Math 472 — Statistical Inference
Midterm Exam — March 8, 2013

Name: KEY

Instructions: You have 50 minutes to complete the exam. Graphing calculators, phones, computers, textbooks, and plagiarism are prohibited. You may use a 5" x 7" card for relevant formulas and notes (front and back). Read the instructions for each problem carefully, show all pertinent work, and explain where necessary. A solution with no work or explanation will be awarded very little credit. Good luck!

Please read and sign: I certify that I have read the above instructions.

Signature: ___________________________ Date: ____________

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**Problem 1.** Consider a random sample $Y_1, \ldots, Y_n$ from the pdf

$$f_Y(y; \theta) = 2y\theta^2, \quad 0 \leq y \leq \frac{1}{\theta},$$

where $\theta$ is an unknown parameter.

(1) Find the maximum likelihood estimator for $\theta$.

$$L(\theta) = \prod (2y_i\theta^2) = 2^n\theta^{2n} \prod y_i.$$  
Increasing function of $\theta \Rightarrow$ Take $\theta$ as large as possible.

Since $0 \leq y_1, \ldots, y_n \leq \frac{1}{\theta}$, have

$$\theta \leq \frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_n}$$

$$\Rightarrow \theta \leq \min \frac{1}{y_i} = \frac{1}{y_{\text{max}}}$$

$$\hat{\theta}_e = \frac{1}{y_{\text{max}}}$$

(2) Find the method of moments estimator for $\theta$.

**Exponential pdf**

Theoretical Mean: $E(Y) = \int_0^\infty y (2y\theta^2) dy = \frac{2\theta^2}{3} y^3 \bigg|_0^\infty = \frac{2}{3\theta}$

Sample Mean: $\bar{y}$

Set $\bar{y} = \frac{2}{3\theta} \Rightarrow \theta = \frac{2}{3\bar{y}}$

$$\hat{\theta}_e = \frac{2}{3\bar{y}}$$
Problem 2. Consider a random sample \( Y_1, \ldots, Y_n \) from the exponential pdf
\[
f_Y(y; \theta) = \frac{1}{\theta} e^{-y/\theta}, \quad y \geq 0,
\]
where \( \theta \) is an unknown parameter. Consider the following estimators:
\[
\hat{\theta}_1 = Y_1 \quad \hat{\theta}_2 = \bar{Y}.
\]

(1) Which of these estimators is unbiased? Explain.
\[
\bar{Y} \text{ exponential } \Rightarrow E(\bar{Y}) = \theta
\]
\[
E(\hat{\theta}_1) = E(Y_1) = E(\bar{Y}) = \theta \quad \text{unbiased}
\]
\[
E(\hat{\theta}_2) = E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \theta = \theta \quad \text{unbiased}
\]

(2) Which of these estimators is more efficient? Explain.
\[
\bar{Y} \text{ exponential } \Rightarrow \text{Var}(\bar{Y}) = \theta^2
\]
\[
\text{Var}(\hat{\theta}_1) = \text{Var}(Y_1) = \theta^2
\]
\[
\text{Var}(\hat{\theta}_2) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(Y_i), \text{ by independence}
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n} \theta^2 = \frac{1}{n} \theta^2 < \text{Var}(\hat{\theta}_1) \text{ if } n > 1.
\]
So \( \hat{\theta}_2 \) is more efficient when \( n > 1 \).
(3) Which of these estimators is sufficient? Explain.

\[ L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{y_i}{\theta}} = \left( \frac{1}{\theta} \right)^n \exp \left( -\frac{1}{\theta} \sum y_i \right) = \left( \frac{1}{\theta} \right)^n \exp \left( -\frac{n}{\theta} \bar{y} \right) = \left( \frac{1}{\theta} e^{-\frac{n}{\theta} \bar{y}} \right)^n \]

As this is a function of \( \bar{y} \) and \( \theta \) (and not of the individual \( y_1, \ldots, y_n \)), we conclude \( \hat{\theta}_2 = \bar{y} \) is sufficient.

Intuitively, it is clear that \( L(\theta) \) cannot be factored as

\[ L(\theta) = F(y_1, \ldots, y_n; \theta) \cdot G(y_1, \ldots, y_n), \]

where \( G \) is independent of \( \theta \). For a formal proof that \( \hat{\theta}_2 = \bar{y} \) is not sufficient, observe that

\[ \ln L(\theta) = \ln F(y_1, \ldots, y_n; \theta) + \ln G(y_1, \ldots, y_n) \]

\[ \Rightarrow \frac{\partial \ln L(\theta)}{\partial y_2} = \frac{\partial \ln G(y_1, \ldots, y_n)}{\partial y_2} \text{ is independent of } \theta. \]

But

\[ L(\theta) = \left( \frac{1}{\theta} \right)^n \exp \left( -\frac{1}{\theta} \sum y_i \right) \]

\[ \Rightarrow \ln L(\theta) = -n \ln \theta - \frac{1}{\theta} \sum y_i \]

\[ \Rightarrow \frac{\partial \ln L(\theta)}{\partial y_2} = -\frac{1}{\theta}, \quad \text{which is clearly not independent of } \theta. \]
Problem 3. Suppose that $n = 10$ samples $y_1, \ldots, y_{10}$ are taken from a normal distribution with variance $\sigma^2 = 1$ and unknown mean, and that they are used to construct the 95% confidence interval $(1.00, 2.24)$ for the mean $\mu$. Answer the following questions.

1. What can you infer from this confidence interval? Explain carefully.

   We can be reasonably certain that
   \[ \mu \in (1.00, 2.24) \]
   since 95% of intervals constructed in this way will contain the true mean.

2. What is the probability that the true mean $\mu$ lies in the interval $(1.00, 2.24)$?

   There is nothing random here. Either $\mu$ is in this interval, or it is not. (So the probability is 0 or 1, if you like.)

3. Suppose that the true mean is $\mu = 0$. Is this consistent with the confidence interval $(1.00, 2.24)$ for the mean?

   Yes, it is consistent since 5% of intervals constructed in this way will fail to contain the true mean. We simply conclude that $(1.00, 2.24)$ is one of the atypical intervals.
Problem 4. Suppose that \( n = 4 \) samples \( x_1, \ldots, x_4 \) are taken from a normal distribution with mean \( \mu \) and variance \( \sigma^2 \), and that
\[
\sum_{i=1}^{4} x_i = -3.10, \quad \sum_{i=1}^{4} x_i^2 = 16.18.
\]
In each of the following problems, construct an appropriate 95% confidence interval.

1. Construct a confidence interval for the standard deviation \( \sigma \).
\[
\tilde{s} = \frac{1}{n-1} \left( \sum x_i^2 - \frac{1}{n} (\sum x_i)^2 \right) \approx 4.58
\]
Confidence Interval = \( \left[ \sqrt{\frac{(n-1)s^2}{\chi^2_{0.975, 3}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{0.025, 3}}} \right] = [1.21, 7.99] \)

2. Construct a confidence interval for the mean.
Confidence Interval = \( \bar{x} \pm t_{0.025, 3} \frac{s}{\sqrt{n}} \) = \([-4.18, 2.63]\)
(We use \( t_3 \) as our sampling distribution since \( \sigma \) is unknown.)

3. Suppose that the standard deviation is known to be \( \sigma \approx 1.5 \). Construct a confidence interval for the mean \( \mu \).
Confidence Interval = \( \bar{x} \pm z_{0.025} \frac{\sigma}{\sqrt{n}} \) = \([-2.245, 0.695]\)
Problem 5. Suppose that \( n = 9 \) samples \( y_1, \ldots, y_9 \) are taken from a normal distribution with unknown mean \( \mu \) and variance \( \sigma^2 \), and that

\[
\sum_{i=1}^{9} y_i = 10.54, \quad \sum_{i=1}^{9} y_i^2 = 32.99.
\]

Construct an appropriate decision rule for each of the following hypothesis tests.

(1) Test \( H_0 : \mu = 0 \) against \( H_1 : \mu > 0 \) at the \( \alpha = 0.05 \) level of significance. What do you conclude?

As \( \sigma \) is unknown, we use a \( t \)-test.

\[
s^2 = \frac{1}{n-1} \left( \sum y_i^2 - \frac{1}{n} (\sum y_i)^2 \right) = 2.59
\]

Reject \( H_0 \) if \( t > t_{0.05, 8} = 1.86 \).

We have \( t = \frac{\bar{y} - \mu_0}{s/\sqrt{n}} = 2.19 \), so reject \( H_0 \).

(2) Compute a \( P \)-value for the hypothesis test in part (1). What do you conclude?

\[
P\text{-value} = P(T_8 \geq 2.19 \mid H_0 \text{ is true}) \approx 0.03
\]

This supports our decision to reject \( H_0 \) at \( \alpha = 0.05 \) level of significance, but we would fail to reject at \( \alpha = 0.01 \) level of significance.

(3) Test \( H_0 : \sigma = 1.0 \) against \( H_1 : \sigma \neq 1.0 \) at the \( \alpha = 0.05 \) level of significance. What do you conclude?

\[
\text{Reject } H_0 \text{ if } \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \geq \chi^2_{0.975, 8} \text{ or } \chi^2 < \chi^2_{0.025, 8} = 17.53
\]

We have \( \chi^2 = 20.74 \), so we reject \( H_0 \).
**Problem 6.** You find an ordinary-looking coin in a box of magic supplies. Let $p$ be the probability that a flip of the coin produces heads. You want to determine if it is fair ($p = \frac{1}{2}$) or not. [Hint: The binomial tables I have provided may be useful for this problem.]

(1) Write down an appropriate null hypothesis and alternate hypothesis for this situation.

$$H_0 : p = \frac{1}{2}$$

$$H_1 : p \neq \frac{1}{2}$$

(2) You decide to flip the coin $n = 10$ times and record the number of times heads is showing, say $k$. Complete the following decision rule to test $H_0$ versus $H_1$ at an approximate $\alpha = 0.1$ level of significance:

Reject $H_0$ if the number of heads satisfies … $k \leq 2$ or $k \geq 8$.

We choose this by looking at the table for a binomial pdf with $p = \frac{1}{2}$, $n = 10$. It shows

$$P(k \leq 2) = 0.001 + 0.010 + 0.044 = 0.055$$

$$P(k \geq 8) = 0.001 + 0.010 + 0.044 = 0.055$$

Total $= 0.110$
(3) What is the probability of a type I error in your decision rule?

It is

\[ P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(k < 2 \text{ or } k \geq 8 \mid p = \frac{1}{2}) \]

\[ = 0.110 \quad \text{(by computation in part (2))} \]

(4) What is the probability of a type II error in your decision rule if \( p \geq 3/4 \)?

\[ \beta = P(\text{type II error}) \]

\[ = P(\text{fail to reject } H_0 \mid H_1 \text{ is true}) \]

\[ = P(3 \leq k \leq 7 \mid p = \frac{3}{4}) \]

\[ \leq P(3 \leq k \leq 7 \mid p = \frac{3}{4}) \]

from table

\[ = 0.003 + 0.016 + 0.058 + 0.146 + 0.250 \]

\[ = 0.473 \]
(5) **Bonus.** Suppose that you want to keep the level of significance at about $\alpha = 0.1$ while increasing the power of your hypothesis test to be at least 0.8 when $p \geq 3/4$. Construct a new experiment that accomplishes this.

To improve the power, we increase our sample size $n$. Let's start by taking $n$ large enough to apply a normal approximation to the binomial distribution when $p = \frac{1}{2}$ or $p = \frac{3}{4}$. Recall this means we need

$$n > 9 \cdot \frac{p}{1-p} \quad \text{and} \quad n > 9 \cdot \frac{1-p}{p}$$

$$\Rightarrow n > \max(9, 27, 3) = 27$$

Let's see if $n = 27$ is large enough to accomplish our goal. Take $H_0: p = \frac{1}{2}$ versus $H_1: p \neq \frac{1}{2}$.

We reject $H_0$ if

$$z := \frac{\hat{p} - n \cdot p_0}{\sqrt{n p_0 (1-p_0)}} > Z_{0.05} \quad \text{or} \quad z < -Z_{0.05}.$$ 

Since $p_0 = \frac{1}{2}$, $n = 27$, and $Z_{0.05} = 1.64$, this means

**Reject $H_0$ if $\hat{p} \geq 18$ or $\hat{p} \leq 9$.**

By construction, $\alpha = P(\text{type I error}) \approx 0.1$. (By working directly with the binomial pdf, we find $\alpha = 0.122$.) The power of our test is approximately

$$1 - \beta = 1 - P(9 < \hat{p} < 18 | p \geq \frac{3}{4}) \approx 1 - P\left(9 - n \cdot \frac{3}{4} \frac{1}{\sqrt{n (\frac{3}{4}) (1-\frac{3}{4})}} \leq z \leq \sqrt{\frac{18-n(\frac{3}{4})}{n (\frac{3}{4}) (1-\frac{3}{4})}}\right)$$

$$= 1 - P(-1.59 \leq z \leq 1) = 1 - 0.159 = 0.841.$$ 

**(Working directly with the binomial distribution shows the power is 0.887)**