1. (6 points) Find the limit of the sequence

\[ a_n = \left( \frac{3}{n} \right)^{\frac{1}{n}}. \]

Answer: 1 (commonly occurring limits theorem, section 9.1)

2. (6 points) Does the series \( \sum_{n=1}^{\infty} \frac{n^{\frac{2}{3}} + 2n^{\frac{1}{3}} + 1}{n^{\frac{1}{2}} + 4n^{\frac{1}{3}} + 3} \) converge or diverge? Justify your answer.

Answer: Diverges. Limit comparison test with \( \frac{1}{n^{\frac{5}{6}}} \).

3. (6 points) Show that the series \( \sum_{n=1}^{\infty} \frac{2}{1 + e^n} \) converges.

Answer: Direct comparison test with \( \sum_{n=1}^{\infty} \frac{2}{e^n} \) that converges (integral test).

4. (6 points) Does the series \( \sum_{n=1}^{\infty} \ln\left( \frac{1}{n} \right) \) converge or diverge? Justify your answer.

Answer: Nth term test for divergence. Diverges

5. (6 points) Does the series \( \sum_{n=1}^{\infty} \frac{(\ln(n))^n}{n^n} \) converge or diverge? Justify your answer.

Answer: Root test. Converges

6. (6 points) Is the series \( \sum_{n=1}^{\infty} (-1)^n n^2 \left( \frac{2}{3} \right)^n \) absolutely convergent? Convergent?

Answer: Absolutely convergent (ratio test) so convergent as well.

7. (6 points) Show that the series \( \sum_{n=1}^{\infty} (-1)^n (\sqrt{n + 1} - \sqrt{n}) \) converges conditionally.

Answer: Not absolutely convergent (\( \sum_{n=1}^{\infty} (\sqrt{n + 1} - \sqrt{n}) \) is telescoping series whose sequence of the partial sums \( S_N = \sqrt{N+1} - 1 \) diverges to \( +\infty \)).

8. (6 points) Show that the series \( \sum_{n=1}^{\infty} \frac{1}{n 2^n} \) converges. Use this result to determine whether the series \( \sum_{n=1}^{\infty} \frac{1-2^n}{n 2^n} \) converges or diverges.

Answer: Direct comparison test with \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) that converges (geometric series). If \( \sum_{n=1}^{\infty} \frac{1-2^n}{n 2^n} \) converged then \( \sum_{n=1}^{\infty} \frac{2^n}{n 2^n} - \sum_{n=1}^{\infty} \frac{1}{2^n} \) would converge as well (difference rule) and would equal \( \sum_{n=1}^{\infty} \frac{-1}{n} \) which is known to be a divergent series (p-rule). Contradiction. Hence \( \sum_{n=1}^{\infty} \frac{1-2^n}{n 2^n} \) diverges.

9. For what \( x \) does the series \( \sum_{n=0}^{\infty} (2x)^n \) converge, and what is the sum?

Answer: Converges for \( x \in (-\frac{1}{2}, -\frac{1}{2}) \) (geometric series) and the sum is \( \frac{1}{1-2x} \).
10. (6 points) Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \ln(n + 1) x^n$.

Answer: $R = 1$, $I = (-1, +1)$