Please include a cover sheet that provides a complete sentence answer to each of the following three questions:

(a) In your opinion, what were the main ideas covered in the assignment?
(b) Were there any topics that you believe should have been covered, but were not?
(c) What problems did you have with the assignment, if any?

Write your solutions carefully and neatly, preferably in complete and grammatically correct sentences. Justify all of your arguments.

1. Question 6.4.11, page 378.
2. Question 6.4.12, page 378.
3. Question 7.4.9, page 399–400.
4. Question 7.4.11, page 400.
5. Question 7.4.20, page 405.
6. Question 7.5.8, page 417.
7. Question 7.5.9, page 417.
8. Question 7.5.12, pages 417–418. (You may assume the result in Question 7.3.6.)
9. A company produces a high powered lightbulb that burns out according to an exponential distribution with mean 120 days. Their research team has developed a new lightbulb that is a bit more expensive to produce, but which they believe is yielding longer lifespans. The following table has the burnout times from 20 test bulbs:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>112.6</td>
<td>209.3</td>
<td>238.8</td>
<td>153.1</td>
<td>67.73</td>
</tr>
<tr>
<td>137.5</td>
<td>20.34</td>
<td>150.3</td>
<td>155.4</td>
<td>24.79</td>
</tr>
<tr>
<td>35.18</td>
<td>95.92</td>
<td>572.6</td>
<td>60.27</td>
<td>226.9</td>
</tr>
<tr>
<td>78.48</td>
<td>621.7</td>
<td>36.68</td>
<td>141.1</td>
<td>96.10</td>
</tr>
</tbody>
</table>

Let $\theta$ be the mean lifespan for the new bulbs. Use the generalized likelihood ratio test from class to test the null hypothesis $H_0 : \theta = 120$ days against the alternate hypothesis $H_1 : \theta > 120$ days at the 0.05 level of significance. What do you conclude?
Solutions to Selected Problems.

Solution (7.5.12). (a) We showed in class that the random variable

\[ Y = \frac{(n-1)S^2}{\sigma^2} \]

is a chi-squared random variable with \( n - 1 \) degrees of freedom. Exercise 7.3.6 tells us that the random variable

\[ W = \frac{Y - (n-1)}{\sqrt{2(n-1)}} \]

is approximately normal with mean 0 and variance 1 when \( n \) is large. We solve for \( Y \), insert the above expression for \( Y \) into the resulting equation, and solve for \( \sigma^2 \):

\[ \Rightarrow Y = W \cdot \sqrt{2(n-1)} + (n-1) \]
\[ \Rightarrow \frac{(n-1)S^2}{\sigma^2} = W \cdot \sqrt{2(n-1)} + (n-1) \]
\[ \Rightarrow \sigma^2 = \frac{(n-1)S^2}{W \cdot \sqrt{2(n-1)} + (n-1)} = \frac{S^2}{1 + W \cdot \sqrt{2/(n-1)}}. \]

To get an approximate 100(1 - \( \alpha \))% confidence interval, we use the fact that \( W \sim N(0, 1) \) when \( n \) is large. Therefore

\[ P(-z_{\alpha/2} < W < z_{\alpha/2}) = 1 - \alpha. \]

Manipulating the inequality inside the probability in order to make it look like the above formula for \( \sigma^2 \), we find that

\[ P \left( \frac{S^2}{1 + z_{\alpha/2} \cdot \sqrt{2/(n-1)}} < \sigma^2 < \frac{S^2}{1 - z_{\alpha/2} \cdot \sqrt{2/(n-1)}} \right) = 1 - \alpha. \]

Note that we at least need to take \( n \) large enough that the denominator \( 1 - z_{\alpha/2} \cdot \sqrt{2/(n-1)} \) is positive, for otherwise these manipulations are invalid. It follows that an approximate 100(1 - \( \alpha \))% confidence interval for \( \sigma^2 \) is given by

\[ \left( \frac{s^2}{1 + z_{\alpha/2} \cdot \sqrt{2/(n-1)}}, \frac{s^2}{1 - z_{\alpha/2} \cdot \sqrt{2/(n-1)}} \right). \]

Similarly, an approximate 100(1 - \( \alpha \))% confidence interval for the standard deviation \( \sigma \) is given by

\[ \left( \frac{s}{\sqrt{1 + z_{\alpha/2} \cdot \sqrt{2/(n-1)}}, \frac{s}{\sqrt{1 - z_{\alpha/2} \cdot \sqrt{2/(n-1)}}} \right). \]

(b) To construct a 95% confidence interval for the standard deviation of potassium-argon ages based on Table 7.5.1, we substitute the values \( s^2 = 733.4 \) (calculated for us in Case Study 7.5.1), \( z_{0.025} = 1.96 \), and \( n = 19 \) into the formula we obtained in part (a). This gives (21.1 million years, 46.0 million years). This is certainly close to the confidence interval shown in the Case Study, but it is wider. This suggests that we would want \( n \) to be even larger in order to get a more accurate confidence interval.
Solution (Problem 9). The exponential random variable in question has pdf

\[ f_Y(y; \theta) = \frac{1}{\theta} e^{-y/\theta}, \quad y \geq 0. \]

Here \( \theta \) is the mean lifespan of one of the new lightbulbs. Write \( y_1, \ldots, y_{20} \) for the 20 sample burnout times given in the problem. The test statistic of interest to us is \( \chi^2 = \frac{2n \bar{y}}{\theta_0} \), and our decision rule (which is a generalized likelihood ratio test) states that

“Reject \( H_0 \) if \( \chi^2 \geq \chi^2_{1-\alpha,2n} \).”

(Note that we must modify the hypothesis test given in class since we are testing a 1-sided alternate hypothesis.) Since \( \alpha = 0.05 \) and \( n = 20 \), our decision rule becomes

“Reject \( H_0 \) if \( \chi^2 \geq 55.8 \).”

For the given data, we find that the sample mean is \( \bar{y} = 161.7 \), and the test statistic is

\[ \chi^2 = \frac{2n \bar{y}}{\theta_0} = 53.9. \]

According to our decision rule, we fail to reject the null hypothesis. The new light bulbs, while perhaps better, are not so much better that we should jump to start manufacturing them.

While it was not required for the problem, it is instructive to compute a P-value for this problem. In this case, it is the probability that a set of data yields a test statistic more extreme than the one we observed, given that the null hypothesis is true. That is,

\[ \text{P-value} = P(\chi^2 \geq 53.9 \mid \theta = 120) = 0.07. \]

So it is interesting to note that we would have rejected the null hypothesis at the \( \alpha = 0.1 \) level of significance. This suggests that we may want to tell the R&D people to make the bulbs just a bit better if they want to convince us to switch.