Math 241, Fall 2014, Final Exam

Name and section number:

Solutions
by
D. Yuen

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- You may not use notes or electronic devices on the test.
- Please ask if anything seems confusing or ambiguous.
- Show all your work.
- Good luck!
1. Compute the following limits. Do not use L’Hospital’s rule. If the limit is either positive or negative infinity say which. Simplify your answers.

(a) (4 points) \( \lim_{{x \to 3}} \frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} \) Limit type \( \frac{0}{0} \). Do More Work

\[ \lim_{{x \to 3}} \frac{(x+1) - 4}{(x-3)(\sqrt{x+1} + 2)} = \lim_{{x \to 3}} \frac{x - 3}{(x-3)(\sqrt{x+1} + 2)} \]

\[ = \lim_{{x \to 3}} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{2 + 2} = \frac{1}{4} \]

(b) (4 points) \( \lim_{{x \to \infty}} \frac{4 - x^3 - x}{7(x + 1)^3} \cdot \frac{1}{x^3} \) Limit type \( \frac{-\infty}{\infty} \). Do more work

\[ \lim_{{x \to \infty}} \frac{4x^3 - 1 - \frac{1}{x^2}}{7(1 + \frac{1}{x})^3} = \frac{0 - 1 - 0}{7(1+0)^3} = \frac{-1}{7} \]

(c) (4 points) \( \lim_{{x \to -1^+}} \frac{x^3 + 1}{x^2 + 2x + 1} \) Limit type \( \frac{0}{0} \). Do more work

\[ \lim_{{x \to -1^+}} \frac{(x+1)(x^2-x+1)}{(x+1)^2} = \lim_{{x \to -1^+}} \frac{x^2-x+1}{x - 1} \]

\[ = \left[ \frac{\infty}{0^+} \right] \rightarrow \infty \]

(d) (4 points) \( \lim_{{x \to -1}} \frac{x^2 - 5x + 1}{x + 17} \) Limit type \( \frac{16 - 20 + 1}{21} \) No Issues

\[ = \frac{-3}{21} = \frac{-1}{7} \]
2. (8 points) Using the definition of the derivative, compute $f'(x)$ if $f(x) = x^2 - x$.

(Warning: you get no credit if you use the rules of differentiation).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2-x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x - h - x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 - h}{h}$$

$$= \lim_{h \to 0} (2x + h - 1)$$

$$= 2x + 0 - 1$$

$$= 2x - 1.$$
3. Differentiate the following functions. Do not simplify your answers.

(a) (4 points) \( f(x) = \sqrt[3]{x} \sec(x) \).

\[
f'(x) = \frac{1}{3} x^{-2/3} \sec(x) + \sqrt[3]{x} \sec(x) \tan(x)
\]

(b) (4 points) \( s(t) = \frac{t^3 - 3t}{7t + 4} \).

\[
s'(t) = \frac{(3t^3 - 3)(7t + 4) - (t^3 - 3t)7}{(7t + 4)^2}
\]

(c) (4 points) \( g(\theta) = \cos(\theta + \theta^{-1}) \).

\[
g'(\theta) = -\sin(\theta + \frac{1}{\theta}) \cdot (1 - \frac{1}{\theta^2})
\]

(d) (4 points) \( V(x) = \int_{0}^{x^3} \sqrt{1 + t^2} \, dt \).

\[
V'(x) = \sqrt{1 + (x^3)^2} \cdot 3x^2
\]
4. A (male) tortoise is walking along a straight line with his position at time $t$ given by

$$s(t) = t^2 - 2t + 1, \quad 0 \leq t \leq 3$$

(distance is measured in meters, and time is measured in minutes).

(a) (3 points) At what time(s) between zero and three minutes is he stopped?

$$S'(t) = 2t - 2 \quad \Rightarrow \quad 0$$

$$t = 1$$

Answer: stopped at time $t = 1$ minute

(b) (4 points) At what time(s) between zero and three minutes is his speed decreasing?

$$S''(t) = 2.$$  
positive acceleration always

velocity $S'(t)$  

[---] $+++$

0 1 3

acceleration $S''(t)$  

$+++$

speed is decreasing $0 < t < 1$.

(c) (4 points) What is the distance between the tortoise's position at time zero, and his position after three minutes?

Distance between positions is

$$s(3) - s(0) = 4 - 1 = 3.$$  

As opposed to distance travelled, which would be

$$|s(3) - s(1)| + |s(1) - s(0)| = 14 - 0 + 10 - 1 = 5$$

because of turning around at time $t = 1$.  

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5. The following picture is described by the equation $y^2 + xy = 4$.

(a) (6 points) Find the equation for the tangent line to this graph at the point (3, 1).

Implicit differentiation.

$$2yy' + 1y + xy' = 0$$

At $x = 3, y = 1$, we have

$$2y' + 1 + 3y' = 0$$
$$5y' + 1 = 0 \Rightarrow y' = -\frac{1}{5}.$$

Tangent line: $y - 1 = -\frac{1}{5}(x - 3)$.

(b) (2 points) Find the equation for the normal line to this graph at (3, 1).

Slope of normal line is 5. (negative reciprocal)

Line is $y - 1 = 5(x - 3)$.

(c) (2 points) On the graph above sketch the normal and tangent lines at the point (3, 1) (computed above).
6. (10 points) Martin stands still on the coast. There is a boat two miles straight north of his position. The boat starts moving east, and Martin measures how the angle between the following lines changes: the line pointing straight north from where he stands; and the line from where he stands to the boat.

When the angle Martin is measuring is $\pi/6$, it is changing at a rate of 3 radians/hour. How fast is the boat traveling at this point? (Simplify your answer).

Relation is $\frac{x}{2} = \tan \theta$

$x = 2 \tan \theta$

$$\frac{dx}{dt} = 2 \sec^2 \theta \frac{d\theta}{dt}$$

Evaluate at $\theta = \frac{\pi}{6}$

$$\frac{d\theta}{dt} = 3 \text{ rad/h}$$

$$\frac{dx}{dt} = 2 (\sec \frac{\pi}{6})^2 (3)$$

$$= 2 \left( \frac{2}{\sqrt{3}} \right)^2 3$$

$$= 8 \text{ mi/h}.$$
7. Consider the function \( f(x) = 3x^2 - x^3 \).

(a) (6 points) Find the absolute minimum and maximum of \( f \) on the interval \([0, 4]\), and where they occur.

\[
\text{Use E.V.T.: } f'(x) = 6x - 3x^2 = 0 \\
3x(2-x) = 0 \\
x = 0, 2 \\
x = 0 \Rightarrow f(0) = 0 \\
x = 2 \Rightarrow f(2) = 12 - 8 = 4 \quad \text{MAX} \\
x = 4 \Rightarrow f(4) = 48 - 64 = -16 \quad \text{MIN}
\]

(b) (4 points) Find all inflection points of the graph of \( f \) (just give the \( x \) value(s)).

\[
f''(x) = 6 - 6x = 0 \\
x = 1 \\
f''(x) \quad + + \\>
\\
\]

(c) (4 points) Find all critical points of \( f \) (just give the \( x \) value(s)), and say whether they are local minima, local maxima or neither.

\[
\text{From above, } x = 0, 2
\]

Conclusions

\[
\begin{array}{ccc}
\text{Conclusions } & \text{on } f & \text{Local} \\
\text{on } f & \text{MIN} & \text{MAX} \\
& \text{at } x = 0 & \text{at } x = 2 \\
\end{array}
\]
8. (10 points) On the coordinate system below, sketch the graph of a function \( f \) with the following properties. Make sure all the features listed are clear from your graph.

- \( f \) has \( x \) intercepts at \(-3\) and \(1\);
- \( f \) has a vertical asymptote at \( x = 2 \) (and no other vertical asymptotes);
- \( \lim_{x \to -\infty} f(x) = 1, \lim_{x \to +\infty} f(x) = 1 \);
- \( f \) is continuous on the intervals \((-\infty, 2)\) and \((2, \infty)\) (and not on any bigger intervals);
- \( f \) is differentiable on the intervals \((-\infty, -2)\), \((-2, 2)\) and \((2, \infty)\) (and not on any bigger intervals);
- \( f \) is decreasing on \((-\infty, -2)\) and \((2, \infty)\), and increasing on \((-2, 2)\);
- \( f \) is concave down on \((-\infty, -2)\) and \((-2, 1)\), and concave up on \((1, 2)\) and \((2, \infty)\).
9. (10 points) Emma wants to enclose a rectangular field with total area 200\(m^2\). Along one side of the field, she will use a pre-existing straight wall, but on the other three sides, she needs to buy fence.

If it costs $2 for each meter of fence, what is the least amount she can spend to enclose her field? (Simplify your answer.)

\[
\begin{align*}
\text{pre-existing wall} \\
\underline{\text{y}} & \quad \underline{\text{y}} \\
\end{align*}
\]

Constraint area is \(200 = xy \Rightarrow y = \frac{200}{x}\)

Objective is cost

\[
C = 2(x + 2y) = 2x + 4y
\]

\[
= 2x + 4\left(\frac{200}{x}\right) = 2x + \frac{800}{x}
\]

Domain is \(x > 0\).
10. Consider the function \( f(x) = x^2 - 2x \) as pictured below.

(a) (6 points) Compute a Riemann sum for this function that approximates the integral \( \int_1^3 f(x) \, dx \). Use four equal-width intervals for your Riemann sum, and use the right endpoint of each interval to determine the height of the corresponding rectangle. You do not have to simplify your answer.

The interval is: \( [1, \ 1 \ 1 \ 1 \ 1] \)

\( \Delta x = \frac{1}{2} \)

\( f(\frac{3}{2}) = -\frac{3}{4} \), \( f(2) = 0 \), \( f(\frac{5}{2}) = \frac{5}{4} \), \( f(3) = 3 \)

Riemann sum is

\[ \left( -\frac{3}{4} + 0 + \frac{5}{4} + 3 \right) \frac{1}{2} \]

(b) (2 points) Sketch the rectangles that correspond to part (a) on the graph above.

(c) (1 point) Does your solution to (a) overestimate or underestimate \( \int_1^3 f(x) \, dx \)?

Overestimate, the rectangles' tops are above the graph.
11. Compute the following integrals. Simplify your answers.

(a) (4 points) \( \int_{0}^{\pi/2} \sin(2\theta) \, d\theta \)
\[
= -\frac{1}{2} \cos(2\theta) \bigg|_{0}^{\pi/2} \\
= -\frac{1}{2} \cos(\pi) - \cos(0) = -\frac{1}{2}(-1-1) = 1
\]

(b) (4 points) \( \int \sqrt{3}x^3 - 7x^{-3} \, dx \)
\[
= \frac{3}{8} x^{8/3} - 7 x^{-2} + C = \frac{3}{8} x^{8/3} + \frac{7}{2} x^{-2} + C
\]

(c) (4 points) \( \int_{-1}^{2} \sqrt{1+x^2} \, dx \).  Sub \( u = 1 + x^2 \)  \( x = -1 \Rightarrow u = 2 \)  \( x = 2 \Rightarrow u = 5 \)
\[
= \int_{1}^{2} \sqrt{u} \cdot 2x \, dx \\
= 2 \int_{1}^{5} \sqrt{u} \cdot 1 \, du = \frac{2}{3} u^{3/2} \bigg|_{1}^{5} \\
= \frac{1}{3} \left( 5^{3/2} - 2^{3/2} \right) = \frac{1}{3} \left( 5\sqrt{5} - 2\sqrt{2} \right)
\]

(d) (4 points) \( \int \cos^7(\theta) \sin(\theta) \, d\theta \).  Sub \( u = \cos(\theta) \)
\[
= \int \sin(\theta) \, d\theta = -\cos(\theta) + C \\
= -\frac{u^8}{8} + C = -\frac{1}{8} \cos^8(\theta) + C
\]
12. A function \( f \) of a single variable \( x \) is defined on the interval \([0,5]\) and differentiable on \((0,5)\). The following picture shows the graph of the derivative \( f' \) of \( f \).

In the picture, the portion of the graph on the interval \((0,2)\) identifies with a quarter-circle of radius 2 and center \((0,1)\); the portions of the graph on the intervals \([2,3]\) and \([3,5]\) are line segments.

(a) (2 points) What is \( f''(2.5) \)?

\[
f''(2.5) = 1
\]

(b) (4 points) If \( f(0) = 0 \), what is \( f(3) \)?

\[
f(3) - f(0) = \int_0^3 f'(x)dx = \pi + 3 + \frac{1}{2} \quad \text{(add up areas)}
\]

Thus \( f(3) = f(0) + \pi + \frac{7}{2} = 0 + \pi + \frac{7}{2} = \pi + \frac{7}{2} \).

(c) (4 points) On which interval(s) is the graph of \( f \) concave up, and on which concave down?

\( f \) is concave up when \((f')' > 0\) which by the above is when \( 2 < x < 3 \).

\( f \) is concave down when \((f')' < 0\) which is \( 0 < x < 2 \) and \( 3 < x < 5 \).
13. (10 points) The following picture shows the region between the graphs of $y = 3 - x^2$ and $y = 1 + x^2$.

Find the volume of the shape obtained by revolving this region about the $x$ axis.

Use vertical rectangles (dx-problem)

Get washers.

$$V = \pi \int_{-1}^{1} \left((3-x^2)^2 - (1+x^2)^2\right) dx$$

$$= \pi \int_{-1}^{1} (9 - 6x^2 + x^4 - (1 + 2x^2 + x^4)) dx$$

$$= \pi \int_{-1}^{1} (8 - 8x^2) dx$$

$$= \pi \left[ (8x - \frac{8}{3}x^3) \right]_{-1}^{1}$$

$$= \pi \left( (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) \right) = \pi \frac{32}{3}$$