Math 241, Spring 2015, Final Exam

Name and section number:

Solutions,
by D. Yuen

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- You may not use notes or electronic devices on the test.
- Please ask if anything seems confusing or ambiguous.
- Show all your work.
- Good luck!
1. Compute the following limits. **Do not** use L'Hospital's rule. If the limit is either positive or negative infinity say which. Simplify your answers.

(a) (4 points) \( \lim_{x \to 0^-} \frac{|x + 1|}{2 + \sin^2 x} \)

Type \( \frac{1}{2} \to \frac{1}{2} \)

The function is continuous at \( x = 0 \);

so there is no issue.

(b) (4 points) \( \lim_{t \to -2} \frac{t^2 + 2}{t^2 + 4} \)

\[ = \lim_{t \to -2} \frac{t + 2}{(t - 2)(t + 2)} = \lim_{t \to -2} \frac{1}{t - 2} = -\frac{1}{4} \]

(c) (4 points) \( \lim_{s \to 0} f(s) \), where \( f(s) = \begin{cases} s(3 - \frac{5}{s}), & s \neq 0; \\ 0, & s = 0. \end{cases} \)

irrelevant for \( \lim_{s \to 0} \)

\[ = \lim_{s \to 0} s \left(3 - \frac{5}{s}\right) = \lim_{s \to 0} (3s - 5) = -5 \]

(d) (4 points) \( \lim_{x \to \infty} \frac{2x}{x + 7 \cos x} \)

Type \( \frac{\infty}{\infty} \) Try dividing numerator and denominator both by highest term in denominator which is \( x \)

Note \( \lim_{x \to \infty} \frac{\cos x}{x} = 0 \)

by sandwich theorem because \( -\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x} \)

and \( \lim_{x \to \infty} -\frac{1}{x} = 0 \), \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

\[ = \lim_{x \to \infty} \frac{2}{1 + 7 \cos x} = \frac{2}{1 + 0} = 2 \]
2. (8 points) Let \( r \) be the function \( r(x) = \sqrt{2x + 1} \). Using the definition of the derivative as a limit, show that \( r'(0) = 1 \).

(Warning: you get no credit for using the rules of differentiation).

\[
\begin{align*}
r'(0) &= \lim_{h \to 0} \frac{r(0+h)-r(0)}{h} = \lim_{h \to 0} \frac{r(h)-r(0)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{2h+1} - 1}{h} \cdot \frac{\sqrt{2h+1} + 1}{\sqrt{2h+1} + 1} \\
&= \lim_{h \to 0} \frac{(2h+1) - 1}{h (\sqrt{2h+1} + 1)} \\
&= \lim_{h \to 0} \frac{2h}{h (\sqrt{2h+1} + 1)} \\
&= \lim_{h \to 0} \frac{2}{\sqrt{2h+1} + 1} \\
&= \frac{2}{\sqrt{0+1} + 1} = \frac{2}{2} = 1
\end{align*}
\]
3. Differentiate the following functions. Do not simplify your answers.

(a) (4 points) \( y = 3x^5 - 2x^3 - 7x + \pi \).

\[
y' = 15x^4 - 6x^2 - 7
\]

(b) (4 points) \( z = \frac{2x \sin x}{5x + 3} \).

\[
\frac{dz}{dx} = \frac{(2x \sin x)' (5x + 3) - 2x \sin x (5x + 3)'}{(5x + 3)^2}
= \frac{(2 \sin x + 2x \cos x) 5x + 3 - 2x \sin x (5)}{(5x + 3)^2}
\]

(c) (4 points) \( w = \sqrt{x} + (3 + x^2)^7 \).

\[
\frac{dw}{dx} = \frac{1}{2} x^{-\frac{1}{2}} + 7 (3 + x^2)^6 (2x)
\]

(d) (4 points) \( R(t) = \int_0^{t+2} \cos(1 + x^2) \, dx \).

\[
R'(t) = \frac{d}{dt} \int_0^{t+2} \cos(1 + x^2) \, dx
= \cos(1 + (t+2)^2) \cdot (t+2)'
= \cos(1 + (t+2)^2) \cdot 1
\]
4. Here is the graph of the derivative of a function $f$ with domain $(-1, \infty)$.

\[ y = f'(x) \]

(a) (5 points) Which is larger, $f(2)$ or $f(4)$ (1 of 4 pts)? You must explain (4 of 5 pts).

\[
f(4) \quad \text{because } f \text{ is increasing from } 2 \text{ to } 4 \quad \text{because } f'(x) > 0 \text{ for } 2 < x < 4.
\]

(b) (5 points) At what value(s) of $x$ does $f$ have a local minimum (2 pts)? Explain (3 pts).

1st derivative test \[
\begin{array}{c|c|c|c|c|c}
\hline
x & - & 0 & + & + & 0 & - & + \\
\hline
f' & - & 0 & + & + & 0 & + & + \\
\hline
\end{array}
\]

Local min at $x = 0, 8$.

5. Let $f$ be the function $f(x) = (4 - 5x)^3$.

(a) (6 points) Find the equation of the tangent line to the graph $y = f(x)$ at the point $(1, -1)$.

\[
f(1) = (-1)^3 = -1
\]
\[
f'(x) = 3(4 - 5x)^2 (-5) = -15(4 - 5x)^2
\]
\[
f'(1) = -15(-1)^2 = -15.
\]

Tangent line is: \[ y = f(1) + f'(1)(x-1) \]
\[
\boxed{y = -1 - 15(x-1)}.
\]
6. A point moves along the curve \( y = x^2 + 2x - 2 \) in such a way that when it is at \((1,1)\) the 
x-coordinate is decreasing at a rate of 1 unit/second. At this time what is 
(a) (7 points) the rate of change of the \( y \)-coordinate of the point?

\[
\frac{dy}{dt} = (2x + 2) \frac{dx}{dt}
\]

Given at \( x=1 \), \( \frac{dx}{dt} = -1 \).

So \( \frac{dy}{dt} = (2\cdot1 + 2)(-1) = -4 \)

(b) (3 points) the rate of change of the distance \( d = \sqrt{x^2 + y^2} \) from the point to the origin?

\[
\frac{dd}{dt} = \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} (2x \frac{dx}{dt} + 2y \frac{dy}{dt})
\]

Plug in \( x=1, y=1, \frac{dx}{dt} = -1, \frac{dy}{dt} = -4 \) from part (a)

\[
\frac{dd}{dt} = \frac{1}{2} (1^2 + 1^2)^{-\frac{1}{2}} (2\cdot1(-1) + 2(1)(-4))
\]

\[
= \frac{1}{2} \frac{1}{\sqrt{2}} (-10) = \frac{-10}{2\sqrt{2}} = \frac{-5}{\sqrt{2}}
\]
7. Starting with \((27)^{1/3} = 3\),

(a) (6 points) show how differentials/linear approximation can be used to approximate \((29)^{1/3}\).

Sketch a picture illustrating your computation.

Use function \( f(x) = x^{1/3} \)

\[
\begin{align*}
\frac{\Delta y}{\Delta x} & \approx \frac{29^{1/3} - 27^{1/3}}{2} \\
29^{1/3} & \approx f(29) = f(27) + \Delta y \\
& \approx f(27) + dy \\
& = 3 + dy
\end{align*}
\]

Thus
\[
29^{1/3} \approx 3 + \frac{2}{27}
\]
8. Let \( f(x) = \frac{(x+1)(x+3)}{x^2+3} \). Then \( f'(x) = \frac{4(3-x^2)}{(x^2+3)^2} \) and \( f''(x) = \frac{8x(x^2-9)}{(x^2+3)^3} \). No need to check; you can trust me!

(a) (2 points) List all \( x \)-intercepts and \( y \)-intercepts.
\[
\begin{align*}
f(x) & = 0 \quad \text{when } x = -1, -3 \\
\text{(x-intercepts)}
\end{align*}
\]
\[
\begin{align*}
f(0) & = \frac{3}{3} = 1 \\
\text{(y-intercept)}
\end{align*}
\]

(b) (3 points) List all intervals where \( f \) increases.
\[
\begin{align*}
f'(x) & = 0 \quad \text{when } 3 - x^2 = 0 \\
& \quad \text{or } 3 = x^2 \\
& \quad \pm \sqrt{3} = x
\end{align*}
\]
Increasing on \([-\sqrt{3}, \sqrt{3}]\)

(c) (3 points) List all inflection points and intervals where the graph is concave down.
\[
\begin{align*}
f''(x) & = 0 \quad \text{when } 8x(x^2+9) = 0 \\
& \quad \text{or } x = 0, -3, 3
\end{align*}
\]
Concave down on \((-\infty, -3) \text{ and } (0, 3)\)

(d) (2 points) List any asymptotes (vertical or horizontal).
\[
\begin{align*}
\lim_{x \to \infty} \frac{(x+1)(x+3)}{x^2+3} & = \lim_{x \to \infty} \frac{\frac{1}{x^2}(1+\frac{1}{x})(1+\frac{3}{x})}{1+\frac{3}{x^2}} = \frac{1}{1+0} = 1
\end{align*}
\]
No vertical asymptotes

(c) (3 points) Sketch the graph \( y = f(x) \).
The problems on this page are multiple choice. Circle the letter of the correct answer.

9. (4 points) Consider the integral \( I = \int_0^1 \frac{1}{1 + x^3} \, dx \). Then
   \[ \frac{1}{1 + x^3} < 1 \quad \text{all} \quad x \neq 0 \]
   (a) \( I < 1 \).
   (b) \( I = 1 \).
   (c) \( I > 1 \).

10. (3 points) True or False? If \( f(x) \) is continuous at \( x = a \), then \( f(x) \) is differentiable at \( x = a \).
    (a) True.
    (b) False.

11. (3 points) True or False? Suppose that \( f \) and \( g \) are differentiable on \( (-\infty, \infty) \) and \( f'(x) = g'(x) \) for all \( x \). If \( f(0) > g(0) \), then \( f(x) > g(x) \) for all \( x \).
    (a) True.
    (b) False.
12. (8 points) Find the maximum perimeter of a rectangle that has its bottom two corners on the $x$-axis and its top two corners on the parabola $x^2 + y = 4$. (The perimeter of a rectangle is the sum of lengths of its four edges.)

The perimeter is

$$P = 4x + 2(4-x^2)$$

$$P = 4x + 8 - 2x^2$$

**domain**

$$0 \leq x \leq 2$$

$$\frac{dP}{dx} = 4 - 4x \quad \text{set} \quad \frac{dP}{dx} = 0$$

$$4 = 4x$$

$$1 = x \quad \text{critical point}$$

**1st derivative diagram**

$$\frac{dP}{dx} \begin{array}{ccc} + & + & - \\ x & 0 & 1 & 2 \end{array}$$

**Absolute max**

at $x = 1$,

which yields

$$P = 4 + 2(3) = 10$$
13. Consider the function \( f(x) = \frac{1}{x}, \ 1 \leq x \leq 5 \).

(a) (4 points) Compute a Riemann sum for this function that approximates the integral \( \int_1^5 f(x) \, dx \). Use four equal-width intervals for your Riemann sum, and use the right endpoint of each interval to determine the height of the corresponding rectangle. You do not have to simplify your answer.

\[
\begin{align*}
\eta &= 4 \\
\Delta x &= \frac{b-a}{\eta} = \frac{5-1}{4} = 1 \\
\text{Riemann sum} &= f(2) \Delta x + f(3) \Delta x + f(4) \Delta x + f(5) \Delta x \\
&= \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{5} \cdot 1
\end{align*}
\]

(b) (4 points) Sketch the graph \( y = f(x), \ 1 \leq x \leq 5 \), and the rectangles that correspond to the Riemann sum in part (a).

(c) (1 point) Does your solution to (a) overestimate or underestimate \( \int_1^5 f(x) \, dx \)?

\text{underestimate}
14. Compute the following integrals.

(a) (4 points) \( \int t^2(t - 1) \, dt \) = \( \int (t^3 - t^2) \, dt \) = \( \frac{1}{4} t^4 - \frac{1}{3} t^3 + C \)

(b) (4 points) \( \int_0^1 (2x + 1)^4 \, dx \).

Substitute \( u = 2x + 1 \), \( x = 0 \Rightarrow u = 1 \), \( x = 1 \Rightarrow u = 3 \).

\[
\begin{align*}
\int_1^3 u^4 \frac{1}{2} \, du &= \frac{1}{2} \int_1^3 u^4 \, du \\
&= \frac{1}{2} \left( \frac{1}{5} u^5 \right) \bigg|_1^3 \\
&= \frac{3^5}{10} - \frac{1}{5} = \frac{242}{10} = \frac{121}{5}
\end{align*}
\]

(c) (4 points) \( \int_0^1 \frac{d}{dx} \left[ \frac{x(x+2)}{x^4+5} \right] \, dx \) = \( \frac{x(x+2)}{x^4+5} \bigg|_0^1 \)

\[
\begin{align*}
&= \frac{1 \cdot 3}{6} - \frac{0 \cdot 2}{5} \\
&= \frac{3}{6} = \frac{1}{2}
\end{align*}
\]

15. (6 points) Find an exact formula for \( f(t) \), given that \( f''(t) = 3 - \cos t \), \( f'(0) = 5 \), and \( f(0) = -2 \).

\[
\begin{align*}
f''(t) &= 3t - \sin(t) + C \\
5 &= f'(0) = 0 - \sin(0) + C \\
&\Rightarrow 5 = C \\
f'(t) &= 3t - \sin(t) + 5 \\
f(t) &= \frac{3}{2} t^2 + \cos(t) + 5t + C_2 \\
-2 &= f(0) = 0 + \cos(0) + 0 + C_2 \\
&\Rightarrow -2 = 1 + C_2 \\
&\Rightarrow -3 = C_2 \\
f(t) &= \frac{3}{2} t^2 + \cos(t) + 5t - 3
\end{align*}
\]
16. (a) (5 points) Set-up (do not evaluate!) an integral that represents the volume of the solid generated by revolving about the y-axis the region bounded by the curves \(x = 1\), \(y = 5\), and \(y = x^2 + 1\), \(1 \leq x \leq 2\). You may use any method.

Choose x-problem

Get shells

\[
V = 2\pi \int_1^2 (x - 0)(5 - (x^2 + 1)) \, dx
\]

\[
= 2\pi \int_1^2 x(4 - x^2) \, dx
\]

(b) (5 points) Set-up an integral (do not evaluate!) if the solid is now obtained by revolving the above region about the x-axis.

Choose x-problem

Get washers

\[
V = \pi \int_1^2 \left[ (5 - 0) - (x^2 + 1 - 0) \right] \, dx
\]

\[
= \pi \int_1^2 (5^2 - (x^2 + 1)^2) \, dx
\]