Exam 1 is on Sections 7.1 to 8.1. Here are the topics.

Inverse functions
- Know how to solve for the inverse function
- Know how to sketch the inverse function given a sketch of the original function
- Know how to find the derivative of the inverse function given the original function, namely
  \[(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.
  \]

Exponential and logarithm functions
- Definition: \(y = \log_a x\) means \(a^y = x\)
- Derivatives:
  \[
  \frac{d}{dx} e^x = e^x \quad \frac{d}{dx} \ln x = \frac{1}{x}
  \]
  \[
  \frac{d}{dx} a^x = a^x \ln a \quad \frac{d}{dx} \log_a x = \frac{1}{x \ln a}
  \]
- Technique of Logarithmic Differentiation.
- Domain of \(\ln\) is \((0, \infty)\)
- Limits:
  \[
  e^{+\infty} \to +\infty \quad e^{-\infty} \to 0 \quad \ln(+\infty) \to +\infty \quad \ln(0^+) \to -\infty
  \]
- Integrals:
  \[
  \int e^x \, dx = e^x + C \quad \int \frac{1}{x} \, dx = \ln |x| + C
  \]

Separable Differential Equations
- Attempt to separate the two variables to opposite sides of the equation, and integrate, and remember the \(+ C\) immediately when you integrate.
- Most problems ask to solve for \(y\).
- If it is an initial value problem, solve for \(C\) using the initial condition.

Exponential Growth & Decay (the modeling equation \(\frac{dy}{dt} = ky\))
- This model applies to where the rate of change of some quantity is proportional to its size.
- The solution is \(y = Ae^{kt}\), where \(A\) is the size at \(t = 0\).
- The method is usually to solve for the two unknown constants \(A, k\) using two data points.
- If given a half-life \(t_1\), then a data point is \(\frac{1}{2} A = Ae^{kt_1}\).

Inverse trigonometric functions
- Domains and ranges:
  \[
  \begin{align*}
  \text{arcsin:} & \quad \text{Domain} = [-1,1], \quad -\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}. \\
  \text{arccos:} & \quad \text{Domain} = [-1,1], \quad 0 \leq \arcsin x \leq \pi. \\
  \text{arctan:} & \quad \text{Domain} = (-\infty, \infty), \quad -\frac{\pi}{2} < \arctan x < \frac{\pi}{2}.
  \end{align*}
  \]
- Useful: \(\arcsin(-x) = -\arcsin(x)\), \(\arccos(-x) = \pi - \arccos(x)\), \(\arctan(-x) = -\arctan(x)\).
- Limits:
  \[
  \arctan(+\infty) \to \frac{\pi}{2}, \quad \arctan(-\infty) \to -\frac{\pi}{2}
  \]
- Derivatives:
\[
\begin{align*}
\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx} \arctan x &= \frac{1}{1+x^2}
\end{align*}
\]

Limits and L’Hôpital’s Rule (\( \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} \) for types \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \))

- L’Hôpital’s Rule applies ONLY to limits of type \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \). So always check the type first.
- Indeterminate types require more work. Determinate types have an immediate answer.
- For type \( 0 \cdot \infty \), put one of the factors into the denominator by inverting it. That is,
  \[ AB = \frac{A}{1/B} = \frac{B}{1/A}, \]
  thereby getting a type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).
- For types \( 0^0 \), \( ^0 \infty \), \( 1^\infty \), set the limit to \( L \) and take the logarithm of both sides, yielding \( \ln L \) as a limit of type \( 0 \cdot \infty \) or \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \), which can be attacked using the corresponding methods.
  Don’t forget to do \( L = e^{\ln L} \) as the final answer.
- For the type \( \infty - \infty \), try to combine the terms by factoring, common denominator, logarithm identities, etc.
- Remember to simplify as much as possible before resorting to L’Hôpital’s Rule again.
- Not all limits are best done, or doable, by L’Hôpital’s Rule.

Integration by Parts (\( \int u dv = uv - \int v du \))

- Choose \( dv \) to be something that is easily integrated, such as \( e^{ax} \), \( \sin ax \).
- Choose \( u \) to be something that simplifies when differentiated, such as \( \ln x \), \( \arcsin x \).
- After the above two reasons are used, then \( x^n \) can be chosen to be either \( u \) or \( dv \) as needed.
- The integral \( \int v du \) must be no worse than \( \int u dv \).

Typical integration by parts problems are:

- \( \int x^n f(x) dx \) where \( f(x) \) is easily integrated, such as \( e^{ax} \), \( \sin ax \), \( \cos ax \).
- \( \int x^n g(x) dx \) where \( g(x) \) simplifies when differentiated, such as \( \ln x \), \( \arcsin x \), \( \arctan x \).
- \( \int g(x) dx \) where \( g(x) \) simplifies when differentiated, such as \( \ln x \), \( \arcsin x \), \( \arctan x \).
- \( \int f(x) g(x) dx \) where \( f(x), g(x) \) are from the group \( e^{ax}, \sin ax, \cos ax \); in this case doing integration by parts twice consistently gets back to the original integral, which you can solve for (the “Back to self” method).

Basic integrals:

\[
\begin{align*}
\int x^n dx &= \frac{1}{n+1} x^{n+1} + C \quad \text{for } n \neq -1 \\
\int e^x dx &= e^x + C \\
\int e^{ax} dx &= \frac{1}{a} e^{ax} + C \\
\int \sin x \ dx &= -\cos x + C \\
\int \tan x \ dx &= \ln |\sec x| + C \\
\int \cos x \ dx &= \sin x + C \\
\int \cot x \ dx &= -\ln |\csc x| + C \\
\int \frac{1}{\sqrt{a^2 - x^2}} \ dx &= \arcsin \frac{x}{a} + C \\
\int \frac{1}{a^2 + x^2} \ dx &= \frac{1}{a} \arctan \frac{x}{a} + C \\
\int \frac{1}{ax + b} \ dx &= \frac{1}{a} \ln |ax + b| + C \\
\int \frac{x}{a^2 + x^2} \ dx &= \frac{1}{2} \ln(a^2 + x^2) + C
\end{align*}
\]
Practice Problems

1. Evaluate the following: (a) \( \log_7 49 \) (b) \( \log_{\sqrt{3}} 9 \) (c) \( \log_2 \frac{1}{\sqrt{2}} \) (d) \( \log_{1/2} 4 \) (e) \( \ln \sqrt{e} \)

2. Differentiate the following functions: (a) \( y = (e^x - x)^x \) (b) \( y = \frac{1}{\ln(\cos x)} \) (c) \( y = (x^2 - 1)^{-x} \) (d) \( y = \ln(x^2 + 1) - e^{\sin x} \)

3. Evaluate the following integrals: (a) \( \int_0^1 e^x \sqrt{2e^x - 1} \, dx \) (b) \( \int_0^1 \frac{x}{x^2 + 9} \, dx \) (c) \( \int_1^2 \frac{\ln x}{x} \, dx \) (d) \( \int e^{2x} \, dx \) (e) \( \int \frac{\sin(\ln x)}{x} \, dx \) (f) \( \int \frac{dx}{1 - x} \)

4. Solve the following differential equations. Express \( y \) as a function of \( x \).
   (a) \( y' = x + xy^2 \) (b) \( e^x + y' \cos x = 0 \) (c) \( y' = \frac{1 + x^2}{y} \), \( y(0) = -1 \)

5. Bacteria grow at a rate proportional to its size. The count in a bacteria colony that started at 1000 was 2500 after 3 hours. How long will it take for the population to reach 10000?

6. The half-life of Polonium-210 is 140 days. How much of a sample of 200 mg will be left after 1 year (365 days)?

7. Evaluate the following to exact value.
   (a) \( \arcsin \frac{1}{2} \) (b) \( \arctan \sqrt{3} \) (c) \( \cos(\arctan 7) \) (d) \( \arcsin(\cos \frac{\pi}{3}) \)

8. Find the derivatives of:
   (a) \( y = \arcsin(e^{-x}) \) (b) \( y = e^{\arctan x} \)

9. Evaluate the limits.
   (a) \( \lim_{x \to 0^+} \frac{\arcsin x}{x - 1} \) (b) \( \lim_{x \to \infty} \frac{e^{\arctan x}}{x} \) (c) \( \lim_{x \to 0^+} e^{\arctan(x/2)} \) (d) \( \lim_{x \to \infty} \arctan\left(\frac{e^{-x} + \sqrt{3}}{e^{-x} + 3}\right) \)
   (e) \( \lim_{x \to 0^+} \frac{e^{x^2} - 1}{x^2} \) (f) \( \lim_{x \to 0^+} \left| \ln x \right|^x \) (g) \( \lim_{x \to 0^+} \frac{\cos x}{x^2} \) (h) \( \lim_{x \to 0^+} \frac{\sin x}{\pi - x} \)
   (i) \( \lim_{x \to 0^+} \frac{x - \ln(1 + x)}{x^2} \) (j) \( \lim_{x \to \infty} \frac{x^{\frac{1}{x}}}{} \) (k) \( \lim_{x \to 2^+} \frac{x^3 - 8}{x^2 - 2} \) (l) \( \lim_{x \to 0^+} e^x \ln x \)
   (m) \( \lim_{x \to 0^+} (1 + 3x^2)^{1/x^2} \) (n) \( \lim_{x \to 0^+} \frac{\arctan x}{x} \) (o) \( \lim_{x \to 0^+} x^{\frac{1}{x^2}} \) (p) \( \lim_{x \to 1^+} \frac{1 + \cos(\pi x)}{(x - 1)^2} \)
   (q) \( \lim_{x \to 0^+} (\sin x)^{\sqrt{x}} \) (r) \( \lim_{x \to 0^+} (\ln(2x^2 + 1) - 2 \ln x) \)

10. Find the integrals.
    (a) \( \int (x + \cos 4x) \, dx \), \( \int e^{3x} \, dx \)
    (b) \( \int \frac{dx}{\cos^2 3x} \), \( \int \sec x \tan x \, dx \)
    (c) \( \int \frac{1}{x^2 + 9} \, dx \), \( \int \frac{x}{x^2 + 9} \, dx \)
    (d) \( \int \frac{1}{\sqrt{9 - x}} \, dx \)
    (e) \( \int \frac{1}{\sqrt{9 - x^2}} \, dx \)
(f) \( \int x \ln x \, dx \), \( \int \sqrt{x} \ln x \, dx \), \( \int \frac{\ln x}{x} \, dx \), \( \int \frac{\ln x}{\sqrt{x}} \, dx \)

(g) \( \int xe^x \, dx \), \( \int xe^{3x} \, dx \)

(h) \( \int x \sin x \, dx \), \( \int x \sin(3x) \, dx \)

(i) \( \int \sin(x) \cos(x) e^{\sin x} \, dx \)

(j) \( \int \sin(x) e^{3x} \, dx \)

11. Find a formula for the inverse function of the function \( f(x) = e^{3x+5} \).

12. For the function \( f(x) = e^{3x} + 4x + 1 \), note that \( f(0) = 2 \). Find \( (f^{-1})'(2) \).

13. For the following two graphs, choose the correct graph of its inverse function from the four choices of (I), (II), (III), (IV).

(a) 

(b) 

(I) 

(II) 

(III) 

(IV) 

Solutions to Practice Problems

1. (a) 2  (b) 4  (c) \( \frac{-1}{2} \)  (d) -2  (e) \( \frac{1}{2} \)

2. (a) \( \frac{dy}{dx} = \pi (e^{x^2} - x)^{x-1} (2xe^{x^2} - 1) \)  (b) \( \frac{dy}{dx} = \frac{\sin x}{(\ln x)^2 \cos x} \)

(c) \( \frac{dy}{dx} = (x^2 - 1)^{-\frac{1}{2}} \left( -\ln(x^2 - 1) - \frac{2x^2}{x^2 - 1} \right) \)  (d) \( \frac{dy}{dx} = \frac{2x}{x^2 + 1} - e^{\sin x} \cos x \)

3. (a) Substitute \( u = 2e^x - 1 \). Get \( \frac{1}{2}(2e^a - 1)^{3/2} - 1 \)

(b) Substitute \( u = x^2 + 9 \). Get \( \frac{1}{2}(\ln 10 - \ln 9) \)

(c) Substitute \( u = \ln x \). Get \( \frac{1}{2}(\ln 2)^2 \)

(d) Substitute \( u = 7x \). Get \( \frac{1}{7} e^{3x} + C \)

(e) Substitute \( u = \ln x \). Get \( -\cos(\ln x) + C \)

(f) Substitute \( u = 1-x \). Get \( -\ln |1-x| + C \)

4. (a) \( y = \tan\left(\frac{1}{2}x^2 + C\right) \)  (b) \( y = -\ln|\sec x + \tan x| + C \)  (c) \( y = -\sqrt{2x + \frac{2}{3}x^3 + 1} \)
5. \( \frac{3\ln 10}{\ln 2.5} \) hours (\( C = 1000, k = \frac{\ln 2.5}{3} \))

6. \( 200e^{\frac{\ln 0.5}{140}} \) mg (\( C = 200, k = \frac{\ln 0.5}{140} \))

7. (a) \( -\frac{\pi}{6} \) (b) \( \frac{\pi}{3} \) (c) \( \frac{1}{\sqrt{50}} \) (use method of the triangle) (d) \( \frac{\pi}{6} 

8. (a) \( \frac{dy}{dx} = -\frac{e^{-x}}{\sqrt{1 - e^{-2x}}} \) (b) \( \frac{dy}{dx} = e^{\arctan x} \frac{1}{1 + x^2} \)

9. (a) 0 (b) \( e^x \) (c) \( e^x \) (d) \( \pi/4 \) (e) 1 (f) 1 (g) \( \infty \) (h) -1 (i) \( 1/2 \) (j) \( e \) (k) 0 (l) -\( \infty 

(m) \( e^3 \) (n) 0 (o) \( e \) (p) \( \frac{\pi^2}{2} \) (q) 1 (r) \( \ln 2 \)

10. (a) \( \frac{1}{2} x^2 + \frac{1}{4} \sin 4x + C \), \( \frac{1}{4} e^{3x} + C \)
(b) \( \frac{1}{2} \tan 3x + C \), \( \sec x + C \),
(c) \( \frac{\pi}{3} \arctan \frac{3}{x} + C \), \( \frac{1}{2} \ln(x^2 + 9) + C \) (easy one-formula integrals)
(d) \( -2\sqrt{9 - x} + C \) [Substitute \( w = 9 - x \), get easy power rule]
(e) \( \arcsin \frac{x}{3} + C \) [easy one-formula integral]

(f) \( \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \) [Integration by parts: \( u = \ln x \), \( dv = x \, dx \)]

\[ \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C \] [Integration by parts: \( u = \ln x \), \( dv = \sqrt{x} \, dx \)]

\[ \frac{1}{2} (\ln x)^2 + C \] [Substitute \( u = \ln x \)]

\[ 2\sqrt{x} \ln x - 4\sqrt{x} + C \] [Integration by parts: \( u = \ln x \), \( dv = x^{-1/2} \, dx \)]

(g) \( xe^x - e^x + C \) [Integration by parts: \( u = x \), \( dv = e^x \, dx \)]

\[ \frac{1}{3} xe^{3x} - \frac{1}{9} e^{3x} + C \] [Integration by parts: \( u = x \), \( dv = e^{3x} \, dx \)]

(h) \( -x \cos x + \sin x + C \) [Integration by parts: \( u = x \), \( dv = \sin x \, dx \)]

\[ -\frac{1}{3} x \cos(3x) + \frac{1}{9} \sin(3x) + C \] [Integration by parts: \( u = x \), \( dv = \sin(3x) \, dx \)]

(i) \( e^{\sin x} \sin x - e^{\sin x} + C \) [Substitute \( w = \sin x \), get integral \( \int we^w \, dw \), then integration by parts see (n)]

(j) \( -\frac{1}{10} e^{3x} \cos(x) + \frac{3}{10} e^{3x} \sin(x) + C \) ["Back to self" method: Integration by parts: \( u = \sin(x) \), \( dv = e^{3x} \, dx \), and then again \( u = \cos(x) \), \( dv = e^{3x} \, dx \), and then solve for integral \( \int \sin(x) e^{3x} \, dx \)]

11. \( f^{-1}(x) = \frac{\ln(x) - 5}{3} \)

12. \( (f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{7} \)

13. (a) III (b) II