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Journal Title: Mathematics of the USSR. Sbornik.
Volume: 31(2) Issue: 1977
Pages: 159-170
Article Author:

Article Title: Drinfeld, V.; Elliptic Modules, II
Imprint: [Providence, R.I. ; American Mathematica

ILL Number: 86466126

Call #: QA1 .M35 v.31 (1977)

Location: Owen

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ELLiptic modules. II

v. g. Drinfel'd

Abstract. In this paper the Langlands reciprocity law is proved for representations of GL(2) over the adeles of a function field for which one of the components is cuspidal or special. In part I (Math. USSR Sb. 23 (1974), 561-592), the author considered the case when one of the components is special.

Bibliography: 7 titles.

Let k be a global field of characteristic p, \( \mathfrak{H} \) the ring of adeles of k. According to the Langlands conjecture, for every irreducible admissible representation of GL(2, \( \mathfrak{H} \)) over \( \mathbb{Q} \) in the space of cusps forms there exist a subfield \( E \subset \mathbb{Q} \) of finite degree over \( \mathbb{Q} \) and a compatible system of two-dimensional irreducible \( \lambda \)-adic representations of the Weil group of k (\( \lambda \) runs through the set of non-Archimedean places of E not lying above p) which has the same local L and \( \epsilon \) factors. The purpose of the present paper is to prove the conjecture in the case when one of the local components of the representation of GL(2, \( \mathfrak{H} \)) is either cuspidal or special.

The case when one of the components is special was analyzed in [1]; the desired representations of the Weil group arise in the one-dimensional cohomology of certain curves over k. In the present paper we use the cohomology of certain coverings of these curves. These coverings are constructed in §1. We use the Sato-Ihara-Langlands method to find the representation that arises in the cohomology of these coverings: we describe the set of points on reductions of these curves (§2), and then apply the Lefschetz formula and the Selberg trace formula (§3). In §3 we only sketch the computations (they do not differ much from the computations in [6], and the transformation of the trace formula, which we do not carry out in detail in §3, only uses the methods of §16 in [2]).

The idea of this paper is due to Pierre Deligne and D. A. Kazdan: they convinced me of the possibility of the existence of the coverings used in the paper.

We shall use the following notation: k is a global field of characteristic p; \( \infty \) is a fixed place of k; \( k_v \) is the completion of k at the place v; A is the set of elements of k which are integral outside \( \infty \); \( A_v \) is the completion of A at a place v different from \( \infty \); \( O_{\infty} \) is the ring of integers in \( k_\infty \); \( m_v \) is the maximal ideal of A corresponding to a place v \( \neq \infty \); \( m_\infty \) is the maximal ideal of \( O_\infty \); \( q_v \) is the order of \( A/m_v \); \( \mathfrak{p} \) is the ring of adeles of k; \( \mathfrak{p}_v \) is the ring of adeles without \( \infty \)-component; \( \mathfrak{p}_\infty \) is the ring of adeles without \( \infty \)- and \( v \)-components, where v \( \neq \infty \); and \( k^s \) is the separable closure of k.

§1. Construction of the coverings

In [1] we constructed a scheme \( M^d \) over A on which GL(\( d, \mathfrak{p}_v \)) acts (see [1], §5); \( M^d \) is
the moduli scheme of elliptic $A$-modules of rank $d$ equipped with mutually compatible structures of all levels. We shall construct a scheme $\hat{\mathcal{M}}^d$ over $\mathcal{M}^d$ having an action of $\text{GL}(d, \mathfrak{A}_t)$ and the multiplicative group of a central division ring over $k_\infty$ with invariant $-1/d$.

If $B$ is a ring of characteristic $p$, we let $B\{\tau\}$ denote the ring of “polynomials” in $\tau$ (where $\tau b = b^{\tau p}$ for $b \in B$). If $B$ is perfect (i.e., the Frobenius homomorphism $B \to B^{\tau}$ is an isomorphism), then we can also define the rings $B\{\{\tau^{-1}\}\}$ (the ring of “formal power series in $\tau^{-1}$”) and $B\{\tau\}$ (the ring of “formal Laurent series in $\tau^{-1}$”), which consists of elements of the form $\sum_{-\infty < n < 0} b_n \tau^n$.

Let $B$ be a perfect $A$-algebra, and let $\varphi : A \to B\{\tau\}$ be a homomorphism which is a homomorphism module of rank $d$ (see 1, §2). It is clear that $\varphi$ extends uniquely to a homomorphism $k \to B\{\{\tau^{-1}\}\}$, which we also denote by $\varphi$. Since $\varphi(k \cap O_\infty) \subset B\{\{\tau^{-1}\}\}$ and $\varphi(k \cap m_\infty) \subset \tau^{-1} B\{\{\tau^{-1}\}\}$, it follows that $\varphi$ extends by continuity to a homomorphism $k_\infty \to B\{\{\tau^{-1}\}\}$, which we again denote by $\varphi$. It is obvious that, for any nonzero $a \in k_\infty$, the leading coefficient of $\varphi(a)$ is invertible and has index $d \log_p|a|_\infty$ where $|\cdot|_\infty$ is the normalized absolute value corresponding to $\infty$.

We fix a homomorphism $\varphi_0 : k_\infty \to \overline{\mathbb{F}}_p\{\{\tau^{-1}\}\}$ such that for any nonzero $a \in k_\infty$, the leading coefficient of $\varphi_0(a)$ has index $d \log_p|a|_\infty$. It is easily verified that such a homomorphism exists and is uniquely determined up to an inner automorphism of $\overline{\mathbb{F}}_p\{\{\tau^{-1}\}\}$. We let $D$ denote the centralizer of the image of $\varphi_0$. It is easy to see that $D$ is a central division ring over $k_\infty$ with invariant $-1/d$. We let $\det : D^* \to k_\infty^*$ denote the reduced norm.

We consider the following functor $G_\varphi$ from the category of perfect $B$-algebras to the category of sets: we let an algebra $C$ correspond to the set of pairs consisting of $\lambda \in \mathbb{F}_p \rightarrow C$, 2) an element $u \in C\{\{\tau^{-1}\}\}$ with invertible constant term such that $\varphi_0(u)^{-1} = \lambda \varphi(a)$ for any $a \in k_\infty$, where $\lambda : \mathbb{F}_p\{\{\tau^{-1}\}\} \to C\{\{\tau^{-1}\}\}$ is induced by $\lambda$. $D^*$ acts on this functor as follows: if $g \in D^*$ and $(\lambda, u) \in G_\varphi(C)$, then $g(\lambda, u) = (\lambda \circ \text{Fr}^*, \tau^m \lambda_s(g)u)$, where $\text{Fr}: \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p$ is the Frobenius homomorphism, and $n = -\log_p|\det g|_\infty$. If $\psi : A \to B\{\tau\}$ is another elliptic module of rank $d$, and $s \in B\{\tau\}$ is an isogeny from $\varphi$ to $\psi$ of degree $m$ (i.e., $\varphi(a) = \psi(sa)$), the leading coefficient of $a$ is invertible and has index $m$, then we can define an isomorphism $s_* : G_\varphi \rightarrow G_\psi$ compatible with the action of $D^*$ as follows: $s_*(\lambda, u) = (\lambda \circ \text{Fr}^*, \tau^m s\varphi_0^{-1})$.

**Proposition 1.** The functor $G_\varphi$ is representable by a ring $L$. The stationary subgroup $D^*$ of any element of $L$ is an open subgroup of finite index. If $U \subset D^*$ is an open normal subgroup of finite index, then $\text{Spec} L^U$ is an open Galois covering over $\text{Spec} B$ with group $D^*/U$. If we set $H = \text{Ker}(D^* \to k_\infty^*)$, then $L^H = B \otimes_{O_\infty} \overline{\mathbb{F}}_p$, where the $O_\infty$-algebra structures on $B$ and $\overline{\mathbb{F}}_p$ are induced by $\varphi$ and $\varphi_0$. □

In particular, it follows from the proposition that the action of $D^*$ on $L$ extends to an action of $L^U$ (the profinite completion of $D^*$).

We proceed to the construction of $M^d$. Recall that $M^d$ is an affine scheme; let $M^d = \text{Spec} B'$. There is a universal elliptic module $\varphi : A \to B'\{\tau\}$. Unfortunately, $B'$ is not perfect; we let $B$ denote its “perfect closure” (i.e., the direct limit of an infinite sequence of Frobenius homomorphisms $B' \to B' \rightarrow \ldots$). Proposition 1 allows us, starting from $\varphi$, to construct a $B$-algebra $L$ with an action of $D^*$. For any open subgroup $U \subset D^*$, $L^U$ is etale over $B$. Since the homomorphism $B' \to B$ is radical, it follows that the category of etale $B$-algebras is equivalent to the category of etale $B'$-algebras. We let $L^U$ denote the etale $B'$-algebra over $M^d = \text{Spec} L'$.

The action of $D^*$ on $M^d$ is defined by $g \in \text{GL}(d, \mathfrak{A}_t)$ on $M^d$. Let

$$g \in \text{GL}(d, \mathfrak{A}_t)$$

Recall that in §5 of [1], starting from $\text{Spec} B' \to B'$, but also an isogeny corresponding to $g$ lifts in a canonical way before the statement of Proposition 1 that, for any open normal group $U$ over $M^d$ with group $D^*/U$, $M^d = \text{Spec} L^U$.

**§2. Definition.** Let $v$ be a place of $k$, $v \neq \infty$. We write (together with the action of $GL(d, \mathfrak{A}_t)$) that the orbits of $GL(d, \mathfrak{A}_t) \times \hat{\mathcal{M}}^d$ correspondence with isogeny classes to describe the set of isogeny classes.

**Proposition 2.4.1.** Let $X$ be a Frobenius morphism, and let $E$ be of the pair $(E, \varphi)$ has the following properties:

a) $E \otimes \mathfrak{A}_\infty$ is a field.

b) $[E : k] \mid d$.

c) $|\varphi_\infty| = q^{\frac{d}{d}}$, where $|\cdot|_\infty$ is the absolute value corresponding to $\infty$.

d) $\varphi$ has a zero at a unique place of $k$.

2) $k \otimes \mathcal{O}_\infty$ is a central division ring with invariants equal $[E : k]/d$ at the places of $k$.

3) The $w$-component of the Tate modules of the components of $\text{dim}([E : k]/d)$.

4) The map from the set of isogeny isomorphism classes of pairs $(E, \varphi)$ is bijective.

**Proof.** We let $F_\varphi(\tau)$ denote the central division ring over $F_\varphi(\tau)^{\mathfrak{A}_\infty}$ at $\tau^{\log_q |a|_\infty}$. for $a \in \mathfrak{A}_\infty$, and zero if the rank $d$; we extend it to a homeomorphism...
equipped with mutually compatible \( \hat{M}^d \) over \( M^d \) having an action of division ring over \( k_{\infty} \) with invariant

\[ L^U \] denote the etale \( B^* \)-algebra corresponding to \( L^U \), and we set \( L^U = \lim \rightarrow L^U \) and \( \hat{M}^d = \text{Spec} \ L^U \).

The action of \( \hat{D}^* \) on \( \hat{M}^d \) is defined in the obvious way. We now define the action of \( \text{GL}(d, \mathbb{A}_f) \) on \( \hat{M}^d \).

\[ g \in \text{GL}(d, \mathbb{A}_f) \cap \text{Mat}(d, \prod_{\alpha \in \mathbb{A}_f} A_{\alpha}). \]

Recall that in §5 of [1], starting from \( g \), we defined not only an automorphism \( f^* \) of \( B^* \), but also an isogeny from \( \varphi \) to \( f^*_\ast \). Hence the automorphism of \( M^d \) corresponding to \( g \) lifts in a canonical way to an automorphism of \( \hat{M}^d \) (see the sentence before the statement of Proposition 1). We thereby obtain an action on \( \hat{M}^d \) of the subgroup \( \text{GL}(d, \mathbb{A}_f) \cap \text{Mat}(d, \prod_{\alpha \in \mathbb{A}_f} A_{\alpha}) \).

This action commutes with the action of \( \hat{D}^* \), and, for any nonzero \( a \in A \), the element \( a \in \text{GL}(d, \mathbb{A}_f) \) acts in the same way as \( a^{-1} \in \hat{D}^* \). Hence we actually obtain an action on \( \hat{M}^d \) of the group \( D^* \times \text{GL}(d, \mathbb{A}_f)/k^* \) (\( k^* \) is imbedded diagonally). It follows from Proposition 1 that, for any open normal subgroup \( U \subset D^* \), \( U \setminus \hat{M}^d \) is a Galois covering over \( M^d \) with group \( \hat{D}^* / U \), \( \hat{M}^d = \lim \leftarrow U \setminus M^d \), \( H \setminus \hat{M}^d = \hat{M}^d \otimes_{k_{\infty}} \hat{F}_p \).

§2. Description of the set of points

Let \( \nu \) be a place of \( k \), \( \nu \neq \infty \). We describe the set of \( \hat{A}/m_{\nu} \)-points of the scheme \( \hat{M}^d \)

(continued with the action of \( \text{GL}(d, \mathbb{A}_f) \times \hat{D}^* \) and the Frobenius morphism). It is clear that the orbits of \( \text{GL}(d, \mathbb{A}_f) \times \hat{D}^* \) in the set of \( \hat{A}/m_{\nu} \)-points of \( \hat{M}^d \) are in one-to-one correspondence with isogeny classes of elliptic modules of rank \( d \) over \( \hat{A}/m_{\nu} \). We first describe the set of isogeny classes of elliptic \( A \)-modules of rank \( d \) over \( \mathbb{F}_q \).

PROPOSITION 2.1.

1) Let \( X \) be an elliptic \( A \)-module over \( \mathbb{F}_q \), let \( \varphi : X \to X \) be the Frobenius morphism, and let \( \text{End} \ X \otimes_A k \) be generated by \( \varphi \). Then the pair \( (E, \varphi) \) has the following properties:

(a) \( E \otimes k_{\infty} \) is a field.

(b) \( |E : k| \leq d \).

(c) \( |\varphi|_{\infty} = q^{1/d} \), where \( |\infty| \) is the extension to \( E \) of the normalized absolute value on \( k \) corresponding to \( \infty \).

(d) \( \varphi \) has a zero at a unique place \( w \) lying over \( \nu \).

2) \( k \otimes_A \text{End} \ X \) is a central division ring over \( E \) of dimension \( d([E : k]) \) whose invariants equal \( [E : k]/d \) at the place \( w \), \(- [E : k]/d \) at the place lying over \( \infty \), and 0 at the other places.

3) The \( w \)-component of the Tate module of the elliptic module \( X \) is zero, and the dimensions of the components of the Tate module relative to other places of \( E \) equal \([E : k]/d \).

4) The map from the set of isogeny classes of elliptic \( A \)-modules over \( \mathbb{F}_q \), to the set of isomorphism classes of pairs \( (E, \varphi) \) such that \( \varphi \) generates \( E \) over \( k \) and properties a)-d) hold is bijective.

PROOF. We let \( \mathbb{F}_q(\tau) \) denote the division ring of fractions of the ring \( \mathbb{F}_q(\tau) \). \( \mathbb{F}_q(\tau) \) is a central division ring over \( \mathbb{F}_q(\tau) \) whose parameters equal \( 1/q \) at \( \tau^{1/q} = 0 \), \(-1/q \) at \( \tau^{1/q} = \infty \), and zero elsewhere. Let \( \varphi : A \to \mathbb{F}_q(\tau) \) be an elliptic module of rank \( d \); we extend it to a homomorphism \( k \to \mathbb{F}_q(\tau) \). It is easy to see that \( E \) is the
subfield of $F(p)$ spanned by the center and the image of $k$, $q = \tau_{\log, q}$. Assertions in 1c, 1d and 3) are easily verified. Properties 2) and 1b) follow from the equality

$$[E : F_p(q)] = \log_q q \cdot \frac{[E : k]}{d},$$

which follows from 1c).

To prove 4) we must, starting with a pair $(E, q)$ which satisfies a)–d) and is such that $E'$ generates $E$ over $k$, construct an elliptic $A$-module over $F(q)$. To do this, we first verify that $E$ can be imbedded in $F(q)$ in such a way that $q$ goes to $\tau_{\log, k}$; we then modify the imbedding using an inner automorphism of $F(q)$ so that the image of the ring of integers of $E$ lies in $F(q)$ and so that the given $A$-algebra structure is induced on $F(q)$; the resulting homomorphism $A \to F_q(q)$ is an elliptic module of rank $d$. □

**Definition.** By a $(k, \infty, v)$-type we mean a pair $(E, w)$, where $E$ is a finite extension of $k$ and $w$ is a place of $E$ over $v$ such that 1) $E \otimes k_\infty$ is a field, 2) every element $a \in E$ which has a zero at $w$, a pole at $\infty$, and neither a zero nor pole at other places, generates $E$ over $k$.

Condition 2) may be replaced by the equivalent condition: if $k \subset E' \subset E$, $E' \neq E$, then there is a place $w'$ of $E$ such that $w' \neq w$ but the places of $E'$ corresponding to $w$ and $w'$ coincide.

**Corollary.** The isogeny classes of rank $d$ elliptic $A$-modules over $A/m_0$ are in one-to-one correspondence with $(k, \infty, v)$-types $(E, w)$ such that $[E : k]/d$. This correspondence has the property that the endomorphism ring of the elliptic $A$-module corresponding to the type $(E, w)$ is an order in the central division ring over $E$ whose invariants are $[E : k]/d$ at $w$, $-[E : k]/d$ at $\infty$, and zero elsewhere. The $w$-component of the Tate module is zero, and the dimensions of the components of the Tate module relative to places of $E$ different from $w$ and $\infty$ equal $d/[E : k]$. □

These results are analogous to results in [3] and [5]. The corollary and the definition of the action of $GL(d, \mathfrak{A}_f) \times \tilde{D}^*$ imply the following description of the set of $A/m_0$-points on $\tilde{M}^d$. Let $(E, w)$ be a $(k, \infty, v)$-type such that $[E : k]/d$, and let $\Delta$ be the central division ring over $E$ whose invariants are $[E : k]/d$ at $w$, $-[E : k]/d$ at $\infty$, and zero elsewhere. For each place $\lambda$ of $E$ different from $w$ and $\infty$, we fix an isomorphism between $\text{Mat}(d/[E : k], E_\lambda)$ and $\Delta \otimes_E E_\lambda$. We let $S$ denote the group of matrices in $GL(d/[E : k], E_\lambda)$ whose determinant has modulus 1. We imbed $\Delta^*$ in

$$(\Delta \otimes k_\infty)^* \times GL(d/[E : k], E_\lambda) / S$$

using the obvious homomorphisms

$$\Delta^* \to (\Delta \otimes k_\infty)^* \to GL(d/[E : k], E_\lambda), \quad \Delta^* \to (\Delta \otimes k_\infty)^*$$

and the homomorphism

$$\Delta^* \to E^* \to GL(d/[E : k], E_\lambda)/SL(d/[E : k], E_\lambda) \to GL(d/[E : k], E_\lambda) / S,$$

where $\det: \Delta^* \to E^*$ is the reduced norm. We let $K$ denote the preimage in

$$(\Delta \otimes k_\infty)^* \times GL(d/[E : k], E_\lambda) / S$$

of the image of $\Delta^*$. We fix imbeddings

$$\text{Mat}(d/[E : k], E) \subset \text{Mat}(d, k), \quad \Delta \otimes k_\infty \subset D;$$

then $K$ is a subgroup of $D^* \times \mathfrak{A}/\mathfrak{A}$. We consider the place $w$. Let

$$N = \{g \in \text{GL}(d, \mathfrak{A}): g \Delta \subset \Delta\}$$

and, under this isomorphism, the multiplication by an element of $GL(d, \mathfrak{A})$ is an element of $GL(d, \mathfrak{A})$.

**Proposition 2.2.** The orbits of the scheme $\tilde{M}^d$ are in one-to-one correspondence with $[E : k] / d$. The Frobenius morphism of the type $(E, w)$ is isomorphic to $GL(d, \mathfrak{A})$.

**Corollary.** $\tilde{M}^d$ is the spectrum of the ring of integers of $E_w$. □

Proposition 2.2 gives us (in the case $d = 1$) a representation of $GL(2, \mathfrak{A})$ on the ring of integers of $k$. The action of the group $D^*$ on $\tilde{M}^d$ is a representation of class field theory. □

§

For any open subgroup $U \subset \tilde{D}^*$ we denote the smooth compactification of $Gal(k^s/k) \times \tilde{D}^* / det: \text{GL}(2, \mathfrak{A}) \times \tilde{D}^* \to \mathfrak{A}^*$. Consider the corresponding left $\tilde{D}^*$-representation of $Gal(k^s/k)$ forms with values in $\tilde{Q}$, whose support to the center is a character of $\tilde{D}^*$. A representation runs through the set of places $v$, $I_v$ is a representation of $Gal(k^s/k)$ in a two-dimensional vector space $v, L_v = L(s - 1, \tilde{\sigma})$.

**Proposition 3.2.** 1) $\lim \text{Hom}(\tilde{M}^d_\infty, \mathcal{D}(\mathfrak{A})_f)$ is a representation of $Gal(k^s/k) \times \tilde{D}^* / det: \text{GL}(2, \mathfrak{A}) \times \tilde{D}^* \to \mathfrak{A}^*$. 2) $\lim \text{Hom}(\tilde{M}^d_\infty, \mathcal{D}(\mathfrak{A})_f)$ is a representation of $Gal(k^s/k)$ in a two-dimensional vector space $v, I_v$.

We now formulate the fundamental lemma. Let $U \subset \tilde{D}^* \times GL(2, \mathfrak{A})_f$. We let $\tilde{M}^d_{U, w}$ denote the smooth compactification of the projective $\tilde{D}^* \times GL(2, \mathfrak{A})_f$-space.

**Fundamental lemma.** The representations $\text{Hom}(\tilde{M}^d_\infty, \mathcal{D}(\mathfrak{A})_f)$ and $\text{Hom}(\tilde{M}^d_{U, w}, \mathcal{D}(\mathfrak{A})_f)$ are isomorphic.
then $K$ is a subgroup of $D^* \times \text{GL}(d, \mathfrak{A}_w)$. We let $e$ denote the idempotent of $E \otimes k_v$ corresponding to the place $w$. Let

$$N = \{ g \in \text{GL}(d, k_v) \mid (g^{-1}e = 0, (1-e) (g^{-1}e) = 0) \}.$$

**Proposition 2.2.** The orbits of the action of $\text{GL}(d, \mathfrak{A}_f) \times D^*$ in the set of $\overline{A/m_v}$-points of the scheme $\overline{M}_d^*$ are in one-to-one correspondence with $(k, \infty, v)$-types $(E, w)$ such that $[E : k] \mid d$. The Frobenius morphism takes every orbit to itself. The orbit corresponding to

$$\text{GL}(d/\{E : k\}, \mathfrak{A}_w) \times \overline{D}^*/KN,$$

and, under this isomorphism, the action of the Frobenius morphism is identified with right multiplication by an element of $\text{GL}(d/\{E : k\}, E_w)$, whose determinant is a prime element of the ring of integers of $E_w$. □

Proposition 2.2 gives us (in the same way as in [1], §8) the following description of $\overline{M}_1$:

**Corollary.** $\overline{M}_1$ is the spectrum of the ring of integers of the maximal abelian extension of $k$. The action of the group $D^* \times \mathfrak{A}_f^* = k_v^* \times \mathfrak{A}_f^* = \mathfrak{A}_v^*$ on $\overline{M}_1$ coincides with the action of class field theory. □

### §3. Fundamental theorem

For any open subgroup $U \subset \text{GL}(2, \mathfrak{A}_f) \times \overline{D}^*$ we set $\overline{M}_d^* = U \setminus \overline{M}_2^*$. We let $\overline{M}_d^* \otimes \overline{k}$ denote the smooth compactification of $\overline{M}_d^* \otimes \overline{k}$. Let $l$ denote the central character of $\overline{M}_d^* \otimes \overline{k}$. We let $\overline{M}_d^* \otimes \overline{k}$ act on $\text{Hom}(\text{Gal}(k^*/k) \times \text{GL}(2, \mathfrak{A}_f) \times \overline{D}^*, l)$ (the group $\text{GL}(2, \mathfrak{A}_f) \times \overline{D}^*$ acts on $\overline{M}_d^* \otimes \overline{k}$ on the left, and on the cohomology on the right; we shall consider the corresponding left action on the cohomology). We let $T$ denote the set of indecomposable admissible representations of $\text{GL}(2, \mathfrak{A}_f) \times D^*$ in the space of automorphic forms with values in $\overline{Q}_l$, whose $\infty$-component is special or cuspidal and whose restriction to the center is a character of finite order. For each $\pi \in T$ we set $\pi = \otimes_v \pi_v$, where $v$ runs through the set of places of $k$, and we set $\pi = \otimes_v \pi_v \otimes W_v$, where $W_v$ is a representation of $\text{Gal}(k^*/k)$ in a two-dimensional vector space over $Q_l$ such that, for all but finitely many places $v$, $L(s, W_v) = L(s - 1/2, \pi_v)$ (here the $L$-functions are understood in the sense of Serre and Jacquet-Langlands).

We now formulate the fundamental lemma. Let $v$ be a place of $k$, $v \neq \infty$. For any open subgroup $U \subset \overline{D}^* \times \text{GL}(2, \mathfrak{A}_f)$ containing $\text{GL}(2, \mathfrak{A}_f)$ we set $\overline{M}_d^* = \overline{M}_2^* \otimes A/m_v$. We let $\overline{M}_d^* \otimes \overline{k}$ denote the smooth compactification, and we let $F : \overline{M}_d^* \otimes \overline{k} \to \overline{M}_d^*$ denote the Frobenius morphism. $\text{GL}(2, \mathfrak{A}_f) \times \overline{D}^* \times F^*$ act on

$$\lim_{U \supseteq \text{GL}(2, \mathfrak{A}_f)} \text{Hom}(\overline{M}_d^* \otimes \overline{A}/m_v, \text{Q}_l).$$

**Fundamental Lemma.** The representation of $\text{GL}(2, \mathfrak{A}_f) \times \overline{D}^* \times F^*$ on
\[ \lim_{U \to \text{GL}(A, A_p)} H^p(M_{U,v} \otimes \overline{A}/m_{v}, Q_i) \]
is isomorphic to a representation on
\[ \lim_{U \to \text{GL}(A, A_p)} H^p(M_{U,v} \otimes \overline{A}/m_{v}, Q_i) , \]
and the representation on
\[ \lim_{U \to \text{GL}(A, A_p)} H^1(M_{U,v} \otimes \overline{A}/m_{v}, Q_i) \]
is isomorphic to
\[ \sum_{\pi \in \mathcal{D}} \pi_D \otimes \pi_\pi \otimes X_{\pi}^r. \]

Here \( \pi = \pi_\pi \otimes \pi_\pi^* \); \( X_{\pi}^r = 0 \), if \( \pi_\pi \) is not of class 1, while if \( \pi_\pi \) is of class 1, then \( X_{\pi}^r \) is a
two-dimensional space with action of \( F^* \), such that
\[ L(s-1/2, \pi_v) = \det(1 - q_v^{-s}F^*, X_{\pi}^r)^{-1}. \]

The theorem follows from the lemma and the fact that for every \( U \) we have
\[ H^p(M_{U,v} \otimes \overline{A}/m_{v}, Q_i) = H^p(M_D \otimes \overline{k}, Q_i) \]
for all but finitely many \( v \). The proof of the fundamental lemma uses a local lemma. Let
\( K \) be a local non-Archimedian field, let \( O \) be its ring of integers, let \( \pi \in O \) be a prime
element, let \( v : K^* \to \mathbb{Z} \) be a norm, and let \( q \) be the order of the residue field. Let \( n \in \mathbb{Z}, n > 0 \). We let \( \chi^r \) denote the following function on \( \text{GL}(2, K) \): if \( v(\det g) = n \) and all of
the coefficients of \( g \) lie in \( \pi O \), then \( \chi^r(g) = 1 - q \); if \( v(\det g) = n \) and all of
the coefficients of \( g \) are integral but not all lie in \( \pi O \), then \( \chi^r(g) = 1 \); in the remaining cases
\( \chi^r(g) = 0 \). The Haar measures on the groups in the lemma are normalized so that the
maximal compact subgroups have measure 1.

**Local Lemma.**
1. Let \( \rho \) be a class 1 irreducible admissible representation of \( \text{GL}(2, K) \) such that
\[ L(s-1/2, \rho) = (1 - a_i q^{-s})^{-1} \times (1 - a_i q^{-s})^{-1}. \]
Set \( \rho(\chi^r) = \int \chi^r(g) \rho(g) \, dg \). Then \( \text{Tr} \rho(\chi^r) = a_i^r + a_i^r \).
2. Let \( \lambda, \mu \in K^* \), \( \lambda \neq \mu \). Then
\[ \int_{x \in K^* \times K^* \times \text{GL}(2, K)} \chi^r \left( x^{-1} \left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right) x \right) \, dx = 1, \]
if \( v(\lambda) = n, v(\mu) = 0 \) or \( v(\lambda) = 0, v(\mu) = n \). Otherwise, the integral equals zero.
3. Let \( E \subset \text{Mat}(2, K) \) be a quadratic extension of \( K, \gamma \in E^*, \gamma \notin K^* \). Then
\[ \int_{x \in E \times \text{GL}(2, E)} (x^{-1} \gamma x) \, dx = 0, \quad \text{if} \quad v(\det \gamma) \neq n. \]
Otherwise, the integral is equal to the residue degree of \( E \) over \( K \).

**Proof.**

The first assertion is proved by induction on \( n \) (we use the relation \( \chi^{r} \star \chi^{l} =
\chi^{r+l} + \chi^{r-l} \cdot \delta_{\lambda} \), where \( \delta_{\lambda} \) is the \( \delta \)-function with support in \( \mathbb{G}_{\lambda} \)). Now if we compute
\[ \text{Tr} \rho(\chi^r) \] using the formula in [2], we obtain the second assertion of the lemma. It is easy to see that there exists a \( z \)
which is fixed relative to \( E^* \), and that \( z \) is in the same \( E^* \)-orbit if and only if \( z \) is in the
\( E^* \times \text{GL}(2, K) \times \text{GL}(2, O) \) that contains \( E^* \). The integral obviously equals 0. If \( \Phi \in \Sigma \)
\[ (1 - q) (1 + (q + 1)^{-1}) \]
where \( m = n/2 - 1 + s \), we define \( K^*(1 + n! O_E) \); here \( O_E \) is the ring of integers of \( \text{GL}(2, K) \), \( E^* \times \text{GL}(2, K) \times \text{GL}(2, O) \) is the
Bruhat-Tits, two vertices on the same edge; two vertices on the same edge. Hence, if \( v(\det \gamma) = n, \) then \( a_i = q_m \) for some \( m \), and this is equal to the second assertion of the lemma.

**Proof of the Fundamental Lemma.**

\[ \text{Dom}(\Phi) \times \hat{\mathcal{D}}^* \text{ on } \lim_{U \to \text{GL}(A, A_p)} H^1(M_{U,v} \otimes \overline{A}/m_{v}, Q_i) \]
\[ \times \hat{\mathcal{D}}^* \text{ on } \sum_{\pi \in \mathcal{D}} \pi_D \otimes \pi_\pi \otimes X_{\pi}^r \]
\[ \times \hat{\mathcal{D}}^* \text{ in the space of functions, which is obtained from the function } g \mapsto g^{r-1}. \]
Let \( \Phi \) be a locally constant function.
Let \( \Phi \otimes \hat{\pi}_D \) be the function of
\[ \Phi \otimes \hat{\pi}_D(x, y) = \Phi(x, y). \]

We normalize the Haar measure arbitrarily fix Haar measures on
\[ \text{Dom}(\Phi) = \sum_{i=0}^{2} (-1)^i \text{Tr}[F^i h_i(\Phi)], \]
where \( h_i(\Phi) = \int \Phi(g) h_i(g) \, dg \), and
\[ \text{Tr}[F^i h_i^0(\Phi)] = \text{Tr}[F^i h^0(\Phi)] \]
\[ = \text{Tr}[F^i h^0(\Phi)] \]
\[ - \sum c_i |\beta_i| \]
where the numbers \( c_i, \beta_i, \alpha_i, \alpha_i' \) are given by the first assertion of the local lemma. Fix any \( \Phi \) and \( n \):
\[ \sum c_i |\beta_i| = 1, |\alpha_i| > 1. \]
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\[ \sum c_i a_i^n = \sum c_i \alpha_i \beta_i \]

are the numbers \( c_i, \beta_i, \beta_i, t_i, \) and \( t_i \) depend on \( \Phi \) but not on \( n \). It follows from the first assertion of the local lemma that it suffices to prove the following equations for any \( \Phi \) and \( n \):

\[ \sum c_i a_i^n = \sum c_i \alpha_i \beta_i, \quad \sum t_i \beta_i^n = \sum t_i \beta_i^n. \]

Since \( |\beta_i| = |\beta_i| = 1, |\alpha_i| > 1 \) and \( |\alpha_i| > 1 \), it suffices to verify that
\[ \sup_n |\text{Le}f_n(\Phi) - \text{Sel}_n(\Phi)| < \infty \]

for any \( \Phi \).

We compute \( \text{Sel}_n(\Phi) \) by applying the Selberg trace formula. We set

\[ \text{GL}(2, k_{\infty}) = \lim GL(2, k_{\infty})/V, \]

where \( V \) runs through the set of open subgroups of finite index in \( k_{\infty}^\times \). We choose the Haar measure on \( GL(2, k_{\infty}) \), which is compatible with the measure on \( D^* \) (i.e., so that the corresponding measures on \( GL(2, k_{\infty})/k_{\infty}^* \) and \( D^*/k_{\infty}^* \) are compatible in the sense of [2], §15). We let \( \vartheta_0 \) denote the space of automorphic cusp forms on \( GL(2, k_{\infty}) \), and we let \( \vartheta_{ch} \) denote the space of \( SL(2, \mathfrak{H}) \)-invariant automorphic forms; we set \( \vartheta_+ = \vartheta_0 \oplus \vartheta_{ch} \). The trace formula allows us to compute the trace of \( \psi \) on \( \vartheta_+ \) for any locally constant function \( \psi \) of compact support on \( \text{GL}(2, k_{\infty}) \times \text{GL}(2, \mathfrak{H}) \).

To compute \( \text{Sel}_n(\Phi) \), we choose the function \( \psi \) as follows. For any cuspidal representation \( \rho \) of \( \text{GL}(2, k_{\infty}) \) there exists a locally constant function \( \xi_\rho \) of compact support on \( \text{GL}(2, k_{\infty}) \) such that \( \xi_\rho \) acts trivially on all representations of \( \text{GL}(2, k_{\infty}) \) except for \( \rho \), and \( \text{Tr}(\xi_\rho) = 1 \) (see [2], §16). Let \( \rho \) be a special representation; this means that \( \rho \) occurs in a reducible representation induced from a one-dimensional representation of the Borel subgroup; in addition to \( \rho \), this representation contains a one-dimensional representation \( \xi_{\rho'} \); there exists a locally constant function \( \xi_{\rho'} \) of compact support on \( \text{GL}(2, k_{\infty}) \) such that the trace of \( \xi_{\rho'} \) on \( \rho \) is 1, the trace of \( \xi_{\rho} \) on \( \xi_{\rho} \) is \(-1\), and the trace is 0 on all other irreducible representations of \( \text{GL}(2, k_{\infty}) \) (see [2], §16). The function \( \psi \) is defined as follows: for any \( x_{\infty} \in \text{GL}(2, k_{\infty}) \), \( x_{\infty} \in \text{GL}(2, \mathfrak{H}) \), \( x_{\infty} \in \text{GL}(2, k_{\infty}) \) we set

\[ \psi(x_{\infty}, x_{\infty} \chi, x_{\infty}) = -\int \sum_{\rho \in \vartheta^*} \xi_{\rho}(x_{\infty}) \Phi(y, x_{\infty}) \text{Tr} \rho_D(y) dy; \]

where \( \rho \) runs through the set of cuspidal and special representations of \( \text{GL}(2, k_{\infty}) \), and \( \rho_D \) is the representation of \( D^* \) corresponding to \( \rho \) (in the sense of [2]). Then the trace of \( \psi \) on \( \vartheta_+ \) equals \( \text{Sel}_n(\Phi) \).

After performing transformations similar to those in §16 of [2], we find that the contribution to \( \text{Sel}_n(\Phi) \) from the first term in the trace formula (using the numbering of [2]) equals

\[ -c \sum_{\gamma \in \mathfrak{H}} \hat{\chi}_{\rho}(\gamma) \Phi(\gamma), \]

where \( c \) is the measure of \( \text{GL}(2, k) \setminus \text{GL}(2, k_{\infty}) \times \text{GL}(2, \mathfrak{H}) \). The contribution from the second and third terms of the trace formula equals

\[ \sum_{E} \frac{1}{|\text{Aut}_E|} \sum_{\gamma \in \mathfrak{E}} \int \Phi \otimes \hat{\chi}_{\rho}(\gamma^{-1} \gamma) dx, \]

where \( E \) runs through the set of quadratic extensions of \( k \) which split at \( \infty \). As explained in [2], the sixth and seventh terms in the trace formula for a function \( \psi \) of our type vanish, and the first part of the fifth term also vanishes. The fourth and eighth terms and the remaining part of the fifth term are sums of certain expressions, in which the summation is taken over the places of \( k \); as explained in [2], only the terms corresponding to \( \infty \) can be nonzero. It follows from the second assertion of the local lemma that, if \( n \) is greater than some number depending on \( \Phi \), then the summands corresponding to \( \infty \) in the fourth and fifth terms of the trace formula also vanish. Finally, the summand corresponding to \( \infty \) in the eighth term is bounded in absolute value by a constant depending on \( \Phi \) (in order to see this, one must compute the corresponding integral using residues). Thus,

\[ \text{Sel}_n(\Phi) + c \sum_{E} \frac{1}{|\text{Aut}_E|} \sum_{\gamma \in \mathfrak{E}} \int \Phi \otimes \hat{\chi}_{\rho}(\gamma^{-1} \gamma) dx, \]

is bounded by a constant depending on \( \Phi \).

To compute \( \text{Le}f_n(\Phi) \), we apply the same contribution to \( \text{Le}f_n(\Phi) \) from the \( \infty \) term as we have

\[ \sup_n |\text{Le}f_n(\Phi)|. \]

We shall prove that \( \text{Le}f_n(\Phi) \) is equal to a constant in terms of the trace formula; this follows from

\[ \sup_n |\text{Le}f_n(\Phi)|. \]

We consider \( \text{Le}f_n(\Phi) \) using Proposition 2.4. There are two sorts: 1) orbits corresponding to the regular orbit, corresponding to the type \( \overline{\gamma} \) from an orbit of the first sort. The measure \( m \) on \( \text{Aut}_E \)-orbit in \( E^* \) which is not a 1.

We now consider the contribution to \( \text{Le}f_n(\Phi) \) in the obvious way as a sum over a quaternion division ring \( \Delta \) over \( k \) into two sorts: 1) the classes of elements giving a quadratic extension \( k \). The first sort, which correspond to a
finite index in \( k_\infty \). We choose the measure on \( D^* \) (i.e., so that the \( k_\infty^* \) are compatible in the sense of cuspidal representations on \( \text{GL}(2, k_\infty) \times \text{GL}(2, k_\infty) \)).

Following Proposition 2.2, in our case (when \( d = 2 \)), the orbits are of two sorts: 1) orbits corresponding to types \((E, w)\), where \( |E : k| = 2 \), the “supersingular” orbit, corresponding to the type \((k, v)\), and we first consider the contribution to \( \text{Lef}_n(\Phi) \) from an orbit of the first sort. The field \( E \) has two places over \( v \) and another place \( w \).

We normalize Haar measure on \( E_0^* \) and \( E_0^* \) so that the maximal compact subgroups have measure 1. We let \( \delta_w^m \) denote the function on \( E_0^* \) which equals 1 on elements with norm \( m \) and zero elsewhere; we define \( \delta_w^m \) in the same way. It is not hard to show that the contribution from the orbit corresponding to \((E, w)\) equals

\[
\frac{1}{2} \sum_{x \in E_0^*} \int_{x \in E_0^* \setminus \overline{\delta} \times \text{GL}(2, k_\infty)} \Phi \otimes \delta_w^m \otimes \delta_w^m (x^{-1} v x) dx.
\]

which is equal to

\[
\frac{1}{2} \sum_{x \in E_0^*} \int_{x \in E_0^* \setminus \overline{\delta} \times \text{GL}(2, k_\infty)} \Phi \otimes (\delta_w^m \otimes \delta_w^m + \delta_w^m \otimes \delta_w^m) (x^{-1} v x) dx.
\]

The second assertion of the local lemma shows that this expression equals

\[
\frac{1}{2} \sum_{x \in E_0^*} \int_{x \in E_0^* \setminus \overline{\delta} \times \text{GL}(2, k_\infty)} \Phi \otimes \delta_w^m (x^{-1} v x) dx.
\]

We now consider the contribution from the “supersingular” orbit. It is represented in the obvious way as a sum over conjugacy classes in the multiplicative group of the quaternion division \( \Delta \) over \( k \) which is ramified at \( v \) and \( \infty \). These classes are of two sorts: 1) the classes of elements not in the center (giving such a class is equivalent to giving a quadratic extension \( E \) of \( k \) which does not split over \( v \) and \( \infty \), and an \( \text{Aut}_k \text{-orbit in } E^* \) which is not in \( k^* \); 2) the elements of the center. The classes of the first sort, which correspond to a quadratic extension of \( E \), give
\[
\frac{1}{|\text{Aut}_E|} \sum_{\gamma \in \tilde{\mathbb{D}}^*} \int \Phi \otimes \tilde{\mathcal{D}}^*_{\mathbb{A}_E}(\mathbb{A}_E) d\gamma
\]

(the omitted computations are analogous to those for a nonsupersingular orbit, except that we use the third assertion of the local lemma instead of the second). The classes of the second sort give

\[
c'(q_s-1) \sum_{\gamma \in \mathbb{A}_E} \Phi(\gamma),
\]

where \(c'\) is the measure of \(\Delta^s \setminus \tilde{\mathcal{D}}^* \times \text{GL}(2, \mathbb{A}_E) \times (\Delta \otimes k_o)^*\) (the Haar measure on \((\Delta \otimes k_o)^*\) is normalized so as to be compatible with the measure on \(\text{GL}(2, k_o)\) in the sense of [2], §15). It remains to note that \(c' = c\) (this follows because the Tamagawa measures for \(\text{SL}(2)\) and for the group of quaternions with norm 1 coincide: it is well known that both measures equal 1). \(\square\)

We now refine Proposition 3, using the functional equation for compatible systems of \(l\)-adic representations.

**Theorem.** 1) Let \(\pi = \bigotimes \pi_v\) be an irreducible admissible representation of \(\text{GL}(2, \mathbb{A})\) over \(\mathbb{Q}\) in the space of cusp forms, where \(\pi\) is either cuspidal or special, and its restriction to the center is a character of finite order. Then there exist a subfield \(E \subset \mathbb{Q}\) of finite degree over \(\mathbb{Q}\) and an \([E : \mathbb{Q}]\)-dimensional abelian variety \(\Pi\) with an action of the ring of integers of \(E\), such that, for any non-Archimedean place \(\lambda\) of \(E\) not lying above \(p\) and for any place \(v\) of \(k\),

\[
L \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) = L_0(s, T_\lambda \otimes \omega_\lambda), \quad \varepsilon \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) = \varepsilon_0(s, T_\lambda \otimes \omega_\lambda);
\]

where \(l\) is the prime number divisible by \(\lambda\); \(\omega_v\) is any quasi-character of \(k_v^*\) (or, equivalently, of the Weil group of \(k_v\)).

2) Let \(\pi\) satisfy the conditions in 1), except for finite order of the restriction to the center. Then there exist a number field \(E\) and a compatible system of two-dimensional \(l\)-adic representations \(W_\lambda\) of the Weil group of \(k\) (\(\lambda\) runs through the set of non-Archimedean places of \(E\) not dividing \(p\)) such that

\[
L \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) = L_0(s, W_\lambda \otimes \omega_\lambda), \quad \varepsilon \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) = \varepsilon_0(s, W_\lambda \otimes \omega_\lambda)
\]

for any quasi-character \(\omega_v\) of the group \(k_v^*\).

3) Let \(\pi = \bigotimes \pi_v\) be an irreducible admissible representation of \(\text{GL}(2, \mathbb{A})\) over \(\mathbb{C}\) in the space of cusp forms, such that \(\pi_v\) is cuspidal or special. Then \(\pi\) satisfies the Peterson conjecture (i.e., if \(\pi_v\) lies in the principal series with quasi-characters \(\mu_v\) and \(\nu_v\), then \(\mu_v / \nu_v\) is a character).

**Proof.** Suppose that \(\pi\) satisfies the conditions of 1). Then there exists a locally constant measure \(P\) with compact support on \(\tilde{\mathcal{D}}^* \times \text{GL}(2, \mathbb{A}_E)\), with values in the ring of integers of a number field \(E\), such that the action of \(P\) on \(\hat{\pi}_E \otimes \hat{\pi}_f\) has a one-dimensional image, and, for any other representation \(\pi'\) satisfying the conditions in 1), the action of \(P\) on \(\hat{\pi'}_E \otimes \hat{\pi}_f\) is the zero action. Let \(U\) be an open subgroup of \(\tilde{\mathcal{D}}^* \times \text{GL}(2, \mathbb{A}_E)\) such that \(P\) is invariant relative to left and right translations. There is no guarantee that \(\hat{\pi}_E\) has a small \(L\)-function using a sufficiently large power of the curve over \(k\) which has smooth determinates an element \(\alpha \in L \otimes \mathbb{A}\) with \(g = [E : \mathbb{Q}]\). Fixing an \(L\)-module \(\mathcal{D}\) for the natural homomorphism \(L \to \text{End}(\mathcal{D})\), we let \(l\) denote \(L\).

Proposition 3 shows that dim \(\mathcal{D}\) is not a function of \(k\) for all places of \(k\) holds for all but finitely many places of \(k\), we compute that \(L(\frac{1}{2}, \pi_v \otimes \omega_v) = L(\frac{1}{2}, \pi_v \otimes \omega_v)\)

\[
L \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) = L \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right)
\]

It hence follows immediately that

\[
L_0(s, T_\lambda \otimes \omega_\lambda) = L(s - 1/2, \pi_v \otimes \omega_v)
\]

with the exception of the case when \(k_0\) is the sum of one-dimensional representations, ruled out by a theorem of Grothendieck. The theorem implies that \(\hat{\pi}_E\) is semisimple and the absolute value \(\varepsilon(\pi_v \otimes \omega_v)\) of the Frobenius automorphism of \(\text{Gal}(k_v^* / k_v)\) (this two-dimensional representation) is the tensor product of \(\mathbb{Q}_l\) with a character of finite order. The theorem is proved.

The same theorem of Grothendieck follows another. Thus we have proved the Theorem when \(\pi\) satisfies the conditions of 1) in an obvious way. \(\square\)

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is invariant relative to left and right shifts by elements of $U$. Since $k$ is not perfect, there is no guarantee that $\hat{\Delta}_E^0$ has a smooth compactification; however, after a change of basis using a sufficiently large power of the Frobenius morphism from $k$ to $k$, we obtain a curve over $k$ which has smooth compactification. We let $J$ denote its jacobian. $P$ determines an element $\alpha \in L \otimes \text{End} J$, where $L$ is the ring of integers of $E$. We set $g = [E : \mathbb{Q}]$. Fixing an $L$-module structure on $Z^g$, we obtain an action of $L$ on $J^g$. Using the natural homomorphism $L \otimes \text{End} J \hookrightarrow \text{End} J^g$, we can consider $\alpha$ as an endomorphism of $J^g$. We let $\Pi$ denote the image of $\alpha$.

Proposition 3 shows that $\dim \Pi = g$, and that the property which we have to verify for all places of $k$ holds for all but finitely many places. For any quasi-character $\omega$ of the idele class group of $k$, we compare Deligne’s functional equation for $W_{\lambda} \otimes \omega$ (see [4], Theorem 9.3) and the Jacquet-Langlands functional equation for $\pi \otimes \omega$ (see [2], Theorem 11.1), and we find that, for any place $v$ of $k$ and any quasi-character $\omega_v$ of the group $k_v^*$,

$$L \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) \frac{\epsilon \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right) L \left( 3/2 - s, \pi_v \otimes \omega_v^{-1} \right)}{\epsilon_v \left( s, T_{\lambda} \otimes \omega_v \right) L_v \left( 1 - s, \pi_v \otimes \omega_v^{-1} \right)} = L_\pi \left( s, T_{\lambda} \otimes \omega_v \right) L_v \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right).$$

It hence follows immediately that

$$L_v \left( s, T_{\lambda} \otimes \omega_v \right) = L \left( s - 1/2, \pi_v \otimes \omega_v \right), \quad \epsilon_v \left( s, T_{\lambda} \otimes \omega_v \right) = \epsilon \left( s - \frac{1}{2}, \pi_v \otimes \omega_v \right),$$

with the exception of the case when the representation of $\text{Gal}(k_v^*/k_v)$ in $T_{\lambda}$ is a direct sum of one-dimensional representations which differ by the norm $k_v^* \rightarrow \mathbb{Q}^*$. This case is ruled out by a theorem of Grothendieck (see [7], Theorem IX.4.3). In fact, Grothendieck’s theorem implies that either the representation of $\text{Gal}(k_v^*/k_v)$ in $T_{\lambda}$ is nonsimple and the absolute values of the eigenvalues of any element of $\text{Gal}(k_v^*/k_v)$ inducing the Frobenius automorphism on the residue field are equal to $q_v^{-1/2}$, or else this representation is the tensor product of a standard two-dimensional $l$-adic representation of $\text{Gal}(k_v^*/k_v)$ (this two-dimensional representation is a nontrivial extension of $\mathbb{Q}_l(-1)$ using $Q_l$) with a character of finite order of the group $\text{Gal}(k_v^*/k_v)$. The first assertion of the theorem is proved.

The same theorem of Grothendieck implies that all of the $W_{\lambda}$ are compatible with one another. Thus we have proved the second and third assertions of the theorem in the case where $\pi$ satisfies the conditions of the first assertion. The general case reduces to this one in an obvious way. \hfill $\square$

**BIBLIOGRAPHY**

IMBEDDING THEOREMS IN VARIOUS METRICS

UDC 517.5

ABSTRACT. Let $1 < p < \infty$, and denote by $\lambda_n \in [0, 1]$ a sequence of numbers with $\sum_{n=1}^{\infty} \lambda_n < \infty$. Denote by $E_p(\lambda)$ the error of the trigonometric approximation by trigonometric polynomials $P_N$ of degree $N$ with respect to the metric $L_p$.

In this paper the relation between $E_p(\lambda)$ and $\lambda_N$ is considered. Necessary and sufficient conditions for $\lim_{N \to \infty} E_p(\lambda) = 0$ are given.

Furthermore, it is proved that

$$\sum_{n=1}^{\infty} \lambda_n < \infty$$

is not only sufficient but is also necessary for $E_p(\lambda) = 0$.

The question of imbedding of functions in various metrics is considered.

Bibliography: 7 titles.

Let $1 < p < \infty$, and denote by $f \in L_p$ a measurable function for which

$$\|f\|_p < \infty.$$

The best approximation of a function $f$ in $L_p$ by polynomials $P_N$ of degree $N$ is denoted by $E_p(f)$. If $\lambda_n \in [0, 1]$ are such that $\sum_{n=1}^{\infty} \lambda_n < \infty$, then $E_p(\lambda) = O(\lambda_N)$.

In the present article the relations between various metrics are studied. Relations of Ul'ja and Stečkin [1]. Subsequently, Ul'ja introduced various metrics $H_p^\alpha$ developed by him, proving the theorem of Konjuškov and Stečkin.

THEOREM A. Suppose that $1 < p < \infty$ and

$$\|f\|_q < \infty.$$