1. Are the vectors \( v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) linearly dependent or independent?

**Solution** Let \( c_1 \) and \( c_2 \) be two scalars so that \( c_1 v_1 + c_2 v_2 = 0_{\mathbb{R}^2} \). We want to solve this equation for \( c_1 \) and \( c_2 \).

This equation can be written in a matrix form, by using the augmented matrix \( \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix} \) whose RREF is \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \). Therefore \( c_1 = c_2 = 0 \), which means that \( v_1 \) and \( v_2 \) are linearly independent.

2. Are the degree-3 polynomial functions \( p_1(x) = x^3 \), \( p_2(x) = x^2 + x \), \( p_3(x) = x^2 - x \) and \( p_4(x) = 1 \) linearly dependent or independent?

**Solution** Let \( c_1, c_2, c_3 \) and \( c_4 \) be four scalars so that \( c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) + c_4 p_4(x) = 0 \) for all \( x \). We want to solve this equation for \( c_1, c_2, c_3 \) and \( c_4 \).

Rearranging the terms, we can write this equation in the form \( c_1 x^3 + (c_2 + c_3) x^2 + (c_2 - c_3) x + c_4 = 0 \) for all \( x \). Since two polynomial functions are equal if and only if their coefficients are the same, we get the system \( c_1 = 0 \), \( c_2 + c_3 = 0 \), \( c_2 - c_3 = 0 \) and \( c_4 = 0 \), whose unique solution is \( c_1 = c_2 = c_3 = c_4 = 0 \). Therefore \( p_1(x) \), \( p_2(x) \), \( p_3(x) \) and \( p_4(0) \) are linearly independent.

3. Are the matrices \( M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( M_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) and \( M_4 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \) linearly dependent or independent?

**Solution** Let \( c_1, c_2, c_3 \) and \( c_4 \) be four scalars so that \( c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = 0_{M_2(\mathbb{R})} \). We want to solve this equation for \( c_1, c_2, c_3 \) and \( c_4 \).

This equation can be written \( \begin{pmatrix} c_1 + c_3 + 3c_4 \\ c_2 + c_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) which leads to the system

\[
\begin{align*}
c_1 + c_3 + 3c_4 &= 0 \\
c_2 + c_3 &= 0 \\
0 &= 0 \\
c_1 + 2c_4 &= 0
\end{align*}
\]

This system can be written in a matrix form, by using the augmented matrix

\[
\begin{pmatrix}
1 & 0 & 1 & 3 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0
\end{pmatrix}
\]

whose RREF is

\[
\begin{pmatrix}
1 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Therefore, the system has infinitely many solutions, given by \( \{ (c_1, c_2, c_3, c_4) = (-2c_4, c_4, -c_4, c_4), c_4 \in \mathbb{R} \} \). Hence, there exist infinitely many sets of scalars \( c_1, c_2, c_3 \) and \( c_4 \) not all zero so that \( c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). For instance, setting \( c_4 = 1 \), we get \( c_1 = -2, c_2 = 1 \) and \( c_3 = -1 \), but we can set \( c_4 \) equal to any real number.

4. Find a basis for the subspace \( W \) of \( \mathbb{R}^3 \) that contains all the vectors of the form \( \begin{pmatrix} 2x \\ 2x + y \\ 3y \end{pmatrix} \), for \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \).

Solution: Notice that any arbitrary vector in \( W \) can be written as the linear combination
\[
\begin{pmatrix} 2x \\ 2x + y \\ 3y \end{pmatrix} = x \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}
\]
so \( W = \text{Span}\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\} \). Moreover, the vectors \( \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \) are clearly linearly independent since they are not proportional. Hence, they form a basis for \( W \).

5. Show that the vectors \( v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \), \( v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) form a basis for \( \mathbb{R}^3 \).

Solution: Let \( c_1, c_2 \) and \( c_3 \) be two scalars so that \( c_1v_1 + c_2v_2 + c_3v_3 = 0_{\mathbb{R}^3} \). We want to solve this equation for \( c_1, c_2 \) and \( c_3 \). This equation can be written in a matrix form, by using the augmented matrix
\[
\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}
\]
whose RREF is
\[
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]. Therefore \( c_1 = c_2 = c_3 = 0 \), which means that \( v_1, v_2 \) and \( v_3 \) are linearly independent. Moreover, consider three arbitrary real numbers \( a, b \) and \( c \). Let \( c_1, c_2 \) and \( c_3 \) be two scalars so that \( c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \). We want to solve this equation for \( c_1, c_2 \) and \( c_3 \). This equation can be written in a matrix form, by using the augmented matrix
\[
\begin{pmatrix} 1 & 1 & 1 & a \\ -1 & 0 & 1 & b \\ 1 & 1 & 0 & c \end{pmatrix}
\]
whose RREF is
\[
\begin{pmatrix} 1 & 0 & 0 & a - b - c \\ 0 & 1 & 0 & b - a + 2c \\ 0 & 0 & 1 & a - c \end{pmatrix}
\]. So the equation has a solution for any arbitrary vector \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) which means that \( \text{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3 \). So we can conclude the vectors \( v_1, v_2 \) and \( v_3 \) form a basis for \( \mathbb{R}^3 \).
6. Find a basis for the subset $W$ of $M_2(\mathbb{R})$ that contains all the matrices of the form \[
\begin{pmatrix}
 x & 0 \\
 y & 0 \\
\end{pmatrix}
\] where $x + y = 0$.

**Solution:** Notice that any arbitrary matrix in $W$ can be written in the form \[
\begin{pmatrix}
 x & 0 \\
 -x & 0 \\
\end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}
\] so $W = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right\}$. Therefore, the matrix \[
\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}
\] forms a basis for $W$.

7. Show that the degree-2 polynomial functions $p_1(x) = x^2 + 1$, $p_2(x) = x - 1$ and $p_3(x) = 3$ for a basis for $P_2$.

**Solution** Let $c_1$, $c_2$ and $c_3$ be three scalars so that $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0$ for all $x$. We want to solve this equation for $c_1$, $c_2$, $c_3$. Rearranging the terms, we can write this equation in the form $c_1x^2 + c_2x + c_1 - c_2 + 3c_3 = 0$ for all $x$. Since two polynomial functions are equal if and only if their coefficients are the same, we get the system $c_1 = 0$, $c_2 = 0$ and $c_1 - c_2 + 3c_3 = 0$, whose unique solution is $c_1 = c_2 = c_3 = 0$. Therefore $p_1(x)$, $p_2(x)$ and $p_3(x)$ are linearly independent. Now, consider an arbitrary degree-2 polynomial $ax^2 + bx + c$. We want to solve the equation $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = ax^2 + bx + c$ for all $x$. Again, since two polynomial functions are equal if and only if their coefficients are the same, we get the system $c_1 = a$, $c_2 = b$ and $c_1 - c_2 + 3c_3 = c$, whose unique solution is $c_1 = a$, $c_2 = b$ and $c_3 = \frac{a - b}{3}$. Therefore, $\text{Span}\left\{ x^2 + 1, \ x - 1, \ 3 \right\} = P_2$. So we can conclude that $x^2 + 1, x - 1$ and $3$ form a basis for $P_2$. 