MOCK ANALYSIS QUALIFYING EXAM 1

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set $A$ by $m(A)$.

(1) Let $A, B, C, D, E$ be measurable subsets of $[0, 1]$. Suppose that almost every $x \in [0, 1]$ belongs to at least 4 of these subsets. Prove that at least one of the sets has measure of at least $4/5$.

(2) Suppose $f \in L^1(\mathbb{R})$ and there is $0 < \epsilon$ so that $|\int_A f(x) dx| \leq (m(A))^{1+\epsilon}$ for all measurable sets $A \subset \mathbb{R}$. Prove that $f = 0$ a.e.

(3) (a) Find a sequence of functions $f_n$ in $L^1([0, 1])$ converging a.e. to a limit $g$ so that $\|f_n\|_1 = 2$ for all $n$ and $\|g\|_1 = 1$.

(b) Prove that if $f_n$ and $g$ are in $L^1$ and satisfy the conditions of part (a), then

$$\lim_{n \to \infty} \int_0^1 |f_n(x) - g(x)| dx = 1.$$ 

(4) Suppose $f : [0, 1] \to \mathbb{C}$ is measurable and satisfies $0 < \int_0^1 |f(x)| dx < \infty$. Suppose $0 < \epsilon < 1$. Show that

$$\lim_{n \to \infty} n \int_0^1 \log \left(1 + \left(\frac{|f|}{n}\right)^\epsilon\right) dx = \infty.$$ 

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(5) Consider the “hat” function $\phi \in C(\mathbb{R})$, defined by

$$
\phi(x) = \begin{cases} 
1 - |x| & \text{if } x \in [-1, 1] \\
0 & \text{otherwise} 
\end{cases}
$$

(a) Show that if $g \in C^1(\mathbb{R})$ (i.e. $g$ is continuously differentiable), then $g * \phi$ is continuously differentiable.

(b) Find a function $F \in L_\infty(\mathbb{R})$ so that if $g \in C^1(\mathbb{R})$ then $\frac{d}{dx}(g * \phi) = g * F$.

(c) Show that if $f \in L_1(\mathbb{R})$ then $f * \phi \in C^1(\mathbb{R})$.

(6) Let $C$ denote the Cantor set: i.e., the set of all reals $0 \leq x \leq 1$ that can be expanded in base 3 using only digits 0 and 2. In other words, $x \in C$ if and only if $x = \sum_{j=1}^{\infty} c_j 3^{-j}$ with $c_j \in \{0, 2\}$ for each $j$.

The Cantor function $F : [0, 1] \to \mathbb{R}$ is a continuous function defined as follows: for $x = \sum_{j=1}^{\infty} c_j 3^{-j} \in C$, $F(x) = \sum_{j=1}^{\infty} c_j 2^{-j-1}$, while on each interval in $[0, 1] \setminus C$, $F$ is constant. Evaluate

$$
\int_0^1 F(x) \, dx.
$$