MATH 480: INDUCTION

For each problem, either prove or disprove the statement. If you disprove it, fix the statement (change it to one that is true) and prove the newly fixed version.

(1) \[ \sum_{i=1}^{n} (2i - 1) = n^2. \]

(2) \[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \]

(3) \[ \prod_{i=1}^{n} \left( 1 - \frac{1}{i^2} \right) = \frac{n+1}{2n}. \] Use your answer to find \[ \prod_{i=1}^{\infty} \left( 1 - \frac{1}{i^2} \right). \]

(4) Define the Fibonacci numbers by \( F_0 = 1, F_1 = 1, \) and \( F_n = F_{n-1} + F_{n-2}. \) Prove that \( F_{n+2}F_n - F_{n+1}^2 = (-1)^n. \)

(5) For every natural number \( n, \) show that \[ u_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \] is the \( n \)th Fibonacci number.

(6) Define a double-recursion as follows:
\[
\begin{align*}
x_1 &= y_1 = 1 \\
x_n &= x_{n-1}^2 + y_{n-1}^2 \\
y_n &= x_{n-1}y_{n-1}
\end{align*}
\]
Prove that \( \gcd(x_i, x_n) = 1 \) for \( 1 \leq i < n. \)

(7) For all integers \( n \geq 0, \) show that \( 2^n < 3^n. \)

(8) For all integers \( n \geq 0, \) \( 4n < 2^n. \)
(9) Prove that $7^{2n} - 48n - 1$ is divisible by 2304 for every natural number $n$.

(10) Find a formula for
$$1 + 2 + 3 + \cdots + n$$
and prove that it holds for all $n > 0$.

(11) Find a formula for
$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1)$$
and prove that it holds for all $n > 0$.

(12) Find a formula for
$$0 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n + 1)(n + 2)$$
and prove that it holds for all $n > 0$.

(13) If $x \geq -1$ and $n \geq 0$, then
$$(1 + x)^n \geq 1 + nx.$$  

(14) For any natural number $n$, prove that
$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1.$$  

(15) For any natural number $n$, prove that
$$2\left(\sqrt{n + 1} - 1\right) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$  

(16) For each $n \geq 1$ and each $x \in \mathbb{R}$ such that $\sin x \neq 0$, prove that
$$\cos x \cdot \cos 2x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x}.$$  

(17) **(Tricky!)** Consider the function
$$f(x) = \begin{cases} 
  e^{-1/x^2} & x > 0 \\
  0 & x = 0.
\end{cases}$$
Prove that $f(x)$ is infinitely differentiable at $x = 0$ and that $f^{(n)}(0) = 0$ for all $n \geq 0$. (This is an example of a function that does not equal its Taylor series ... it is not analytic.)