1. Find the domain and the derivative of the function \( f(x) = \frac{\ln x}{x - \ln x} \). On which interval is this function decreasing?

Answer: Domain=(0, \infty). \( f'(x) = \frac{1 - \frac{\ln x}{x - \ln x}}{(x - \ln x)^2} \). The function is decreasing when \( \ln x \geq 1 \) i.e \( x \geq e \).

2. Consider the series \( \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n} \)

(a) Is the series absolutely convergent? Explain your answer.

(b) Is the series convergent? Explain your answer. (Hint: you can use the result of the previous problem)

Answer: Not absolutely convergent since \( \frac{\ln n}{n - \ln n} \geq \frac{\ln n}{n} \) which diverges (limit comparison with \( \frac{1}{n} \)) but convergent (Alternating Series test).

3. Use the method of the integrating factor to find the solution to the differential equation

\[ y'(x) + y(x) \cos x = \cos x. \]

Answer: \( 1 + Ce^{-\sin x} \)

4. Find the solution to the initial value problem

\[
\begin{align*}
    &y'(x) = -10y(x) \\
    &y(0) = -2.
\end{align*}
\]

Answer: \( y(x) = -2e^{-10x} \)

5. Evaluate the integral

\[ \int_{-2}^{2} \frac{dx}{4 + 3x^2} \]

Answer: \( \frac{\pi}{3\sqrt{3}} \)

6. Evaluate the integral

\[ \int \tanh \left( \frac{x}{7} \right) dx \]

Answer: \( 7 \ln |\cosh \left( \frac{x}{7} \right)| + C \)

7. Evaluate the integral

\[ \int x^2 \ln x dx \]

Answer: Integration by part, \( \frac{x^3}{3} \ln x - \frac{x^3}{9} + C \)

8. Evaluate the integral

\[ \int \frac{x}{x^2 - 3x + 2} dx \]

Answer: Sum of partial fractions, \( 2 \ln |x - 2| - \ln |x - 1| + C \)

9. Evaluate the improper integral

\[ \int_{1}^{\infty} \frac{3x - 1}{4x^3 - x^2} dx \]

Answer: Sum of partial fractions, \( 1 + \ln \left( \frac{3}{4} \right) \)
10. Is the improper integral
\[ \int_{1}^{\infty} \frac{e^{-t}}{\sqrt{t}} \, dt \]
convergent or divergent? Explain your answer.
Answer: Direct comparison with \( e^{-t} \) which is convergent (direct integration). Convergent.

11. Find the limit of the sequence \( a_n = (1 - \frac{2}{n})^n (2)^{\frac{1}{n}} \). Justify your answer.
Answer: \( e^{-2} \)

12. Find the limit of the sequence \( a_n = \frac{\ln(n^2)}{n} \). Justify your answer.
Answer: l'Hopital Rule, 0

13. Does the series \( \sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + 1)}{2n^2 + n - 1} \) converge or diverge? Justify your answer.
Answer: Diverges (nth term test for divergence)

14. Consider the function \( f(x) = \frac{1}{1-2x} \).
(a) (5 points) Find the Taylor series at \( x = 0 \) for the function \( f(x) \).
(b) (5 points) For which values of \( x \) does this Taylor series converges?
Answer: \( \sum_{n=0}^{\infty} 2^n x^n \). Interval of convergence= \((-\frac{1}{2}, \frac{1}{2})\)

15. Find the Taylor polynomial of order 3 for the function \( f(x) = \cos x \) at \( a = \frac{\pi}{4} \).
Answer: \( P_3(x) = \sqrt{2} - \frac{\sqrt{2}}{2} (x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4} (x - \frac{\pi}{4})^2 + \frac{\sqrt{2}}{12} (x - \frac{\pi}{4})^3 \)

16. In parts (a) through (d), determine if the series converges absolutely, converges conditionally, or diverges.
(a) \( \sum_{n=1}^{\infty} \frac{2n + 1}{n!} \)
(b) \( \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 2} \)
(c) \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n + 1} \)
(d) \( \sum_{n=1}^{\infty} \left( \frac{3n^2 + 2}{5n^2 - 1} \right)^n \)
(e) \( \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n} + \frac{1}{n^2} \right) \)
(f) \( \lim_{x \to 0} \frac{\arctan(x)}{\tan(x)} \)

17. (a) Find the Taylor polynomial of order 4 generated by \( e^{\sin(x)} \) at \( x = 0 \).
(b) Find the Taylor polynomial of order 2 generated by \( e^x (1 + x)^{7/2} \) at \( x = 0 \).
18. (a) Evaluate \( \int \frac{5x + 2}{x^2 - x - 2} \, dx \).
(b) Evaluate \( \int_0^1 x^2 e^x \, dx \).
(c) Evaluate \( \int x \sin(1 - x) \, dx \).
(d) Evaluate \( \int e^x \cos(e^x) \, dx \).
(e) Evaluate \( \int_1^\infty \frac{1}{x^{3/2}} \, dx \).
(f) Evaluate \( \int_{-\infty}^\infty 4x^3 e^{-x^4} \, dx \).
(g) Determine if \( \int_1^\infty \frac{\sin^2(x)}{x^2} \, dx \) converges or diverges.

19. For each of the two following series, determine if it converges, and if so, determine its sum.
(a) \( \sum_{n=0}^{\infty} \left( \frac{1}{3^n} - \frac{5^n}{7^{n+1}} \right) \)
(b) \( \sum_{n=2}^{\infty} \frac{3}{n(n-1)} \)

20. For each of the following two power series, determine its radius of convergence and its interval of convergence.
(a) \( \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n} \)
(b) \( \sum_{n=1}^{\infty} \frac{x^n}{n^{2n}} \)

21. (a) Solve the initial value problem
\[
\begin{align*}
y' + \tan(x)y &= \sec(x) \quad (\text{for } -\pi/2 < x < \pi/2), \\
y(0) &= 1.
\end{align*}
\]
(b) Find the general solution to the differential equation \( y' + \frac{1}{x}y = \cos(x^2) \) (for \( x > 0 \)).
(c) Find the general solution to the non-linear differential equation \( y' - 2xy = -2xe^{-x^2}y^2 \).

22. (a) Find the general solution to the differential equation \( y'' - y' - 2y = 0 \).
(b) Find the general solution to the differential equation \( y'' - y' - 2y = 4e^x + 6e^{2x} \).
(c) Solve the initial value problem
\[
\begin{align*}
y'' - y' - 2y &= 0, \\
y(0) &= 1, \\
y'(0) &= 5.
\end{align*}
\]
23. (a) Using the Taylor series for \( \ln(1 + x) \) from the formula sheet (or by another method if you wish), find the Taylor polynomial of order 3 generated by \( \ln(1 + \sin(x)) \) at \( x = 0 \).

(b) Solve the initial value problem \[
\begin{align*}
(x^2 + 1)y' + xy &= x, \\
y(0) &= -1.
\end{align*}
\]

24. Integrate. Your answer will require integration skills learned in this class.

(a) \[ \int_{0}^{\infty} 3e^{-2x} \, dx \]

(b) \[ \int_{-3}^{3} -\frac{3 \cosh x}{2} \, dx \]

(c) \[ \int_{0}^{1} (x - 5) \sin x \, dx \]

(d) \[ \int_{0}^{\infty} \frac{d\theta}{\sqrt{4\theta^2 + 16\theta + 20}} \]

(e) \[ \int_{0}^{1} x^2 e^{-x} \, dx \]

(f) \[ \int_{0}^{1} \left[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right] \, dx \]

25. Recall the difference between a sequence and a series. A sequence is a list of numbers \( \{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \ldots\} \), whereas a series is a sum of a list of numbers \( \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots \).

(a) What does it mean for a series \( \sum_{n=0}^{\infty} a_n \) to converge? Give a definition or explanation. Note: a statement of a convergence test is not sufficient.

(b) Give an example of a convergent sequence \( \{a_n\}_{n=0}^{\infty} \) whose series \( \sum_{n=0}^{\infty} a_n \) diverges.

(c) Is it possible to have a series \( \sum_{n=0}^{\infty} a_n \) which converges but \( \lim_{n \to \infty} a_n \) diverges? Explain why or why not.

26. Recall that an alternating series is a series of the form \( \sum_{n=0}^{\infty} (-1)^n a_n \).

(a) How can you tell if an alternating series is convergent?

(b) What does it mean for a (alternating series) to be absolutely convergent? What is the connection between absolute convergence and convergence for series?

(c) Give an example of an alternating series \( \sum_{n=0}^{\infty} (-1)^n a_n \) which is convergent, but is not absolutely convergent.

27. Does the infinite series converge or diverge? Explain how you can tell.

(a) \[ \sum_{n=0}^{\infty} \frac{n}{n^2 + 1} \]

(b) \[ \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n! 2^n} \]
28. State three convergence tests for an infinite series $\sum a_n$, and list the assumptions and conclusions of those tests.

29. Find the Taylor series for the function centered at $x = 0$ (that is, the Maclaurin series). Write your answer in $\Sigma$-notation, $\sum a_n x^n$ (you may use the list of common series on the last page, if it helps).
   
   (a) $f(x) = x^2 - 1$
   (b) $g(x) = e^{-2x}$
   (c) $f(x) = \cos(2\pi x)$
   (d) $g(x) = \frac{\sin x - 1}{x}$
   (e) $h(x) = \cosh(2x)$
   (f) $j(x) = \frac{e^{-2x} - 1}{x}$

30. Find the sum of the infinite series $\sum_{n=0}^{\infty} \frac{\pi^2 n}{16^n (2n)!}$ (hint: think of it as Taylor/Maclaurin series).

31. Let $f(x) = \sin 2\pi x$. The third-order Taylor polynomial is $P_3(x) = 2\pi x - \frac{8\pi^3}{6} x^3$
   
   (a) What is the maximum possible error $E = |\sin 2\pi x - P_3(x)|$ when $|x| \leq 1$? Show your work.
   (b) What is the largest interval $[-x, x]$ on which the error $E \leq 0.1$? Show your work.

32. Recall that a second order linear homogeneous ODE is an equation of the form
   \[ ay'' + by' + cy = 0 \]
   
   (a) Prove the principle of superposition, that is: show that if $y_1(x)$ and $y_2(x)$ are solutions to $ay'' + by' + cy = 0$, then $y(x) = y_1(x) + y_2(x)$ is also a solution.
   (b) Suppose $y_1(x) = e^{-2x}$ and $y_2(x) = 5e^{-2x}$ are solutions to such an ODE. Do $y_1$ and $y_2$ form a set of fundamental solutions? How can you tell?
   (c) Explain (but don’t “prove”) why the principle of superposition does not apply to the non-homogeneous equation $ay'' + by' + cy = g(x)$, except in the special case where $g(x) = 0$.

33. Solve the initial value problem (IVP) $y' + \frac{1}{2x} y = \sqrt{x}$, $x > 0$, $y(1) = 2$.

34. Evaluate the following integrals:
   
   (a) $\int \frac{1}{(1 + x^2)^{\frac{3}{2}}} \, dx$
   (b) $\int x\sqrt{x} + 3 \, dx$
   (c) $\int 4x^2 + 2 \, dx$

35. For each of the following improper integrals, determine whether or not it converges. If it diverges, explain how you know. If it converges, evaluate it.
   
   (a) $\int_{1}^{\infty} \frac{\ln x}{x^2} \, dx$
36. Which of the following series converge? Give reasons for your answers.

(a) \[ \sum_{n=1}^{\infty} \frac{n+\sqrt{n}+1 - \sqrt{n}}{n} \]

(b) \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \]

(c) \[ \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}} \]

37. Consider the power series \[ \sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n \sqrt{n}}. \]

(a) What is its center and radius of convergence?

(b) Where does it converge absolutely and where does it converge conditionally? Show all your reasoning.

38. It's 2064. Civilization is in ruins and the human race is all but extinct. Mankind's survival hinges on your ability to perform the following calculation\(^1\): You must compute an approximation for \( \int_{0}^{1/2} e^{-x^2} \, dx \) that is accurate to within 0.01. Fortunately, your math professor walked you through exactly this problem in an exam once:

(a) Write down the Taylor series \( T(x) \) generated by \( f(x) = e^{-x^2} \) at 0. For what values of \( x \) does \( T(x) \) converge to \( f(x) \)?

(b) Next, obtain an infinite series that converges to \( \int_{0}^{1/2} e^{-x^2} \, dx \).

(c) How many terms of this series do we need to obtain an approximation that achieves the desired accuracy? Explain how you know. Then compute the approximation (you may leave your answer as a fraction).

39. Now you have to obtain an approximation for \( \sqrt{4.1} \) using the Taylor series for the function \( f(x) = \sqrt{x} \) centered at \( a \).

(a) What is a good choice for \( a \)? Use the degree 1 Taylor polynomial generated by \( f \) at your choice of \( a \) to compute an approximation for \( \sqrt{4.1} \).

(b) Find the worst possible error in this approximation according to the remainder estimate given by Taylor's theorem.

40. A tank initially contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at a rate of 1 gal/min, and the mixture is pumped out at a rate of 3 gal/min.

(a) Set up a differential equation for \( y(t) \), the amount of fertilizer in the tank after \( t \) minutes have elapsed. Also specify the initial condition.

(b) Solve the differential equation. Your answer should not have any unspecified constants in it.

\(^1\)And no, WolframAlpha isn’t around anymore.
41. Find the particular solution to the differential equation
\[ y'' - 6y' + 13y = 0 \]
satisfying the initial conditions \( y(0) = 1 \) and \( y'(0) = 1 \).

42. Determine whether or not the series \( \sum_{n=1}^{\infty} a_n \) converges or diverges, when
\[ a_1 = \frac{1}{2}, \quad a_{n+1} = a_n^{1/n}, \quad n \geq 1 \]
Hint: find \( \lim_{n \to \infty} a_n \). Solution: find \( \lim_{n \to \infty} a_n = 1 \).

43. Is it true that for any sequence of positive number \( a_n \) such that \( \sum_{n=1}^{\infty} a_n \) converges, there is another sequence of positive numbers \( b_n \), with \( b_n > a_n \) for all \( n \), such that \( \sum_{n=1}^{\infty} b_n \) also converges?
(If you think the answer is “yes”, explain how to obtain \( b_n \) given \( a_n \). If you think “no”, give an example of a sequence \( a_n \) such that there is no such sequence \( b_n \).) Solution: Yes, let \( b_n = 2a_n \).

44. Determine whether the following alternating series converges.
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{7} \right)^n \]

45. Let \( g(x) \) be the sum of the first three terms (constant, linear, and quadratic) of the Taylor series generated by
\[ f(x) = 2x^3 + x^2 - 2, \quad \text{at } a = -1. \]
In other words,
\[ g(x) = \sum_{n=0}^{2} \frac{f^{(n)}(a)}{n!} (x - a)^n. \]

Determine whether or not \( f(0) > g(0) \).
Solution: for \( x = -1 \),
\[ f(x) = -2 + 1 - 2 = -3 \]
\[ f'(x) = 6x^2 + 2x = 6 - 2 = 4, \]
\[ f''(x) = 12x + 2 = -12 + 2 = -10 \]
so
\[ g(x) = -10x^2 + 4x - 3 \]
and
\[ g(0) = -3 \]
\[ f(0) = -2 \]
so indeed \( f(0) > g(0) \).
46. Suppose $-1 < x < 1$. Can the error in the approximation

$$e^x \approx 1 + x + x^2/2$$

be more than $1/2$? In other words, is it possible that

$$|e^x - \left(1 + x + \frac{x^2}{2}\right)| > \frac{1}{2}?$$

Hint: use Taylor’s formula.

Solution: let $f(x) = e^x$. The error is at most, for some $c$ between $x$ and 0,

$$\left|\frac{f^{(3)}(c)}{3!} (x - 0)^3\right| = \frac{e^c}{6} |x|^3 \leq \frac{e^{|c|}}{6} \leq \frac{e}{6} < \frac{1}{2}$$

so the answer is No.

47. Find a solution $y(x)$ of the differential equation

$$2y'' + 4y' + 2y = 0$$

satisfying $y(0) = 2$. (There is actually more than one correct answer, but you don’t have to find the most general solution.) Solution:

$$2r^2 + 4r + 2 = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r + 1)^2 = 0$$

$$y = C_1 e^{-x} + C_2 xe^{-x}$$

$$2 = y(0) = C_1$$

$C_2$ can be anything, say 0.

48. Find a power series for

$$\frac{2x}{(1 - x^2)^2}.$$

Hint: first find the derivative

$$\frac{d}{dx} \left(\frac{1}{1 - x^2}\right).$$

49. Solve the differential equation

$$xy' + 3y = \frac{\cos x}{x^2}, \quad x > 0.$$ 

Solution for $\cos$ replaced by $\sin$:

$$y' + 3 \frac{y}{x} = \frac{\sin x}{x^3}$$

$$P(x) = \frac{3}{x}$$

$$Q(x) = \frac{\sin x}{x^3}$$

$$r(x) = e^{\int P(x) \, dx} = x^3$$

$$y(x) = x^{-3} \int Q(x) r(x) \, dx = \frac{C - \cos x}{x^3}$$
50. Circle either true or false.

(a) Every function \( f(x) \) has an inverse \( f^{-1}(x) \).

**TRUE** \hspace{1cm} **FALSE**

(b) \( \ln x - \ln y = \frac{\ln x}{\ln y} \) for all numbers \( x, y \).

**TRUE** \hspace{1cm} **FALSE**

(c) \( \frac{d}{dx} 2^x = (\ln 2)2^x \).

**TRUE** \hspace{1cm} **FALSE**

(d) The domain of \( \tan^{-1} x \) is \([-1, 1]\).

**TRUE** \hspace{1cm} **FALSE**

(e) If \( 0 \leq f(x) \leq g(x) \) for all \( x \) and \( \int_0^\infty g(x) \, dx \) converges, then we can conclude that \( \int_0^\infty f(x) \, dx \) converges.

**TRUE** \hspace{1cm} **FALSE**

(f) If \( \lim_{n \to \infty} a_n = 0 \), then we can conclude that \( \sum_{n=1}^\infty a_n \) converges.

**TRUE** \hspace{1cm} **FALSE**

(g) The following series converges.
\[
\sum_{n=1}^\infty 1 = 1 + 1 + 1 + 1 + 1 + \ldots
\]

**TRUE** \hspace{1cm} **FALSE**

(h) The following series converges.
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{7}} + \ldots
\]

**TRUE** \hspace{1cm} **FALSE**

(i) Every telescoping series
\[
\sum_{n=1}^\infty (b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \ldots
\]

converges and the sum is \( b_1 \).
51. Evaluate the following limits.

(a) \( \lim_{x \to 2^+} \ln(x^2 - 4) \)

(b) \( \lim_{x \to \infty} e^{-x^2} \)

(c) \( \lim_{n \to \infty} \frac{4^n + n!}{10^n + 46n^{32}} \)

(d) \( \lim_{x \to 0} \frac{e^{2x} - 1}{\ln(1 - x)} \)

(e) \( \lim_{x \to \infty} (x^2 + 1)^{1/\ln x} \)

52. Evaluate the following integrals.

(a) \( \int_{e}^{e^2} \frac{\ln x)^2}{x} \, dx \)

(b) \( \int x \cos(2x) \, dx \)

(c) \( \int_{-\infty}^{\infty} \frac{2}{1 + x^2} \, dx \)

(d) \( \int \frac{2x + 5}{x^2 - 2x + 1} \, dx \)

(e) \( \int \frac{1}{(1 + x^2)^{5/2}} \, dx \)

53. Determine whether each of the following converges or diverges. If it converges, you **DO NOT** need to calculate the sum. Clearly state which test for convergence/divergence you are using, and show the necessary work in executing the test.

\( \sum_{n=0}^{\infty} \frac{e^n}{2e^n + n^2} \)

\( \sum_{n=0}^{\infty} \frac{n!}{(2n)!} \)

\( \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{n/2}} \)

54. For each of the following series, determine if it is absolutely convergent, conditionally convergent, or divergent. Show all your work and clearly indicate any tests you are using.
55. Suppose we define a function $f$ by the formula

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$  

(Notice this is not a power series since it is not a sum of powers of $x$.) The domain of $f$ is the set of all numbers $x$ for which the above series converges. What is the domain of $f$?

56. Compute the Taylor series for $f(x) = xe^x$ centered at $a = 1$. Formulas for the derivatives of $f$ are given below.

$$f(x) = xe^x$$
$$f'(x) = xe^x + e^x$$
$$f''(x) = xe^x + 2e^x$$
$$f'''(x) = xe^x + 3e^x$$
$$\vdots$$
$$f^{(k)}(x) = xe^x + ke^x$$

57. Calculate the sum of the following series.

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$$

58. For what $x$ does the series $\sum_{n=0}^{\infty} (2x)^n$ converge, and what is the sum?

59. Determine the interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}(x + 1)^n}{3^n}$$

60. Calculate the Taylor polynomial of order 2 for $f(x) = e^{3x} \cos x$ centered at $a = 0$. 

61. Let \( f(x) \) be a function with graph as pictured.

Estimate \( f^{-1}(1) \).

62. Say \( g(x) = 2x - 3 \). Find, and simplify, a formula for \( g^{-1}(x) \).

63. Say \( h \) is a differentiable function such that \( h(2) = 3 \) and \( h'(2) = 4 \). Find \( (h^{-1})'(3) \).

64. Compute the derivatives of the following functions. You do not have to simplify your answers.
   (a) \( f(x) = 2\ln(2x) \).
   (b) \( f(x) = x^x \).

65. Compute the following limits. Justify your answers using algebraic manipulations or L'hôpital's rule (writing 'as powers beat logs', for example, is not enough for full credit). Simplify your answers.
   (a) \( \lim_{n \to \infty} \frac{n^2}{e^n} \).
   (b) \( \lim_{n \to \infty} \left( \log_4(16n) - \log_4(n^2) + \log_4(n) + \frac{\log_4(n)}{n} \right) \).

66. (a) Use the trapezoidal rule with \( n = 4 \) to find an approximation to the integral
   \[ \int_0^4 x^2 \, dx. \]

   (b) Using the error formula for the trapezoidal rule, estimate how far your answer to (a) is from the actual value of the integral.

67. Compute the following integrals, or say if they diverge. For full credit, follow the instructions below.
   • If you change the integrand to a power series, just write down the first three non-zero terms (plus the constant of integration, if any).
   • If the integral is improper, make sure any limiting arguments that you use are explicit.
   • If your computation involves a trig substitution, do not include any trig functions in your final answer.
• Simplify your answers.

(a) \[ \int \sin^2(x) \cos^3(x) \, dx. \]
(b) \[ \int e^x \cos(2x) \, dx. \]
(c) \[ \int \frac{2}{\sqrt{x^2 + 4}} \, dx. \]
(d) \[ \int e^{-x^2} \, dx. \]
(e) \[ \int_3^\infty \frac{4}{x^2 - 4} \, dx. \]

68. Compute the following series. Simplify your answers.

(a) \[ \sum_{n=0}^{\infty} \frac{1 - 3^n}{4^n}. \]
(b) \[ \sum_{n=2}^{\infty} \frac{1}{n(n + 1)}. \]

69. For each of the following series, say whether they converge or diverge. Make sure you justify your solutions, and state clearly which test(s) you are using (if any).

(a) \[ \sum_{n=1}^{\infty} \frac{-2}{\sqrt{n}}. \]
(b) \[ \sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}. \]
(c) \[ \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{1 + n^2}}{1 + n}. \]
(d) \[ \sum_{n=0}^{\infty} \frac{4^n n^4}{(n + 4)!}. \]

12 Find the values of \( x \) for which the power series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n3^n} (x - 4)^n \]

(a) converges absolutely; (b) converges conditionally; (c) diverges. Justify your answers.

70. Find the first three non-zero terms in the Taylor series for \( \cos(x) \) centered at \( a = \pi/3 \).

71. Let \( f(x) = \sin(2x) \). Find an estimate for the error \( R_3(2) \), when \( f \) is approximated by its order 3 Taylor polynomial \( P_3(x) \) centered at 0. Simplify your answer.
72. Consider the slope field pictured below.

(a) Which of the differential equations below matches this slope field?

(a) \( \frac{dy}{dx} = \frac{x}{y} \), (b) \( \frac{dy}{dx} = x^2 \), (c) \( \frac{dy}{dx} = xy \), (d) \( \frac{dy}{dx} = y^2 \).

(b) Sketch the solution to this differential equation that satisfies \( y(2) = 1 \) on the slope field.

73. Which (possibly none, or more than one) of the following are solutions to the differential equation \( \frac{dy}{dx} = 2xy \)?

(a) \( y = -e^{x^2} \) (b) \( y = -e^{-x^2} \), (c) \( y = e^{x^2} \), (d) \( y = e^{-x^2} \).

74. Solve the following differential equations (either give the general solution, or solve for a particular solution satisfying the given initial conditions). If you use the power series method, just write down the first three non-zero terms of the solution.

(a) \( y'' - 2y + 5 = 0 \).

(b) \( y'' + xy = 0 \), \( y(0) = 2 \), \( y'(0) = 3 \).

(c) \( y' + 2y = 4x \).