Exam 1 Review
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Exam 1 is from Sections 2.1 to 3.1. Here are the topics.

Limits
- Know how to evaluate a limit given the graph of a function.
- Know how to evaluate a limit when given a formula for the function. This includes checking for the type of limit and applying an appropriate technique.
  - Type \( \frac{0}{0} \): Try factoring or multiplying by the conjugate or using the special limit
    \[
    \lim_{x \to 0} \frac{\sin x}{x} = 1.
    \]
  - Type \( \frac{\infty}{\infty} \): Try dividing both numerator and denominator by the highest power in the denominator.
  - Type \( \frac{\text{bounded}}{\infty} \): This has a limit of zero by using the Sandwich theorem.
  - Type \( \frac{1^+}{0^+} \): This has a limit of \( \infty \). If negative signs are present, just count how many negatives are in the product or quotient.
- Know one-sided limits.
- Know how to handle a compound function that is defined by cases.

Continuity
- The definition is: \( f \) is continuous at input \( a \) if both \( f(a) \) and \( \lim_{x \to a} f(x) \) exist and are equal.
- Know the types of discontinuity: removable, jump, infinite, oscillating. The type of discontinuity is determined by the three values \( \lim_{x \to a^+} f(x) \), \( \lim_{x \to a^-} f(x) \), and \( f(a) \).

Derivatives
- Know how to calculate a derivative using the definition: \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \). That is, not from using derivative rules; know the difference! Know how to handle polynomials and fractions and square roots.
- Know how to sketch the graph of the derivative function given the graph of the original function; or at least how to pick from a multiple choice selection. This includes the con

Applications
- Tangent lines. The main idea is that given the curve \( y = f(x) \), the value of the derivative \( f'(a) \) evaluated at input \( x = a \) is the slope of the tangent line at the point \( (a, f(a)) \). A useful formula for the equation of a line with slope \( m \) through a point \( (x_0, y_0) \) is the point-slope formula: \( y - y_0 = m(x - x_0) \).
- Velocity. If \( s(t) \) is the position, where \( t \) is time, then \( s'(t) \) is velocity.
Practice Problems

1. Evaluate the following limits, showing all work and thinking, including the usage of the Sandwich Theorem if it applies. If the answer is \( \infty \) or \(-\infty\), say so.

   (a) \( \lim_{x \to 2} \frac{x-2}{3x^2 - 8x + 4} \)

   (b) \( \lim_{x \to 2} \frac{x-\pi}{x-2} \)

   (c) \( \lim_{x \to -1} \frac{1+x^2}{1-x^2} \)

   (d) \( \lim_{x \to 0} \frac{1+x^2}{x^{101}} \)

   (e) \( \lim_{x \to 0} \frac{1+x^2}{x^{3} - x^{2}} \)

   (f) \( \lim_{x \to \pi} \sqrt{x+2} - 1 \)

   (g) \( \lim_{x \to \infty} \frac{5x - 2}{1 - 3x} \)

   (h) \( \lim_{x \to \infty} \frac{1+2\sin x}{x} \)

   (i) \( \lim_{x \to \infty} \frac{3-x^{3}}{x^{2}} + 1 \)

   (j) \( \lim_{x \to \infty} \frac{1+x^2}{x^{3} - x^{2}} \)

   (k) \( \lim_{x \to \infty} \sin x \)

   (l) \( \lim_{x \to \pi/2} \frac{\sin x}{x} \)

2. Consider the following graph of a function \( y = f(x) \).

   (a) Discuss the limit behavior as \( x \) approaches 1, 2, 3, 4. (Give the two-sided limit and also the one-sided limits if they are different from the two-sided limit.)

   (b) Where is the function discontinuous, and what are the types of discontinuity?

3. Let \( g(x) = \begin{cases} 
3x+1 & \text{if } x < 0 \\
x^2 - 3x & \text{if } 0 \leq x < 2 \\
-x & \text{if } 2 \leq x < 3 \\
11 & \text{if } x = 3 \\
6 - x^2 & \text{if } 3 < x 
\end{cases} \)

   Where is \( g \) discontinuous and what are the types of discontinuity?

4. Using only the definition, find the derivative of \( f(x) = 13x^2 - 4x + 3 \).

5. Using only the definition, find the derivative of \( f(x) = \frac{1}{2-13x} \).

6. Using only the definition, find the derivative of \( f(x) = \sqrt{3x+1} \).

7. Consider the following graph of a function \( y = f(x) \).

   Sketch the graph of its derivative function.

8. Use the result of #6 to find an equation of the tangent line to the graph of \( y = \sqrt{3x+1} \) at the point \((1,2)\).

9. The position function of an object is \( s(t) = t^2 - 6t + 5 \). Suppose you are given that \( s'(t) = 2t - 6 \). Find when and where (if any) the velocity of the object is zero.
10. Let \[ f(x) = \begin{cases} 
3x + c & \text{if } x < 0 \\
x^2 + 5c + 2 & \text{if } 0 \leq x.
\end{cases} \]
For what value(s) of \( c \) is the function continuous everywhere?

11. Use the Intermediate Value Theorem to show that \( x^5 = x^2 + 5 \) has at least one solution.

12. Given the following graphs, which choice best matches the graph of its derivative?

The choices:

(a) \[ \begin{array}{c}
\end{array} \]
(b) \[ \begin{array}{c}
\end{array} \]
(c) \[ \begin{array}{c}
\end{array} \]

The choices:

(i) \[ \begin{array}{c}
\end{array} \]
(j) \[ \begin{array}{c}
\end{array} \]
(k) \[ \begin{array}{c}
\end{array} \]

Solutions to Practice Problems

1. (a) \( \frac{1}{4} \) \hspace{1cm} (b) \( -\infty \) \hspace{1cm} (c) \( \infty \)

(d) does not exist \hspace{1cm} (e) \( -\infty \) \hspace{1cm} (f) \( \frac{1}{2} \) \hspace{1cm} (g) \( -\frac{5}{3} \)

(h) 0 (Sandwich Theorem needed use inequality \( \frac{-1}{x} \leq \frac{1 + 2\sin x}{x} \leq \frac{3}{x} \))

(i) \( -\infty \) \hspace{1cm} (j) 0 \hspace{1cm} (k) does not exist \hspace{1cm} (l) \( \frac{2}{\pi} \)

2. (a) \( \lim_{x\to 1} f(x) = 1, \lim_{x\to 2} f(x) = 0, \lim_{x\to 3^+} f(x) = d.n.e. \text{ with } \lim_{x\to 3^-} f(x) = 0 \text{ and } \lim_{x\to 3} f(x) = \infty \), \( \lim_{x\to 4} f(x) = 1/2 \).
(b) The function is discontinuous at 1, 3, 4, with removable, infinite, and removable discontinuities, respectively.

3. The only candidates for discontinuities are at 0, 2, 3, the boundaries between the subdomains. There is a discontinuity at 0 (jump discontinuity). There is no discontinuity at 2. There is a removable discontinuity at 3.

4. \[ f'(x) = \lim_{h \to 0} \frac{13(x + h)^2 - 4(x + h) + 3 - (13x^2 - 4x + 3)}{h} = \cdots \text{(use technique of limits: expand and factor and cancel)} \cdots = 26x - 4. \]

5. \[ f'(x) = \lim_{h \to 0} \frac{\frac{2 - 13(x + h)}{2 - 13x} - \frac{4}{2 - 13x}}{h} = \cdots \text{(use common denominator and turn into a simple fraction, then expand and factor and cancel)} \cdots = \frac{13}{(2 - 13x)^2}. \]

6. \[ f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x + h)} + 1 - \sqrt{3x + 1}}{h} = \cdots \text{(multiply by the conjugate top and bottom and simplify)} \cdots = \frac{3}{2\sqrt{3x + 1}}. \]

7. \[ \text{The slope at } x = 1 \text{ is } y' = \frac{3}{2\sqrt{4}} = \frac{3}{4}. \text{ The tangent line is } y - 2 = \frac{3}{4}(x - 1). \]

8. The object has velocity zero when \( t = 3 \) and at position \( s(3) = -4 \).

10. \( c = -1/2 \)

11. Let \( f(x) = x^5 - x^2 - 5 \). By trial and error, find \( a \) and \( b \) such that \( f(a) \) and \( f(b) \) have opposite signs. Then use the Intermediate Value Theorem carefully.

12. (a) II  (b) III  (c) III.