   a) The matrices \( (a_{ij}) \in B \subset B' \) for which \( |d_{11}| = 1 \) and the lower right corner \( (i, j > 1) \) coincides with the identity matrix acts trivially on \( W_0 \).

   b) Let \( \pi \) be a simple element of \( A_\nu \). The matrix

\[
\begin{pmatrix}
\pi & 0 \\
0 & 1
\end{pmatrix}
\]

acts on \( W_0 \) like the Frobenius of the field \( A_\nu/\pi \).

Proof. Statement 1) is clear (the density of \( W \) follows from Proposition 5.4).

2) To each point \( \omega \in W \) corresponds an elliptic module \( X \) and a homomorphism \( (k_\nu/A_\nu)^d \rightarrow X \) whose kernel is isomorphic to \( k_\nu/A_\nu \). This defines a \( \text{GL}(d, \mathbb{A}_f) \)-invariant mapping of the set \( W \rightarrow \text{GL}(d, \mathbb{A}_f)/B' \). Let \( I \subset A \) be an ideal, \( I \neq 0 \). The mapping \( U_I \setminus W \rightarrow \text{GL}(d, \mathbb{A}_f)/B' \) is clear.

\section*{§6. Unipotent \( \mathbb{A} \)-groups}

A) Three analogs of the unipotent \( \mathbb{A} \)-groups \( O \subset K \) its ring of integers, and \( \text{GL}(d, K) \).

1) "Analytic" analog. Let \( (g, \mathbb{A}_f) \) be a codimension analytic space. The group \( \text{GL}(d, \mathbb{A}_f) \)

\[
(g, \mathbb{A}_f)
\]

Let \( \Omega^d \) be a set of points of \( \text{Spec } \Omega \). A subset \( \Omega^d \) is a \( \text{GL}(d, K) \)-invariant set.

2) "Homogeneous" analogs on the left on \( \mathbb{A} \). Let \( M \subset K^d \), \( N \subset K^d \), and \( \text{GL}(d, M) \approx \text{GL}(d, N) \).

An \( \mathbb{A} \)-simplex is a subset \( \Delta \subset \mathbb{A} \times \cdots \times \mathbb{A} \)

\[
\Delta = \{(a_1, \ldots, a_d) : a_1 \leq \ldots \leq a_d\}
\]

\( s^d \) is a simplicial scheme of codimension \( d \) on \( \mathbb{A} \) denoted \( s^d(\mathbb{R}) \), and the set of points is denoted \( s^d(\mathbb{Q}) \).

3) "Topological" analogs, denote the extension of this \( s^d \) integral (rational) if for every \( \rho \in \text{GL}(d) \) \( \log_q \rho(x) \in \text{Q} \). It can be shown that \( s^d(\mathbb{R}) \) the class of integral corresponds \( \cdots \rightarrow \text{M}_k \rightarrow \text{M}_1 \rightarrow \text{M}_0 \rightarrow \Delta \rightarrow \cdots \)

\[
\rho(v_i),
\]

If \( (z_1, \ldots, z_d) \in \Omega^d \), then \( \rho \) defines a norm on \( \mathbb{A}^d \). This defines a \( \text{GL}(d, \mathbb{A}_f) \) to check that \( \lambda \) is surjective.

Proposition 6.1. 1) Let \( x \in \Delta \), \( \Delta \).

§6. Uniformization of modular manifolds

A) Three analogs of the upper half-plane. Let $K$ be a local nonarchimedean field, $O \subset K$ its ring of integers, and $p \in O$ a prime element. Let $d$ be a natural number.

1) "Analytic" analog. Let $P_{K}^{d-1}$ be projective space, considered as a rigid analytic space. The group $GL(d, K)$ acts on $P_{K}^{d-1}$ according to the formula

$$\begin{align*}
(g; (z_1, \ldots, z_d)) &\mapsto (z_1, \ldots, z_d; g^*).
\end{align*}$$

Let $\Omega^d$ be a set of points of $P_{K}^{d-1}$ not lying in any hyperplane defined over $K$. The set $\Omega^d$ is a $GL(d, K)$-invariant subset of $P_{K}^{d-1}$.

2) "Homogeneous" analog [5]. Let $S^d = GL(d, K)/GL(d, O)$. The group $GL(d, K)$ acts on the left on $K^d$, and so the elements of $S^d$ can be interpreted as similarity classes of free $d$-dimensional $O$-submodules in $K^d$. We introduce a metric $\rho$ on $S^d$. Let $M \subset K^d$, $M = N \oplus O^d$, $M \supset N \supset \gamma^k M$, $N \not\supset \gamma^{k-1} M$. Let $[M]$ and $[N]$ be the corresponding elements of $S^d$. We set $\rho([M], [N]) = k$.

A simplex is a subset $\Delta \subset S^d$ that satisfies one of the following equivalent conditions: a) $\rho(x, y) \leq 1$ for $x, y \in \Delta$; b) there exist submodules $M_i \subset K^d$ ($1 \leq i \leq k$) such that $M_i \supset O^d$, $M_i \supset M_{i+1}$ for $i < k$, $M_k \supset \gamma k M_1$, and $\Delta = \{[M_1], \ldots, [M_k]\}$. Thus $S^d$ is a simplicial scheme of codimension $d - 1$. The corresponding polyhedron is denoted $S^d(R)$, and the set of points of $S^d(R)$ with rational barycentric coordinates is denoted $S^d(Q)$.

3) "Topological" analog [9]. We fix a norm $\| \cdot \|$ on $K$, and use the same symbol to denote the extension of this norm to $\overline{K}$. We set $q = |\pi^{-1}|$. We call the norm $\nu$ on $K^d$ integral (rational) if for every $x \in K^d$, $x \neq 0$, we have $\log_q \nu(x) \in \mathbb{Z}$ (respectively $\log_q \nu(x) \in \mathbb{Q}$). It can be shown that the set of norms on $K^d$ up to similarity can be identified with $S^d(R)$; the class of rational norms corresponds to $S^d(Q)$ and the class of integral norms corresponds to $S^d$. Here is the construction. Let $K^d \supset M_1 \supset M_2 \supset \ldots \supset M_k \supset \gamma k M_1$, $a_1, \ldots, a_k \geq 0$, $\Sigma a_i = 1$. Let $\nu_i$ be the norm on $K^d$ corresponding to $M_i$. To the point of $S^d(R)$ with coordinates $(a_1, \ldots, a_k)$ corresponds the class of norms $\nu = \max_{1 \leq i \leq k} [q^{a_1 r_1 + a_2 r_2 + \ldots + a_k r_k}]$. The metric $\rho$ on $S^d$ extends to a metric on $S^d(R)$ which we shall also denote by $\rho$: If $\nu_1$ and $\nu_2$ are norms on $K^d$, then

$$\rho((\nu_1), (\nu_2)) = \frac{1}{\log_q \frac{\nu_1}{\nu_2}} \left( \sup_{K^d} \frac{\nu_1}{\nu_2} \right).$$

If $(z_1, \ldots, z_d) \in \Omega^d$, then the function $(a_1, \ldots, a_k) \mapsto [\sum a_i z_i]$ is a rational norm on $K^d$. This defines a $GL(d, K)$-invariant mapping $\lambda: \Omega^d \rightarrow S^d(Q)$. (It is easy to check that $\lambda$ is surjective.)

Proposition 6.1. 1) Let $x_1, \ldots, x_k \in S^d$ and $c \in Q$. Let
Then $X_\varepsilon$ is an open affine subset of $\mathbb{P}^{d-1}_K$. If $c_1 < c_2$, then $X_{c_1}$ is in the interior of $X_{c_2}$.

2) $\Omega^d$ is an admissible open subset of $\mathbb{P}^{d-1}$.

Proof. Let $n$ be a natural number, $n > c$. For each $i$ ($1 \leq i \leq k$), let $v_i$ be the norm on $K^d$ whose class is $u_i$, let $C_i = \{y \in K^d | v_i(y) = 1\}$, and let $P_i \subset C_i$ be a finite $q^{-n}$-net. To each pair $(a, b)$, where $a = (a_1, \ldots, a_d) \in K^d$ and $b = (b_1, \ldots, b_d) \in K^d$, there corresponds a rational function $r^a_b$ on $\mathbb{P}^{d-1}_K$:

$$r^a_b(z_1, \ldots, z_d) = \frac{\sum a_i z_i}{\sum b_i z_i}.$$ 

Let $W \subset \mathbb{P}^{d-1}_K$ be the intersection of the domains of definition of the functions $r^a_b$, where $a, b \in P_i$, $1 \leq i \leq k$. It is easy to check that the set of functions $\Pi_{i=1}^k r^a_b$ (where $u_i, v_i \in P_i$) induces a closed imbedding of $W$ into an affine space. Clearly $X_\varepsilon$ is the preimage of the polydisc of radius $q^{-\epsilon}$ under this mapping. From this, 1) follows.

To deduce 2) from 1), it is sufficient to prove that if $B$ is a Tate algebra over $K$, and $\phi: \text{Max } B \to \mathbb{P}^{d-1}_K$ is a morphism such that $\text{Im } \phi \subset \Omega^d$, then $\text{Im } (\lambda \circ \phi)$ is a bounded subset of $S^d(Q)$. Indeed, let $\phi$ be defined by functions $t_i \in B$ ($1 \leq i \leq d$), $|t_i| \leq 1$. Let

$$\psi(a_1, \ldots, a_d) = \inf_{x \in \text{Max } B} \left| \frac{d}{i=1} \sum a_i t_i (x) \right|.$$ 

Then $\psi$ is a continuous function on $K^d$, and $\psi$ vanishes only at zero. Let

$$t = \max_{1 \leq i \leq d} |a_i|,$n

and let $|t|$ be the corresponding element of $S^d$, and $\epsilon = \inf_{i=1} \psi(a_i)$. Clearly $\text{Im } (\lambda \circ \phi)$ is contained in a ball with center $|t|$ and radius $\log_q \epsilon$. □

Let $\bar{S}^d$ be the barycentric subdivision of $S^d$ (i.e., the points of $\bar{S}^d$ are the simplices of $S^d$). If $\Delta_1, \ldots, \Delta_m$ are simplices of $S^d$ such that $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_m$, then $|\Delta_1, \ldots, \Delta_m|$ is a simplex of $\bar{S}^d$.

Proposition 6.2. Let $c \in \mathbb{Q}$, $0 < c < 1$. For each simplex $\Delta \subset S^d$ of codimension $k - 1$, set

$$V^c_\Delta = \{y \in S^d(Q) | \rho(y, x) \leq 1 - \frac{3 - c}{4^k} \text{ for } x \in \Delta, \sum_{x \in \Delta} \rho(y, x) \leq k - 1 + \frac{1 + c}{4^k}\},$$

$$U^c_\Delta = \bar{S}^d \setminus (V^c_\Delta).$$

The sets $U^c_\Delta$ generate a $GL(d, K)$-invariant admissible affine covering of $\Omega^d$ with nerve $\bar{S}^d$. If $c_1 < c_2$, then $U^c_\Delta \subset U^{c_2}_\Delta$.

Proof. We shall show that the sets $V^c_\Delta$ form a covering of $S^d(Q)$ with nerve $\bar{S}^d$. (This is sufficient by Proposition 6.1.)

Let $\Phi \subset S^d$ be a simplex of dimension $d - 1$, and let $\Phi(Q)$ be the corresponding closed subset of $S^d(Q)$. If $x \in \text{Max } B$, it is easy to check that the distance $\rho(x, y)$ is $< 1$, $\rho(x, z)$ is $\leq 1$, and so $\rho(x, z)$ is $\leq 1$. Let $\Delta_1 \subset \Phi$ and $\Delta_2 \subset \Phi$ be simplices. Let the $i$th point of $\Phi$ lie in $\Delta_1$ if $i \leq 1$, and $\Delta_2$ if $i > 1$. Then

$$a_i \geq 1.$$ 

This verifies the condition $\Sigma^d_{1=1} V^c_i = \Phi(Q)$.

Denote by $M_\Phi$ the set of those points in $\Phi(Q)$ with coordinates $(a_1, \ldots, a_d)$, $a_i \geq (3 - c)/4^k$ and $a_i \leq (3 - c)/4^k$.

B) Factorization of rigid $\Phi \subset \text{Max } B$. Let $\Phi$ be a field which is complete over $K$.

Proposition 6.3. Let $B$ be a Tate algebra over $K$. Then $B^G$ is a Tate algebra over $K$.

Proof. We represent $B$ as a tensor product of completion $K$-algebras.

Let $C$ be a tensor product of completion $K$-algebras, and $G$ acts on $C$, and there exists a $K$-algebra $B^G$ which is finite over $C$. Let $B^G$ be finite over $B$. □

Proposition 6.4. Let $B_1$ and $B_2$ be Tate algebras over $K$. Let the finite group $G$ acts on $B_1$. Then the morphism $\text{Max } B_1 \to \text{Max } B_2$ extends to a finite algebra isomorphism $B_1^G \to B_2^G$.

Proof. Clearly the morphism is an extension of sets of points and of $\text{Max } B_1 \to \text{Max } B_2$. □

If $B$ is a Tate algebra over $K$ and $G \subset \text{Max } B$ in place of $\text{Max } B^G$, the finite group $G$, let $Y = \text{Max } B^G$ be a morphism invariant under the action of $G$, $Y \to Z$ such that $\phi = \psi \circ \pi$, where
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closed subset of $S^d(Q)$. If $x \in \Phi(Q)$ has barycentric coordinates $(a_1, \ldots, a_d)$, then it is easy to check that the distance from $x$ to the $i$th point of $\Phi$ equals $1 - a_i$. If 
$$V^c_\Delta \cap \Phi(Q) \neq \emptyset,$$
then $\Delta \subset \Phi$. Indeed, if $y \in \Delta$, $z \in \Phi$ and $x \in V^c_\Delta \cap \Phi(Q)$, then $\rho(x, y) < 1$, $\rho(z, x) \leq 1$, and so $\rho(y, z) < 2$. Consequently $\Delta \cup \Phi$ is a simplex, and $\Delta \subset \Phi$. Let 
$\Delta_1 \subset \Phi$ and $\Delta_2 \subset \Phi$ be simplices of dimensions $k_1 - 1$ and $k_2 - 1$, $k_2 \leq k_1$, with $\Delta_2 \not\subset \Delta_1$.

Let the $j$th point of $\Phi$ lie in $\Delta_1 \setminus \Delta_2$. If a point of $\Phi(Q)$ with coordinates $(a_1, \ldots, a_d)$ lies in $V^c_{\Delta_1} \cap V^c_{\Delta_2}$, then

$$a_j \geq \frac{3 - d}{4k_1} \quad \text{and} \quad \sum_{i \neq j} a_i \geq 1 - \frac{1 + d}{4k_1},$$

This verifies the condition $\sum_j a_j = 1$.

Denote by $M_k$ the set of the first $k$ points of the simplex $\Phi$. Clearly a point of $\Phi(Q)$ with coordinates $(a_1, \ldots, a_d)$, where $a_i = (3 - c)/4^i$ for $i > 1$, will lie in 
$$\bigcap_{k=1}^d V^c_{M_k}.$$ It remains to show that the sets $V^c_{\Delta}$ cover $S^d(Q)$. Let $x \in \Phi(Q)$ have coordinates $(a_1, \ldots, a_d)$, $a_1 \geq a_2 \geq \cdots \geq a_d$. There exists $n \geq 1$ such that $a_n \geq (3 - c)/4^n$ and $a_i < (3 - c)/4^n$ for $i > n$. Then $x \in V^c_{M_n}$. □

B) Factorization of rigid analytic spaces by the action of a discrete group. Let $K$ be a field which is complete with respect to a nonarchimedean absolute value.

**Proposition 6.3.** Let $B$ be a Tate algebra over $K$, and let the finite group $G$ act on $B$. Then $B^G$ is a Tate algebra and $B$ is finite over $B^G$.

**Proof.** We represent $B$ as a factor of the algebra of convergent power series $R$.

Let $C$ be a tensor product of copies of $R$ which correspond to elements of $G$. Then $G$ acts on $C$, and there exists a $G$-invariant epimorphism $C \rightarrow B$. It is easy to check that $C^G$ is a Tate algebra and $C$ is finite over $C^G$. Therefore $B$ is finite over $C^G$, and $B^G$ is finite over $C^G$. It follows from this that $B^G$ is a Tate algebra and $B$ is finite over $B^G$. □

**Proposition 6.4.** Let $B_1$ and $B_2$ be Tate algebras, and Max $B_2 \subset$ Max $B_1$ an open affine subset. Let the finite group $G$ act on $B_1$, with $B_2$ invariant with respect to $G$. Then the morphism Max $B_2^G \rightarrow$ Max $B_1^G$ is an open imbedding. If Max $B_2 \subset$ Max $B_1$, then Max $B_2^G \subset$ Max $B_1^G$.

**Proof.** Clearly the morphism Max $B_2^G \rightarrow$ Max $B_1^G$ induces a one-to-one correspondence between sets of points and isomorphisms of complete local rings. By a basis theorem [8] this morphism is an open imbedding. If Max $B_2 \subset$ Max $B_1$, then there exists a closed imbedding $\phi$: Max $B_1 \rightarrow$ Max $K[t_1, \ldots, t_j]$ such that $\phi$($Max$ $B_2$) is contained in the set $|t_i| \leq 1 - c, c > 0$. Let $s_{ij}$ be the $i$th symmetric function in translations $\phi^*(t_j)$ by elements of $G$. The functions $s_{ij}$ define a finite morphism from Max $B_1^G$ into the identity polydisc which sends Max $B_2^G$ into the polydisc of radius $1 - c$. □

If $B$ is a Tate algebra under the action of a finite group $G$, we shall write $G \backslash Max$ $B$ in place of Max $B^G$. Let $X$ be an affine analytic space under the action of the finite group $G$, let $Y = G \backslash X$, and let $Z$ be a separable space. Let $\phi$: $X \rightarrow Z$ be a morphism invariant under the action of $G$. Then there exists a unique morphism $\psi$: $Y \rightarrow Z$ such that $\phi = \psi \circ \pi$, where $\pi$: $X \rightarrow Y$ is the projection. Indeed, let $Z = \bigcup Z_i$. 

...
be an admissible affine covering. Then the sets \( X_i = \psi^{-1}(Z_i) \) are affine and \( G \)-invariant, and a finite number of sets \( X_i \) cover \( X \). We set \( Y_i = G \backslash X_i = \pi(X_i) \). The sets \( Y_i \) form an admissible covering of \( X \). It remains to check that for each \( i \) there exists an identity morphism \( \psi_i : Y_i \to Z \) consistent with \( \psi \).

**Definition.** Let \( X \) be a separable rigid analytic space. The action of a group \( \Gamma \) on \( X \) is called **discrete** if for each \( \gamma \in \Gamma \), \( \gamma \) is finite. The group \( \Gamma_i \) of \( \gamma \) is finite for each \( i \in I \).

1. \( \gamma(X_i) = X_i \) for \( \gamma \in \Gamma \) and \( i \in I \).
2. Let \( i \in I \) and \( \Gamma_i = \{ \gamma \in \Gamma | \gamma(X_i) = X_i \} \). Then the group \( \Gamma_i \) is finite.
3. If \( \gamma \in \Gamma_i \), then \( X_i \cap X_{\gamma(i)} = \emptyset \). If \( i,j \in I \) and \( j \in \Gamma_i \), then \( X_i \cap X_{\gamma(j)} \emptyset \) for all but finitely many \( i \in I \).
4. Let \( i \in I \). Then the covering of \( \bigcup Y_i \cap X_{\gamma(i)} \) is admissible. In the situation described by the definition, let \( Y = \cap \Gamma Y_i \) and let \( Y_i \) be the image of \( X_i \) in \( Y \). It is easy to check that in a unique way \( Y \) can be made into a separable analytic space, and the mapping \( X \to Y \) into a morphism such that the covering of \( Y \) by the sets \( Y_i \) is admissible and affine, and \( \gamma \) coincides with \( \Gamma_i \backslash X_i \) as a space. If \( \phi \) is a \( \Gamma \)-invariant morphism from \( X \) into the separable space \( Z \), it is easy to check that there exists a unique morphism \( \psi : Y \to Z \) consistent with \( \phi \). It follows from Proposition 6.2 that the discrete subgroup \( \Gamma \subset GL(d, K)/K^* \), where \( K \) is a local nonarchimedean field, acts discretely on \( \Omega^d \).

**C. Uniformization of modular manifolds.** In this subsection \( \kappa = k_\infty \). If the ideal \( I \subset \mathcal{A} \) satisfies the conditions of Proposition 5.3, we set \( \Omega^d_I = (M^d \subset k_\infty) \). Indeed, as is clear from the proof of Proposition 5.3, the manifold \( \Omega^d_I \) makes sense for any nonzero ideal \( I \neq A \).

Let \( \Omega^d = GL(d, k) \backslash \Omega^d \times GL(d, \mathcal{U}) \). (Here \( GL(d, \mathcal{U}) \) is considered as a discrete set.) We introduce a left action of the group \( GL(d, \mathcal{U})/K^* \) on \( \Omega^d \). Also, \( \Omega^d \) is an open-closed subset of \( \Omega^d \). In the same way we define \( \tilde{S}^d, \tilde{\mathcal{S}}^d(R), \) etc.

A projective \( \mathcal{A} \)-module \( P \) of dimension \( d \) together with a norm \( \nu \) on \( P \otimes_A \mathcal{A} \) will be called a **metrized \( \mathcal{A} \)-module** of dimension \( d \). Metrized \( \mathcal{A} \)-modules \( (P_1, \nu_1) \) and \( (P_2, \nu_2) \) will be called **similar** if there exists an isomorphism \( f : P \cong P_2 \) such that \( \nu_1 = f^*(\nu_2) \) and \( f^*(\nu_2) \) are proportional. Let \( I \subset \mathcal{A} \) be an ideal, \( I \neq A \), \( I \neq 0 \). A structure of level \( I \) on a \( d \)-dimensional metrized \( \mathcal{A} \)-module \( (P, \nu) \) is an isomorphism \( \psi : (I^{-1}P)/\mathcal{A} \cong \mathcal{A} \). Clearly the set of similarity classes of \( d \)-dimensional metrized \( \mathcal{A} \)-modules with structure of level \( I \) can be identified with \( U_I \setminus \Sigma^d(R) \).

Let \( (P, \nu) \) be a metrized \( \mathcal{A} \)-module. We define a function \( \nu \) on \( P \otimes k_\infty \) by the formula \( \nu(u) = \inf_{\nu(v)} \nu(v) \). Let \( x, y \in (I^{-1}P)/I \), \( x \neq 0 \), \( y \neq 0 \). We set \( \mu_{x,y} = \nu(\psi(x))/\nu(\psi(y)) \). Then \( \mu_{x,y} \) can be considered as a function on \( U_I \setminus \Sigma^d(R) \).

**Proposition 6.5.** The subset \( H \subset \Sigma^d(R) \) is bounded if and only if the functions \( \mu_{x,y} \) are bounded for \( x, y \in (I^{-1}P)/I \), \( x \neq 0 \), \( y \neq 0 \).

**Proof.** The necessity is clear. We shall prove sufficiency. Let \( (P, \nu) \) be a metrized \( d \)-dimensional \( \mathcal{A} \)-module, and let \( e_1, \ldots, e_d \) be a basis for \( I^{-1}P/P \). Let \( e_i^* \) be an element in the class of \( e_i \), which is smallest in norm, and let \( u \) be an element of \( I^{-1}P \) which is smallest in norm, where \( c_\nu \) depends only on \( \nu \).

By Proposition 6.6, let \( \nu(u) \geq c_\nu \). Then \( \rho(\nu(u)) \leq \rho(c_\nu) \). By boundedness, \( \rho(\nu(u)) \leq \rho(c_\nu) \). By Proposition 6.5, the definition of \( \rho \), see \( \rho(\nu(u)) \leq \rho(c_\nu) \).

**Proposition 6.6.** Let \( I \subset \mathcal{A} \) be a discrete subgroup of \( GL(d, \mathcal{U}) \).

**Proof.** a) In \( \mathcal{A} \) we essentially consistent with the projections at \( z_1, \ldots, z_d \) coordinates of level \( I \). It is easy to check the \( \Omega^d \). Since this morphism \( \phi : U_I \setminus \mathcal{S}^d(R) \to \mathcal{U}^d \).

b) Let \( x \in U_I \setminus \mathcal{S}^d(R) \), \( y = \phi(x) \). If \( y \) is the rings of the manifolds \( U_I \setminus \mathcal{S}^d(R) \), then \( O_y/O_x \) is an isomorphism. Let \( O_y/m_y \) be a maximal unramified in the ring \( O_y/m_y \). Then \( m_y \) is an injective, and its image is a. Let \( O_y/m_y \) that is invariant with the homomorphism \( \psi : O_y \to O_y \) splits the manifolds of the same dimension.

**Lemma.** Let \( \Sigma \subset \Omega^d \) be a closed set.

**Proof.** It suffices to consider \( x \neq 0 \), \( y \neq 0 \), let \( x, y \in \mathcal{A} \), and \( y \) in the universal elliptic ellipsoid \( \Sigma \). There corresponds an isomorphism \( \psi : (I^{-1}P)/I \cong \mathcal{A} \). Let \( u \neq 0 \). Then \( |f(\psi(x))/\psi(y)| \leq c(f) \).

Therefore \( |\psi(x)/\psi(y)| \leq c(f) \).

**Proposition 6.5.**
of $I^{-1}P$ which is smallest in norm. Let $\mu_{x,y} \leq c$ for all $x$ and $y$. Then $\epsilon_i'/|u| \leq c_i'$, where $c_i'$ depends only on $c$. We define a norm $\nu'$ on $P \otimes \kappa_{\infty}$ by the formula $\nu'(\Sigma e_i') = \max_i |a_i|$. Then $\nu'(u) \geq 1/c_1'$. By Minkowski's lemma, the order of $I^{-1}P/\Sigma e_i'$ is bounded by a constant depending only on $c$. On the other hand, it follows from Minkowski's lemma that $p(\nu', |\nu'|)$ is bounded by a constant depending only on $c$. (For the definition of $p$, see §6A). □

**Proposition 6.6.** Let $I \subset \mathfrak{a}$ be an ideal, $I \neq 0, I \neq \mathcal{A}$. Then $U_{I}^{\wedge} \mathfrak{h} = \mathfrak{m}_{I}^{d}$. This identification is consistent with the projection $\mathfrak{m}_{I}^{d} \rightarrow \mathfrak{m}_{I}^{d}$ for $f \in I$, and also with the action of the group $GL(d, \mathcal{A})$.

**Proof.** a) In §3 we essentially obtained the identity $\mathfrak{m}_{I}^{d}(\mathfrak{m}_{I}) = U_{I}^{\wedge} \mathfrak{h}(\mathfrak{m}_{I})$ consistent with the projections and the action of $GL(d, \mathcal{A})$. (That is, a point of $\mathfrak{h}^d$ with coordinates $(z_1', \ldots, z_d')$ corresponds to the lattice $\Sigma z_i \mathcal{A}$ with the natural structure of level $I$.) It is easy to check that the mapping $\mathfrak{h}^d(\mathfrak{m}_{I}) \rightarrow \mathfrak{m}_{I}^{d}(\mathfrak{m}_{I})$ induces a morphism $\mathfrak{h}^d \rightarrow \mathfrak{m}_{I}^{d}$. Since this morphism is invariant with respect to $U_{I}^{\wedge}$, there is a corresponding morphism $\phi: U_{I}^{\wedge} \mathfrak{h} \rightarrow \mathfrak{m}_{I}^{d}$.

b) Let $x \in U_{I}^{\wedge} \mathfrak{h}$, $y = \phi(x)$, and let $O_x$ and $O_y$ be the completions of the local rings of the manifolds $U_{I}^{\wedge} \mathfrak{h}$ and $\mathfrak{m}_{I}^{d}$ at the points $x$ and $y$. We shall prove that $\phi^*: O_x \rightarrow O_y$ is an isomorphism. Let $m_y \subset O_y$ be a maximal ideal, and let $n$ be a natural number. There is an elliptic A-module $X$ with structure of level $I$ over the ring $O_y/m^n_y$. Let $f \in O_y/m^n_y[[x]]$ be a formal isomorphism from the additive $A$-module into $X$ (cf. Proposition 1.2). Just as in §3, it can be verified that $f$ is an entire function. Let $O_{y'}$ be a maximal unramified extension of $O_y$, and let $\Lambda$ be the set of zeros of $f$ in the ring $O_{y'}/m^n_y$. Then $\Lambda$ is an $A$-module, the homomorphism $\Lambda \rightarrow O_{y'/m^n_y}$ is injective, and its image is a lattice. In this way we obtain a homomorphism $O_x \rightarrow O_{y'/m^n_y}$ that is invariant with respect to $Gal(O_{y'/m^n_y}/O_x)$. Thus there is constructed a homomorphism $\psi: O_x \rightarrow O_y$ such that $\phi^* \psi = \text{id}$. Since $\mathfrak{m}_{I}^{d}$ and $U_{I}^{\wedge} \mathfrak{h}$ are smooth manifolds of the same dimension (cf. Proposition 5.4), it follows that $\phi^*$ is an isomorphism.

**Lemma.** Let $\Sigma \subset \mathfrak{m}_{I}^{d}$ be an affine open subset. Then the image of $\Sigma$ in $U_{I}^{\wedge} \mathfrak{h}(\mathcal{Q})$ is bounded.

**Proof.** It suffices to consider the case $I = (\beta)$ with $|\beta| > 1$. Let $x, y \in (I^{-1}A)^d, x \neq 0, y \neq 0$, let $t_{x,y} = \text{the function on } \mathfrak{m}_{I}^{d}$ equal to the quotient of the images of $x$ and $y$ in the universal elliptic $A$-module; and let $c = \sup_{x,y} \sup_{\psi \in \psi_0} t_{x,y}$. To each element of $\Sigma(\kappa_{\infty})$ there corresponds a $d$-dimensional lattice $\Gamma \subset \kappa_{\infty}$ together with an isomorphism $\psi: (I^{-1}A)^d \rightarrow I^{-1}I/\Gamma$. Let $f(x) = z_{\Sigma} \psi(1 - z/\alpha)$. Let $L_{y}(x)$ be the smallest (in modulus) element in the class of $\psi(x)$, and let $u$ be the smallest (in modulus) element of $\Gamma$, $u \neq 0$. Then

$$|f(\psi(x))| \leq |\psi(x)|, \quad |f(\psi(x))| = \frac{|u|}{\beta} = \inf_{x \neq 0} \psi(x).$$

Therefore $|\psi(x)/\psi(y)| \leq c$ for $x, y \in (I^{-1}A)^d, x \neq 0, y \neq 0$. It remains only to apply Proposition 6.5. □
It follows from the lemma that $\phi^{-1}(\Sigma)$ can be covered by a finite number of affine sets. It follows from the fundamental theorem of [8] that $\phi$ is an isomorphism. □

§7. Tate uniformization

Let $O$ be a complete discrete normed ring over $A$, $m \subset O$ a maximal ideal, $K$ the field of fractions of $O$, and $K^s$ the separable closure of $K$. In this section $\| \|$ denotes the norm on $K$ and not on $k_\infty$.

Let $\phi: A \to K[r]$ be an elliptic module of rank $d$. We shall say that $\phi$ has stable reduction if there exists an elliptic $A$-module $\phi': A \to K[r]$ such that $\phi' \simeq \phi$, $\phi'(A) \subset O[r]$, and the reduction of $\phi'$ modulo $m$ is an elliptic $A$-module (i.e. there exists an $a \in A$ such that the degree of the reduction of $\phi'(a)$ is greater than 1). Clearly the rank of the reduction $\phi'$ is not greater than $d$. In case of equality we shall say that $\phi$ has good reduction. Then $\phi$ has good reduction if and only if $\phi$ is obtained from some elliptic $A$-module over $O$ by an extension of the ring of scalars.

**Proposition 7.1.** Every elliptic $A$-module over $K$ has potentially stable reduction.

**Proof.** Let $v$ be an (additive) valuation in $K$. We set

$$ w(\sum a_i x^i) = \inf_{i > 0} \left( \frac{1}{p^i} - v(a_i) \right). $$

Let $y_1, \ldots, y_k$ be generators of the ring $A$, and let $x = \inf_{1 \leq i \leq k} w(\phi(y_i))$. If $K'$ is a finite extension of $K$ such that $x = \epsilon(K'/K) \in \mathbb{Z}$ (where $\epsilon(K'/K)$ is the index of ramification), then $\phi$ has stable reduction over $K'$.

**Remark.** For modules of rank 1 the concepts of stable and good reduction coincide.

Let $X$ be an elliptic $A$-module over $K$. By a lattice in $X$ we mean a projective $\text{Gal}(K^s/K)$-invariant $A$-submodule of finite type $\Gamma \subset X(K^s)$ such that a finite number of elements of $\Gamma$ are contained in every disc.

**Proposition 7.2.** The isomorphism classes of elliptic modules of rank $d$ over $K$ are in one-to-one correspondence with the isomorphism classes of pairs $(X, \Gamma)$, where $X$ is an elliptic $A$-module over $K$ of rank $d$ ($d_1 \leq d$) with potentially good reduction, and $\Gamma$ is a lattice in $X$ of dimension $d - d_1$.

**Proof.** 1) Just as in §5, one can construct an elliptic $A$-module of rank $d$ for the pair $(X, \Gamma)$.

2) Let $\phi: A \to O[r]$ be an elliptic $A$-module of rank $d$ over $K$, and let the reduction of $\phi$ modulo $m$ be an elliptic $A$-module of rank $d_1$. The existence and uniqueness of a pair $(\psi, a)$ follows from Proposition 5.2, where $\psi: A \to O[r]$ is an elliptic $A$-module over $O$ of rank $d_1$,

$$ u = 1 + \sum_{i=1}^{\infty} a_i x^i \in O \{\xi\}, \quad w(\psi(a)) = w(a) u $$

for $a \in A$, $a_i \in m$, $a_i \to O$.

**Lemma.** $u$ is an analytic homomorphism.

**Proof.** It follows from the relation $\phi(a) u = w(\psi(a))$ that there exist $k$ and $s$ such that $k \geq 1$, $|a_i| \leq \max |a_i|$ for $i > s$. Let $c_i = \sup_{s \geq 1} |a_i|$. Then $d_i$ is not 1.

Let $\Gamma$ be the kernel of the rational mapping $\phi$, then $|\gamma| > 1$. Let $a \in A$, $|d_0| > 1$, set $f = \psi(a)$. Clearly $f^{-1}(\Gamma)$ is isomorphic to $\Gamma/f(\Gamma)$. If $z \in K^s$, $|z| > 1$, then $|f(z)| = |\Gamma|$. Thus we have obtained classes of elliptic $A$-modules up to isomorphism classes of pairs $(X, \Gamma)$.

**Theorem 1.** The scheme $\text{abell}^0$, the abelian extension of $k$, compactifies $\text{abell}^0$ with the action in class field theory.

**Proof.** Let $I \subset A$ be an ideal. It follows from Proposition 7.1 that $\text{abell}^0$ is a finite extension of $\text{V}(I)$. Let $v \in \mathbb{R}$ be the valuation that follows from Proposition 5.5. Let $\text{abell}^0$ be the field $A_y(n)$ of the field $A_y(n)$ with respect to $\mathbb{R}$. On the other hand, the compactification is consistent with the action of $\text{abell}^0$ on $M^1$ is exact, and $\text{abell}^0$ is an exact theory.

**Corollary.** Over any algebraic $A$-module of any rank.

§9. Comparison

A) Construction of the bundle of projective tangent section. The multiplication

The functor that associates to an elliptic $A$-module of level 1 (for all $I \subset A$, by a scheme $I^1$.

The group $B$, consisting of elements acting on $I^1$. Let $I \subset A$ be an
Elliptic Modules

$k \geq 1$, \(|a_i| \leq \max(|a_{i+k}|, \ldots, |a_{i+k+1}||p^{k+1} \ldots, |a_{i+k}|^{p^{k+1}})

for \(i > s\). Let \(c_i = \sup_{j \geq s} |a_j|^p\). Since \(a_i \to 0\), we have \(c_i \leq \max_{1 \leq j \leq i} c_{i-j}\) for \(i > s\).

Let \(d_i = c_i^{p^{-i}}\). Then \(d_i \leq \max_{1 \leq j \leq i} d_j^{p^{i-j}}\). \(d_i < 1\) for \(i > s\), and so \(d_i \to 0\). □

Let \(\Gamma\) be the kernel of the homomorphism \(u\). Clearly, \(\Gamma \subset K^\times\). If \(y \in \Gamma\), \(y \neq 0\), then \(|y| > 1\). Let \(a \in A\), \(|a|_\infty > 1\), be such that the image of \(a \in O\) is not zero. We set \(f = \psi(a)\). Clearly \(f^{-1}(\Gamma)/\Gamma \cong [A/(a)]^{d-1}\), and the kernel of the homomorphism \(f^{-1}(\Gamma)/\Gamma \to \Gamma/\Gamma\) is isomorphic to \([A/(a)]^d\). Therefore \(\Gamma/\Gamma \cong [A/(a)]^{d-1}\). If \(z \in K^\times\), \(|z| > 1\), then \(|f(z)| = |z|^{d-1}\). It follows from this that \(\Gamma\) is a lattice of dimension \(d - d_1\).

3) Thus we have obtained a one-to-one correspondence between the isomorphism classes of elliptic \(A\)-modules of rank \(d\) over \(K\) having stable reduction, and the isomorphism classes of pairs \((X, \Gamma)\), where \(X\) is an elliptic \(A\)-module over \(K\) of rank \(d_1\) (\(d_1 < d\)) with good reduction, and \(\Gamma\) is a lattice in \(X\) of dimension \(d - d_1\). It remains only to apply Galois descent. □

§8. Complex multiplication \((d = 1)\)

Theorem 1. The scheme \(M^1\) is the spectrum of the ring of integers of the maximum abelian extension of \(k\), completely split at \(\infty\). The action of \(\mathfrak{H}^*/k^*\) on \(M^1\) coincides with the action in class field theory.

Proof. Let \(I \subset A\) be an ideal satisfying the conditions of Proposition 5.3. It follows from Proposition 5.1 that the morphism \(M^1 \to \text{Spec} A\) is finite. It follows from Proposition 3.4 that \(M^1\) is a smooth curve, and that the morphism \(M^1 \to \text{Spec} A\) ramifies only over \(V(I)\). Let \(v \in \text{Spec} A\), \(v \notin V(I)\), and let \(n\) be a prime element in \(A_v\). It follows from Proposition 5.5 that \(n\) acts on the fiber of \(M^1_v\) over \(v\) like the Frobenius of the field \(A_v/(n)\). Therefore each connected component of \(M^1\) is invariant with respect to \(\mathfrak{H}^*/k^*\). On the other hand, \(M^1(\kappa_v) = M^1(\overline{\kappa_v}) = \mathfrak{H}^*/k^*\) (cf. §3), and this identification is consistent with the action of \(\mathfrak{H}^*/k^*\). Therefore \(M^1\) is connected, the action of \(\mathfrak{H}^*/k^*\) on \(M^1\) is exact, and \(\mathfrak{H}^*/M^1 = \text{Spec} A\). It remains only to apply class field theory. □

Corollary. Over any algebraically closed field over \(A\) there exist elliptic \(A\)-modules of any rank.

§9. Compactification of modular surfaces \((d = 2)\)

A) Construction of the boundary. Let \(X\) be an elliptic module over \(S\). We denote by \(\overline{X}\) the bundle of projective lines over \(S\) obtained by joining to \(X\) an infinitely distant section. The multiplicative semigroup \(A\) acts on \(X\).

The functor that associates to the scheme \(S\) over \(A\) the set of isomorphism classes of elliptic \(A\)-modules \(X\) of rank 1 over \(S\), together with a consistent structure of level \(I\) (for all \(I \subset A\), \(I \neq 0\)) and homomorphism \(k \to \text{Mor}(S, \overline{X})\), is represented by a scheme \(N^I\).

The group \(B\), consisting of matrices of the form \(\begin{pmatrix} x & \gamma \\ 0 & y \end{pmatrix}\) with \(x \in \mathfrak{H}^*\) and \(y \in \mathfrak{H}^*_p\) acts on \(N^I\). Let \(I \subset A\) be an ideal satisfying the conditions of Proposition 5.3.
Let $V_I$ be the group of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, where $x, y \in \mathbb{A}, x \equiv 1 \mod I, y \equiv 0 \mod I$ and $N_I^1 \overset{def}{=} V_I \setminus N^1$. It is easy to check that the morphism $N_I^1 \to \hat{M}_I^1$ is smooth. Let $\hat{N}_I^1$ be the completion of $N_I^1$ along an infinitely distant section. (We shall consider $N_I^1$ as a scheme and not as a formal scheme.) Let $\hat{N}_I^1 \subseteq N_I^1$ be the complement at the infinitely distant section, and let $\hat{N}_I^1 = \lim \hat{N}_I^1$ and $\hat{N}_I^1 = \lim \hat{N}_I^1$. Let $\hat{M}_I^2$ and $\hat{M}_I^2$ be the schemes induces by the schemes $\hat{N}_I^1$ and $\hat{N}_I^1$ with respect to the embedding $B \subseteq GL(2, \mathbb{A}/k^\times)$.

Let $I \subseteq A$ be an ideal satisfying the conditions of Proposition 5.3, and let $\hat{M}_I^2 = U_I \setminus \hat{M}_I^2$ and $\hat{M}_I^2 = U_I \setminus \hat{M}_I^2$. Let $O$ be a complete discrete normed ring over $A$ with field of fractions $K$.

Proposition 9.1. 1) The morphism $\text{Spec } K \to \hat{M}_I^2$ extends to a morphism $\text{Spec } O \to \hat{M}_I^2$ if and only if the corresponding elliptic module over $K$ has potentially good reduction.

2) There exists a unique $GL(2, \mathbb{A})$-invariant morphism $s: \hat{M}_I^2 \to M^2$ that induces a one-to-one mapping from the set of those morphisms $S \to \hat{M}_I^2$ such that the pre-image of $\hat{M}_I^2$ is Spec $K$ into the set of those morphisms $S \to M^2$ which do not extend to morphisms $S \to \hat{M}_I^2$.

Proof. Assertion 1) is clear. Let a morphism $\text{Spec } K \to \hat{M}_I^2$ not extend to a morphism $\text{Spec } O \to \hat{M}_I^2$. To this morphism corresponds an elliptic $A$-module over $K$ of rank $2$, having potentially bad reduction, with structure of level $I$. This is the same (cf. Proposition 7.2) as an elliptic $A$-module $X$ over $K$ of rank $1$ together with a one-dimensional lattice $\Gamma$ in $X$ such that $\text{Gal}(K^g/K)$ acts on $I^{-1}\Gamma/\Gamma$ trivially and an epimorphism $(I^{-1}/A)^2 \to I^{-1}\Gamma/\Gamma$. (By definition, $I^{-1}\Gamma$ is the set of points of $X(K^g)$ that fall in $\Gamma$ under "multiplication" by any element $a \in I$.) Every automorphism of the $A$-module $I^{-1}\Gamma$ which is trivial on $I^{-1}\Gamma/\Gamma$ is the identity. Therefore $I^{-1}\Gamma \subseteq X(K)$. The bijection 2) is thus constructed.

It is easy to check that it induced by a morphism $\hat{M}_I^2 \to M^2$. □

Corollary. Let $u \in \text{Spec } A$, $u \notin V(I)$, and let $Y$ be the fiber of $\hat{M}_I^2$ over $u$ and $\hat{Y}$ a smooth compactification of $Y$. The completion of $\hat{Y}$ along $\hat{Y} - Y$ is canonically isomorphic to the fiber $\hat{M}_I^2$ over $u$.

B) Compactification of $\hat{M}_I^2$.

Proposition 9.2. Let $X_1$ and $X_2$ be normal surfaces, $\pi_i: X_i \to \text{Spec } A$ proper morphisms, $D_i \subseteq X_i$ closed subschemes finite over Spec $A$, and $\varphi: X_1 - D_1 \to X_2 - D_2$ a finite morphism over Spec $A$. Then $\varphi$ extends to a finite morphism $X_1 \to X_2$.

Proof. Let $G \subseteq X_1 \times X_2$ be the graph of $\varphi$, and let $\overline{G}$ be its closure. Since $G$ is finite over $X_1 - D_1$ and $X_2 - D_2$, it follows that $G$ is close in $(X_1 - D_1) \times X_2$ and $X_1 \times (X_2 - D_2)$. Therefore $\overline{G} = G$ is finite over Spec $A$. It follows from this that the projection $G_1 \to X_1$ is a finite morphism and birational isomorphism, i.e. $\overline{G} \cong X_1$. □

Proposition 9.3. 1) Let $I \subseteq A$ be an ideal satisfying the conditions of Proposition 5.3. There exists a unique (up to isomorphism) smooth surface $\hat{M}_I^2$ containing $M_I^2$ as an open everywhere dense set such that the morphism $\hat{M}_I^2 \to \text{Spec } A$ extends to a proper morphism $\hat{M}_I^2 \to \text{Spec } A$, and $\hat{M}_I^2 - M_I^2$ is finite over Spec $A$.

2) If $J \subseteq I$, then the project $\hat{M}_I^2 = \lim \hat{M}_J^2$. The action of $GL(2, \mathbb{A}/k^\times)$ is trivial.

3) The completion of $\hat{M}_I^2$ in particular, the morphism $\hat{M}_I^2 \to \text{Spec } A$.

Proof. Let $a \in A$, $|a| > 1$ and $\Sigma a_i x^i$ in the universal elliptic ring. We set $t = a_k^{\pm 1}a_{2k}^{\pm 1}$.

Proposition 9.1 that $\phi$ is a finite morphism. On the other hand, it is easy to construct a morphism $\phi \circ s$ that coincides with the morphism $A((1/t))$ is the localization of $A$ at $t$ (with respect to a nonarchimedean valuation).

Lemma. Let $X = \{1, 2, 3, 4\}$.

Let $O_X$ be a sheaf of rings on $X$.

$H^0(\{1, 2, \mathbb{A}/k^\times)$

The category of quasicoherent sheaves on $X$.

Proof. 1) Let $M$ be a module over $\mathbb{A}/k$. Then $\hat{M}$ is a quasicoherent sheaf, and the natural homomorphisms $M \to H^0(X, \hat{M})$ are isomorphisms.

2) $A[y] \to A[y, y^{-1}]$. Therefore $H^0(X, \hat{M}) = M$.

3) The restriction of the sheaf $\hat{M}$.

4) If $F \subseteq I_1$, then $\hat{M} \to F$.

5) If $H^0(X, F) = 0$, then $G_1 = 0$.

6) Let $0 \to G_1 \to H^0(X, F) \to 0$, and so $G_1 = 0$. Since $H^1(X, F) = 0$.

It remains to check that $\deg n$. (It is necessary to use Proposition 9.1 that the fibers of $\hat{M}_I^2$ are finite.)

§10. Conclusion.

A) Étale cohomology of $X$ with respect to a nonarchimedean valuation.
mod \ l, y \equiv 0 \mod \ l \text{ and } l_{i1} = 0. \text{ Let } \tilde{M}^1_i \text{ be smooth. Let } \tilde{M}^1_i \text{ be smooth.}

We shall consider \( \tilde{N}^1_i \) as a complement at the infinitely prime \( \tilde{M}^2 \) and \( \tilde{M}^2 \) be the schemes embedding \( B \subset \text{GL}(2, \mathbb{Z})/k^n \).

Proposition 5.3, and let \( \tilde{M}^2 \) be a scheme ring over \( A \) with field

to a morphism \( \text{Spec } O \to A[1/n] \).

a potentially good reduction. We need a \( \tilde{M}^2 \to M^2 \) that induces a \( \tilde{M}^2 \to M^2 \) such that the pre-images \( K \to M^2 \) which do not exist.

\( M^2 \) not extend to a morphism module over \( K \) of rank 2, having the form (Proposition 7.2) as an integral lattice \( \Gamma \) in \( X \) such that \( x^{-1}(-\Gamma) \to -\Gamma \). (By multiplication” \( \cdot \), which is trivial on \( \Gamma \).) The extension of basis.

There arises a morphism \( \sigma : \tilde{M}^2 \to \tilde{M}^2 \times \text{Spec } A(1/n) \).

Lemma. Let \( X = \{1, 2, 3\} \), and take the sets \( \phi, X, \{1, 2, 3\} \) and \( \{2, 3\} \) to be open.

Let \( O_X \) be a sheaf of rings on \( X \) such that

\[ H^0(\{1, 2\}, O_X) = A[[y]], \quad H^0(\{3, 2\}, O_X) = A[[y, y^{-1}]], \quad H^0(\{2\}, O_X) = A((y)) \]

The category of quasicoherent \( O_X \)-modules is equivalent to the category of \( A[y] \)-modules.

Proof. 1) Let \( M \) be a module over \( A[y] \). We set \( H^0(U, \tilde{M}) = M \otimes A[y] \), \( \tilde{M} \) is a quasicoherent sheaf, and \( M \to M \) is a functor. It remains to prove that the functorial homomorphisms \( M \to H^0(X, \tilde{M}) \) and \( H^0(X, F) \to F \) (where \( F \) is a quasicoherent sheaf) are isomorphisms.

2) \( 0 \to A[y] \to A[y, y^{-1}] \to A[y] \to 0 \) is an exact sequence of flat \( A[y] \)-modules. Therefore \( H^1(X, \tilde{M}) = 0 \text{ and } H^1(X, \tilde{M}) = 0 \).

3) The restriction of the morphism \( H^0(X, F) \to F \) on \( \{2, 3\} \) is an isomorphism.

4) If \( \text{Supp } F \subset \{1\} \), then \( H^0(X, F) \cong F \).

5) If \( H^0(X, F) = 0 \), then \( F = 0 \). (This follows from 3) and 4.)

6) Let \( 0 \to G_1 \to H^0(X, F) \to F \to G_2 \to 0 \) be an exact sequence. Then \( H^0(X, G_1) = 0 \), and so \( G_1 = 0 \). Since \( H^1(X, H^0(X, F)) = 0 \), we have \( H^0(X, G_2) = 0 \). Therefore \( G_2 = 0 \).

It remains to check that \( \sigma \) is an isomorphism. It is easy to check that \( \deg \phi = \deg \pi \). (It is necessary to use the Corollary to Proposition 9.1.) It follows from Proposition 9.1 that the fibers of \( \sigma \) consist of one point (and not necessarily reduced). On the other hand, one can check that the restriction of \( \sigma \) to each connected component of \( \tilde{M}^2 \) is a closed embedding.

§ 10. Connection with automorphic forms \((d = 2)\)

A) Étale cohomology of rigid analytic spaces. Let \( K \) be a field complete with respect to a nonarchimedean absolute value, let \( K^S \) be the separable closure of \( K \),
and let $n$ be a natural number such that $1/n \in K$. Let $X$ be a rigid analytic space over $K$.

**Definition.** 1) The set of pairs $(L, \phi)$, where $L$ is an invertible sheaf over $X$ and $\phi: O_X \cong L^\times$, is denoted $H^1_L(X, \mu_n)$. (There is a group structure on $H^1_L(X, \mu_n)$.)

2) $H^0_{et}(X, Z/(n)) \cong H^0_{rigid}(X, Z/(n))$ and $H^2_{et}(X, \mu_n) \cong H^2_{rigid}(X, \mu_n)$.

(On the right side of these equalities $Z/(n)$ is the constant sheaf and $\mu_n$ is the sheaf of $n$th roots of unity; this is a sheaf in the rigid topology.)

3) $\mathcal{H}^2_{et}(X \otimes K^s, \mu_n) \cong \lim_{\xrightarrow{\text{et}}} H^2_{et}(X \otimes L, \mu_n)$,

where $L \subset K \subset L^\times$ and $[L: K] < \infty$. In the same way we define $H^0_{et}(X \otimes K^s, \mu_n)$ and $H^2_{et}(X \otimes K^s, Z/(n))$. (Gal($K^s$, K) acts on all of these groups.)

4) $H^0_{et}(X \otimes K^s, Z/(n)) \cong H^0_{et}(X \otimes K^s, \mu_n) \otimes \mu_n^{-1}$.

(Here $\mu_n^{-1}$ is a Gal($K^s$, K)-module homomorphism from the group of $n$th roots of unity of the field $K$ into $Z/(n)$.

**Properties of étale cohomologies.**

1) If $X = X_{an}$, where $Y$ is a projective scheme over $K$, then the "cohomology" of $X$ coincides with the cohomology of $Y$. (This follows from theorems of type GAGA; cf. [11] and [12].)

2) There exists an exact sequence

$$0 \to H^0(X, \mu_n) \to H^0(X, O_X^*) \to H^1(O_X, \mu_n) \to H^1(X, \mu_n) \to H^2(X, O_X^*) \to H^2(X, \mu_n).$$

(Here $H^1(X, O_X^*)$ is the rigid cohomology.)

3) Let the sets $S_i$ form an admissible open covering of $X$ whose nerve has dimension not greater than 1. There exists an exact sequence

$$0 \to H^0(X, \mu_n) \to \prod_{i \neq j} H^0(X_i, \mu_n) \to \prod_{i \neq j} H^0(X_i \cap X_j, \mu_n) \to \prod_{i \neq j} H^1(X_i \cap X_j, \mu_n).$$

4) If $X_1 \to X_2$ is a finite étale morphism of rigid analytic spaces with Galois group $G$ whose order is prime to $n$, then $H^1(X_2, \mu_n) = H^1(X_1, \mu_n)^G$.

**Proposition 10.1.** Let $x, x_{i} \in K$, $x_{i} \in K^s$, $1 \leq i \leq k$, and $c \in K^s$. Let $z_{i} \in K^s$ such that $|z_{i}| < |c|$, $0 < |c_{i}| < |c|$, and let $|z_i - z_j| > |c_{i}|$ for $i \neq j$. Set $X = \{z|z| \leq |c|, |z_i - z_j| > |c_{i}| \}$. Then

$$H^0(X \otimes K^s, Z/(n)) = Z/(n), \quad H^1(X \otimes K^s, \mu_n) \cong (Z/(n))^k.$$

(Gal($K^s$, K) acts trivially.)

**Proof.** a) The circle of radius $|c|$ is the union of circles of radius $|c_{i}|$ and sets $X$. Since the circle is connected, and the circles which intersect $X$ and circles of radius $|c_{i}|$ are connected, it follows that $X$ is connected. Analogously, $X \otimes L$ is connected if $L$ is a finite extension of $K$.

b) $H^1(X, O_X^*) = 0$ since every divisor on $X$ is principal. The set of rational functions without poles on $X$ is everywhere dense in $H^0(X, O_X^*)$. Clearly, if $f \in H^0(X, O_X^*)$ and $\sup_{z \in K} |(z)| < 1$, then $\phi_{z}(O_X^*) \otimes Z/(n)$ can be chosen so that if $f$ is a rational function at the center of a circle or is outside of the last circle,

We obtain an epimorphism $\phi: \mu_n \to \phi_{z}(O_X^*) \otimes Z/(n)$,

and so $H^1(X \otimes K^s, \mu_n) \cong Z/(n)$.

**B. Cohomology of $\Omega^1$.**

In the case of $\Omega^1$, $\mu_n$-coefficients in $C$ is an anticyclic graph with values in $C$. A1-annihilating function on a single vertex.

**Proposition 10.2.** The group

$$H^0(\Omega^1 \otimes K^s, \mu_n)$$

is isomorphic to the group of $\mu_n$-

(Gal($k_{\infty}$, $k_{\infty}$) is isomorphic to $\mu_n$-

(The isomorphism is consistent with $\mu_n$-

Gal($k_{\infty}$, $k_{\infty}$) is isomorphic to $\mu_n$-

Proof (cf. [56A].) Let $0 \to F = \bigcup_{i \in \mathbb{Z}} U_{\Delta_i}^c \to \bigcup_{i \in \mathbb{Z}} U_{\Delta_i}^c$,

where $F$ is a finite-dimensional $\mathbb{A}$-module. Since $F$ is absolutely irreducible, we shall describe $H^1$ for a valuation in $k_{\infty}$ such that $|\mathbb{A}|$.

Let $F$ be a holomorphic invertible function on $X_{\overline{\mathbb{A}}}$ such that $\mathbb{F} \rightleftharpoons \phi \circ \nu \circ f$. The following statements are true:

1) $\bar{f}$ is linear on every $

2) If $x \in \lambda(F) \bigcap S^2$, then the value

from $x$ is equal to 0.

Every function on $\lambda(F)$ is a holomorphic function $f$ on $F \otimes L$. A function $f \in H^1(F \otimes k_{\infty}$, $\mu_n)$ if and only if $f$.

**C. Cohomology of $\Omega^2 \otimes K^s$.**

**Proposition 10.3.** Let $\Pi$ be the constant functions $\mathbb{P}^1(k_{\infty}) \to \Omega^1(k_{\infty})$.
a rigid analytic space

invertible sheaf over $X$ and sheaf on $H^1_c(X, \mu_n)$. In particular, $H^0_\text{rigid}(X, \mu_n)$ is the sheaf and $\mu_n$ is the sheaf

$H^0_\text{rigid}(X, \mu_n)$.

for some $H^0_\text{et}(X \otimes K^\times, \mu_n)$ and $H^0_\text{et}(X \otimes \mu_n^{-1})$.

intersection of $n$th roots of unity

the \textit{cohomology}'' theorem of type GAGA;

$H^1(X, O_X)$

$H^1(X, O_X) \rightarrow H^1(X, O_X)$.

spaces with Galois action on $K^\times$, $\mu_n$.

$K^\times$. Let $|z| < |c|$, $|c| > |z-z_i| \geq |c_i|$. Then

$(Z/\langle n \rangle)^k$. The set of rational functions $f$ on $X$ and circles of radius $|c_i|$ and sets $X$ and circles of radius $|c|$. Analogously, $X \otimes L$ is

Clearly, if $f \in H^0(X, O_X)$ and $\sup_{x \in X} |f(z) - 1| < 1$, then $f = g^n$, where $g \in H^0(X, O_X)$. Therefore every element of $H^0(X, O_X^\times) \otimes \mathbb{Z}/\langle n \rangle$ can be represented as a rational function. It is easy to check that if $f$ is a rational function whose divisor on $P^1(k)$ is contained in one of the $k$ small circles or is outside of the large circle, then there exists $c \in K$ such that

$\sup_{z \in X} |cf(z) - 1| < 1$.

We obtain an epimorphism $\phi: (Z/\langle n \rangle)^k \rightarrow H^1(X, \mu_n)/(K^\times/(K^\times)^n)$. Let $X_i$ be a circle of radius $|c_i|$. We consider the homomorphism $H^1(X, \mu_n) \rightarrow H^1(X_i, \mu_n)$. In order to prove that $\phi$ is an isomorphism, it is sufficient to check that $H^1(X_i, \mu_n)/(K^\times/(K^\times)^n) \cong (Z/\langle n \rangle)^k$ for $1 \leq i \leq k$. This follows from the fact that $z^m$ is not an $n$th power in $K[z, z^{-1}]$ for $n < m$. Indeed, if $f = \sum_{n \geq 1} a_n z^n \in K[z, z^{-1}]$ and $f^n = z^m$, then $\max_{n \geq 0} |a_n| = 1$. We reduce the equation $f^n = z^m$ modulo the maximal ideal and obtain a contradiction. Thus we obtain an exact sequence

$0 \rightarrow K^\times/(K^\times)^n \rightarrow H^1(X, \mu_n) \rightarrow (Z/\langle n \rangle)^k \rightarrow 0$,

and so $H^1(X \otimes K^\times, \mu_n) \cong (Z/\langle n \rangle)^k$.

B) Cohomology of $\Omega^2$. Let $C$ be an abelian group. A 1-cochain on a graph with coefficients in $C$ is an antisymmetric function on the set of oriented edges of the graph with values in $C$. A 1-cochain is \textit{harmonic} if the sum of its values on all edges emanating from a single vertex of the graph is equal to 0.

**Proposition 10.2.** The group

$H^0(\Omega^2 \otimes k_\infty, \mathbb{Z}/\langle n \rangle) = (Z/\langle n \rangle) \cdot H^1(\Omega^2 \otimes K^\times, \mu_n)$

is isomorphic to the group of harmonic 1-cochains on $S^2$ with coefficients in $\mathbb{Z}/\langle n \rangle$.

(The isomorphism is consistent with the action of $GL(2, k_\infty)$; the action of $Gal(k_\infty/k_\infty)$ is trivial.)

**Proof (cf. §6A).** Let $0 < c < 1$, and let $\Delta_1 \subseteq S^2$ and $\Delta_2 \subseteq S^2$ be simplices. Let $F = U_{\Delta_1} \cap U_{\Delta_2}$, where $F$ is isomorphic to the space $X$ studied in Proposition 10.1. Since $F$ is absolutely irreducible and $S^2(\mathbb{R})$ is connected, we see that $\Omega^2$ is connected. We shall describe $H^1(F \otimes k_\infty, \mu_n)$ in invariant terms. Let $\nu$ be an (additive) valuation in $k_\infty$ such that $\nu(k_\infty) = Z$. Let $k_\infty \subseteq L \subseteq k_\infty$ with $[L : k_\infty] < \infty$, and let $f$ be a holomorphic invertible function on $F \otimes L$. There exists a function $\tilde{f} : \lambda(F) \rightarrow \mathbb{Q}$ such that $\tilde{f} \circ \lambda = \nu \circ f$. The function $\tilde{f}$ has the following properties:

1) $\tilde{f}$ is linear on every edge intersecting $\lambda(F)$, and its \textit{derivative} is an integer.

2) If $x \in \lambda(F) \cap S^2$, then the sum of its \textit{derivatives} on directions emanating from $x$ is equal to 0.

Every function on $\lambda(F)$ satisfying 1) and 2) is equal to $\tilde{f}$ for some $L$ and some function $f$ on $F \otimes L$. A function $f$ on $F \otimes L$ corresponds to the zero class in $H^1(F \otimes k_\infty, \mu_n)$ if and only if the \textit{derivative} of $\tilde{f}$ on every edge is divisible by $n$. □

C) Cohomology of $\bar{M}^2 \otimes K^\times$.  

**Proposition 10.3.** Let $\Pi$ be a representation of $GL(2, k_\infty)$ in the space of locally constant functions $P^1(k_\infty) \rightarrow \mathbb{Q}$, factored out by the constants, and let $A_0$ be the space
of parabolic automorphic forms on $GL(2, \mathbb{k}) \backslash GL(2, \mathbb{A})$ (cf. [10]). Let

$$V = \text{Hom}_{\mathbb{A}(\mathbb{k}, k)}(\pi, A_L).$$

Let $l \neq p$, and let $W^\infty$ be a 2-dimensional l-adic representation of $\text{Gal}(k_{\infty}^C/k_{\infty})$ such that there exists a non-split exact sequence $0 \to \pi_{\Omega} \to W^\infty \to Q' \to 0$. Then $H^1(\mathbb{M} \otimes k_{\infty}^C, \pi_{\Omega}) \cong V \otimes W^\infty$ (isomorphism of $GL(2, \mathbb{k}) \times \text{Gal}(k_{\infty}^C/k_{\infty})$-modules) and $H^0(\mathbb{M} \otimes k_{\infty}^C, \pi_{\Omega})$ is isomorphic to the space of locally constant functions $k_{\infty}^C \otimes \mathbb{F}_p^* \to Q'_{\Omega}$ (Gal($k_{\infty}^C/k_{\infty}$) acts trivially, and $g \in GL(2, \mathbb{k})$ acts according to the formula $(gf)(x) = f(x \det g)$).

**Proof.** a) Let $I \subset A$ be an ideal, $l \neq 0$, $l \neq A$. The images of the sets $U_A^C$ and their $GL(2, \mathbb{A})$-translations form an admissible open covering of $\bar{\Omega}^2$. Clearly, the stabilizer in $U_I$ of any vertex or edge of $S^2$ is a p-group. Using properties 3) and 4) of the “étale cohomologies”, we obtain an isomorphism

$$H^0(U_I \otimes k_{\infty}^C, \mathbb{Z}/(2), \mathbb{Z}/(2)) = H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n))$$

and an exact sequence

$$0 \to H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n)) \to H^0(U_I \otimes k_{\infty}^C, \mathbb{Z}/(2)) \to \left[ H^1(\mathbb{O} \otimes k_{\infty}^C, \mathbb{Z}/(2), \mathbb{Z}/(n)) \right] U_I \otimes \mathbb{Z}/(2) \to 0,$$

where $H^1(\mathbb{O} \otimes k_{\infty}^C, \mu_n)$ coincides (cf. Proposition 10.2) with the group of harmonic 1-cochains on $\bar{S}^2$ with coefficients in $\mathbb{Z}/(n)$.

b) Let $I_0 \subset \text{Gal}(k_{\infty}^C/k_{\infty})$ be the inertial group. The mapping

$$I_0 \times H^0(U_I \otimes k_{\infty}^C, \mathbb{Z}/(n)) \to H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n))$$

deefined by the formula $(x, \alpha) \mapsto x \alpha - x$ is bilinear, and $Gal(k_{\infty}^C/k_{\infty})$ is invariant.

It is easy to check that this homomorphism associates to a $U_I$-invariant harmonic 1-cochain on $\bar{S}^2$ the cohomology class of the corresponding cochain on $U_I \backslash \bar{S}^2$.

c) Let $K$ be a complete archimedean field, $Y$ a smooth projective analytic curve over $K$, and $D \subset Y$ a finite subspace. It is easy to show that

$$H^0(Y, \mathbb{Z}/(n)) = H^0(Y - D, \mathbb{Z}/(n)), H^1(Y \otimes k_{\infty}^C, \mu_n) \subset H^1((Y - D) \otimes k_{\infty}^C, \mu_n).$$

More precisely, if $Y - D = \bigcup_{i \in N} Y_i$ is an admissible affine covering such that $Y_i$ for each $i \in N$ intersects only a finite number of sets of the covering, then

$$H^0(Y \otimes k_{\infty}^C, \mu_n) = \bigcup_{\Phi \subset N} \text{Ker} \left[ H^0((Y - D) \otimes k_{\infty}^C, \mu_n) \to H^0((\bigcup_{i \in N - \Phi} Y_i) \otimes k_{\infty}^C, \mu_n) \right]$$

(where $\Phi$ runs through the finite subsets of $N$). Therefore

$$H^0(U_I \otimes k_{\infty}^C, \mathbb{Z}/(n)) = H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n)),$$

and $H^1(U_I \otimes k_{\infty}^C, \mu_n)$ consists of those elements of $H^1(U_I \otimes k_{\infty}^C, \mu_n)$ for which the corresponding harmonic 1-form is finite modulo $U_I$ (or, equivalently, parabolic).

d) Going to the limit, we obtain

$$H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n))$$

and an exact sequence

$$0 \to H^1(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n)) \to H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n)) \to H^0(U_I \otimes k_{\infty}^C, \mathbb{Z}/(n)) \to H^1(U_I \otimes k_{\infty}^C, \mu_n),$$

where $V_I$ is the space of parabolic coefficients in $\mathbb{Q}$. It is easy to construct in b) is an isomorphism

$$\tilde{\mathcal{V}} = \lim \mathcal{V}_I$$

where $\mathcal{V}_I$ is the set of p-parabolic coefficients in $\mathbb{Q}$. This isomorphism coincides with the corresponding left action on $\mathcal{V}_I$. If $\mu$ is a harmonic 1-cochain on $\bar{S}^2$, then $\mathcal{V}_I$ is isomorphic to the set of harmonic 1-cochains on $\bar{S}^2$.

The group $Gal(k_{\infty}^C/k_{\infty}) \times \hat{G}(\mathbb{Z}/(n)) \times Gal(\mathbb{Z}/(n)) \times \hat{G}(\mathbb{Z}/(n)) \times \hat{G}(\mathbb{Z}/(n))$ acts on $\bar{S}^2$ and $\mathcal{V}_I$ is the space of p-parabolic coefficients in $\mathbb{Q}$. It is easy to construct in b) is an isomorphism

$$\tilde{\mathcal{V}} = \lim \mathcal{V}_I$$

where $\mathcal{V}_I$ is the set of p-parabolic coefficients in $\mathbb{Q}$. This isomorphism coincides with the corresponding left action on $\mathcal{V}_I$. If $\mu$ is a harmonic 1-cochain on $\bar{S}^2$, then $\mathcal{V}_I$ is isomorphic to the set of harmonic 1-cochains on $\bar{S}^2$.

Theorem 2. 1) $H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n))$

(The group $GL(2, \mathbb{k}) \times \hat{G}(\mathbb{Z}/(n)) \times Gal(\mathbb{Z}/(n)) \times \hat{G}(\mathbb{Z}/(n)) \times \hat{G}(\mathbb{Z}/(n))$ acts on $\mathcal{V}_I$)

2) For any $i \in T$ and $v \in W_i^T$ "coincides" with $\mathcal{V}^T_i$ (in the sense of the group $B \subset GL(2, \mathbb{k})$).

**Proof.** 1) In §9A we describe the homomorphism

$$H^0(U_I \backslash \bar{S}^2(\mathbb{R}), \mathbb{Z}/(n))$$

is an isomorphism.

2) Let $L(\mathcal{V}_I)$ be the restriction of $\mathcal{V}_I$ to $B$. Just as in [14], it is proved...
d) Going to the limit, we obtain an isomorphism

$$H^0(\overline{M}_f \otimes k^e, \mathbb{Q}_l) \cong H^0(U_f \setminus \mathcal{S}(\mathbb{R}), \mathbb{Q}_l)$$

and an exact sequence

$$0 \to H^1(U_f \setminus \mathcal{S}_2(\mathbb{R}), \mathbb{Q}_l) \to H^1(\overline{M}_f \otimes k, \mathbb{Q}_l) \to V_f \otimes \mathbb{Q}_l \to 0$$

where $V_f$ is the space of parabolic $U_f$-invariant harmonic 1-cochains on $\mathcal{S}_2$ with coefficients in $\mathbb{Q}$. It is easy to check that the homomorphism $U_f \to H^1(U_f \setminus \mathcal{S}_2(\mathbb{R}), \mathbb{Q})$ constructed in b) is an isomorphism. It follows from this that $H^1(\overline{M}_f \otimes k, \mathbb{Q}_l) \cong V \otimes \mathbb{W}_l$, where $V = \lim_i V_i$.

e) It remains to show that $\overline{V} = V$. For this it suffices to verify that the space of harmonic 1-cochains on $\mathcal{S}_2$ with coefficients in $\mathbb{Q}$ is canonically isomorphic to $\text{Hom}(\mathbb{W}_l, \mathbb{Q})$. This isomorphism is constructed in the following way. Let $x$ and $y$ be adjacent points in $\mathcal{S}_2$, and let $P_{x,y}$ be the set of points $z \in \mathcal{P}(k)$ such that $y$ is between $x$ and $z$. If $\mu$ is a distribution on $\mathcal{P}(k)$ with values in $\mathbb{Q}$ such that $\int_{P_{x,y}} \mu = 0$, then the 1-cochain $1_{\mu}$ which has the value $\int_{P_{x,y}} \mu$ on $x,y$ harmonic.

§11. Fundamental theorem

The group $\text{Gal}(k^e/k) \times GL(2, \mathbb{Q}_l)$ acts on $H^*(\overline{M}_f \otimes k^e, \mathbb{Q}_l)$. (The group $GL(2, \mathbb{Q}_l)$ acts on $\overline{M}_f$ on the left, and on the cohomology on the right; we shall consider the corresponding left action on the cohomology.) We set $V \otimes \mathbb{Q} = \bigoplus_{i \in T} V_i$, where $V_i$ is an irreducible representation of $GL(2, \mathbb{Q}_l)$. (It is proved in [10] that $V_i \not\cong V_j$ for $i \neq j$.) It follows from Proposition 10.3 that

$$H^1(\overline{M}_f \otimes k, \mathbb{Q}_l) \cong \bigoplus_i V_i \otimes \mathbb{W}_l,$$

where $W_i$ is a representation of $\text{Gal}(k^e/k)$ in the space $V_i$. Let $V_i = \bigotimes_{\text{Spec } A} V_i^e$, and let $V_i^e$ be the restriction of $W_i$ on $\text{Gal}(k^e/k)$. Let $V_i^e$ be the representation contragredient to $V_i^e$.

Theorem 2. 1)

$$H^0(\overline{M}^e \otimes k^e, \mathbb{Q}_l) = H^0(M^e \otimes k^e, \mathbb{Q}_l).$$

(2 The group $GL(2, \mathbb{Q}_l)$ acts on $M_1^e$ by the homomorphism det: $GL(2, \mathbb{Q}_l) \to \mathbb{Q}_l^*$.)

2) For any $i \in T$ and $v \in \text{Spec } A$ such that $V_i^e$ is a representation of class 1, $W_i$ "coincides" with $V_i^e$ (i.e. $W_i$ is unramified and $L(s, W_i^e) = L(s - 1/2, V_i^e)$).

Proof. 1) In §9A we defined an imbedding $M_1^e \to \overline{M}_f$ consistent with the action of the group $B \subset GL(2, \mathbb{Q}_l)$. It follows from Proposition 10.3 that the corresponding homomorphism

$$H^0(\overline{M}_f \otimes k^e, \mathbb{Q}_l) \to H^0(M^e \otimes k^e, \mathbb{Q}_l)$$

is an isomorphism.

2) Let

$$L(s, V_i^e) = (1 - \mu_i t^{-1})(1 - \mu_i t^{-1})^{-1}.$$
(arithmetic) on $W^\nu$ coincides either with $\mu_1^{-1}q^{-\frac{1}{2}}$ or with $\mu_2^{-1}q^{-\frac{1}{2}}$. We shall show that if one of the eigenvalues of the Frobenius coincides with $\mu_1^{-1}q^{-\frac{1}{2}}$, then another is equal to $\mu_2^{-1}q^{-\frac{1}{2}}$. It is known that $\tilde{V}_i \simeq V_j$ for some $j \in T$. If the restriction of $V_i$ to the center of the group $GL(2, \mathcal{U})$ is the multiple character $\omega \in \det$, then $\tilde{V}_i \simeq V_j \otimes \omega^{-1}$ (cf. [10]). It follows from properties of the cup product that one of the eigenvalues of the Frobenius on $W^\nu_i$ coincides with $\mu_1^{-1}q^{-\frac{1}{2}}$. Let $s \in H^0(\mathcal{M}^2 \otimes k, \mathcal{O}_i)$, $s \neq 0$, and $gs = \omega(\det g)s$ for $g \in GL(2, \mathcal{U})$. Multiplication by $s$ induces an automorphism of $H^1(\mathcal{M}^2 \otimes k, \mathcal{O}_i)$. It follows from assertion 1) and Theorem 1 that one of the eigenvalues of the Frobenius on $W^\nu_i$ is $\mu_1^{-1}q^{-\frac{1}{2}}\omega(\pi)$, where $\pi$ is a prime element of $A_i$. But $\omega(\pi) = \mu_1^{-1}\mu_2^{-1}$. □

Remark. The action of $GL(2, \mathcal{U}_i)$ in [14] is different from the action described in §5D) on the outer automorphisms of $GL(2, \mathcal{U}_i)$. This explains the different formulations of Theorem 2 and the corresponding theorem in [14].

Received 17/DEC/73

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Translated by M. B. NATHANSON

Mat. Sbornik Tom 94 (136) (1974), No. 4

UDC 519.44

Abstract: This paper considers the problem of finding stable groups of automorphisms.

Bibliography: 26 items

§ 1. Introduction

A class $\mathcal{X}$ of groups is said to be maximal automorphic images, and if from a group $\mathbb{G}$ in $\mathcal{X}$ (the empty set is also considered as such) a group $\mathbb{G}'$ is obtained by a set of subgroups $\{\mathbb{G}(x, y)\}$, the subgroups $\mathbb{G}(x, y)$ of which were set forth by G. E. Hupett in the course of the last ten years; see also Hupett's book [2] and the subsequent results are related to only one such class of groups. The reason for this is that the main focus of this paper is on the program of studying groups with a stable group of automorphisms.

By taking different definitions of automorphisms of a group, we can obtain different classes of subgroups. One of the general principles according to the type of automorphism of subgroups. The essence of this idea is as follows:

Suppose we are studying groups $\mathbb{G}$ in $\mathcal{X}$, and the taking of subgroups, homomorphisms $\mathbb{G} \rightarrow \mathbb{G}'$ in $\mathcal{Y}$ by a group in $\mathcal{Y}$ again belong to some class $\mathcal{Z}$. Then a group $H \in \mathcal{Z}$ is isomorphic to a group $B \in \mathcal{X}$ if some $f \in \mathcal{Z}$ isomorphic to a group $B \in \mathcal{X}$, then we say that $f$ acts identically on $B$, if $A/C_A(B)$ is central in $A$. A series of notable series of groups with this property have finite length in $\mathcal{X},$ and the class of all those groups is finite.

Thus, by choosing different definitions of automorphisms of a group, we can obtain different classes of subgroups. One of the general principles according to the type of automorphism of subgroups. The essence of this idea is as follows:

AMS (MOS) subject classification: 20D15, 20D20.