You must be able to:

- Determine whether a function $T : V \rightarrow W$ is a linear transformation.
- Find a basis for the kernel of a linear transformation.
- Calculate additions, differences and composites of linear transformations.
- Determine the matrix $[T]_\alpha^\beta$ of a linear transformation $T : V \rightarrow W$ with respect to a basis $\alpha$ for $V$ and a basis $\beta$ for $W$.
- Determine the coordinate vector $[T(v)]_\beta$ given a linear transformation $T : V \rightarrow W$, a vector $v$ in $V$, a basis $\alpha$ for $V$ and a basis $\beta$ for $W$.
- For two basis $\alpha$ and $\beta$ of a vector space $V$, find the change of basis matrix from $\alpha$ to $\beta$.
- Find the eigenvalues and basis for the associated eigenspaces of an $n \times n$ matrix $A$.
- Determine if an $n \times n$ matrix $A$ is diagonalizable and, if so, find the diagonal matrix $D$ and the invertible matrix $P$ so that $D = P^{-1}AP$.
- Find the eigenvalues and basis for the associated eigenspaces of a linear transformation $T : V \rightarrow V$.

Important definitions/results/formulas to remember:

- If $V$ and $W$ are vector spaces, a function $T : V \rightarrow W$ is called a linear transformation if, for all vectors $u$ and $v$ in $V$ and all scalars $c$, the following two properties are satisfied:
  1. $T(u+v)=T(u)+T(v)$.
  2. $T(cv)=cT(u)$.
- If $A$ is an $n \times m$ matrix, the matrix transformation
  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$
  $X \rightarrow AX$

  is a linear transformation.
- Suppose $T : V \rightarrow W$ is a linear transformation.
  1. $T(-v) = -T(v)$ for any $v$ in $V$.
  2. $T(u - v) = T(u) - T(v)$ for any $u$ and $v$ in $V$.
3. \( T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k) \)
   for any scalars \( c_1, c_2, \ldots, c_k \) and any vectors \( v_1, v_2, \ldots, v_k \) in \( V \).

- If \( T : V \rightarrow W \) is a linear transformation, the kernel of \( T \), denoted \( \text{Ker}(T) \) is the set of all the vectors in \( V \) so that \( T(v) = 0_W \). In set notation
  \[ \text{Ker}(T) = \{ v \in V \mid T(v) = 0_W \} \]
- In the case of a matrix transformation
  \[ T : \mathbb{R}^m \rightarrow \mathbb{R}^n \]
  \[ X \rightarrow AX \]
  then \( \text{Ker}(T) = \text{NS}(A) \).
- If \( T : V \rightarrow W \) is a linear transformation then \( \text{Ker}(T) \) is a subspace of \( V \).
- If \( T : V \rightarrow W \) and \( S : V \rightarrow W \) are linear transformations and \( c \) is a scalar then we can define the linear transformations
  \[ S + T : V \rightarrow W \]
  \[ v \rightarrow S(v) + T(v) \]
  and
  \[ cT : V \rightarrow W \]
  \[ v \rightarrow cT(v) \]
- If \( T : U \rightarrow V \) and \( S : V \rightarrow W \) are linear transformation, then the composite
  \[ S \circ T : U \rightarrow W \]
  \[ u \rightarrow S(T(u)) \]
  is a linear transformation.
- Provided the indicated operations are defined, the following properties hold when \( R, S \) and \( T \) are linear transformations and \( c \) and \( d \) are scalars.
  1. \( S + T = T + S \)
  2. \( R + (S + T) = (R + S) + T \)
  3. \( c(dT) = (cd)T \)
  4. \( c(S + T) = cS + cT \)
  5. \( (c + d)T = cT + dT \)
  6. \( R \circ (S \circ T) = (R \circ S) \circ T \)
  7. \( R \circ (S + T) = R \circ S + R \circ T \)
  8. \( (R + S) \circ T = R \circ T + S \circ T \)
  9. \( c(S \circ T) = cS \circ T = S \circ (cT) \)
Suppose \( T : V \rightarrow W \) is a linear transformation. Denote \( \alpha = \{v_1, \ldots, v_m\} \) a basis for \( V \) and \( \beta = \{w_1, \ldots, w_n\} \) a basis for \( W \). Assume that

\[
T(v_1) = a^1_1 w_1 + a^1_2 w_2 + \cdots + a^1_n w_n
\]

\[
T(v_2) = a^2_1 w_1 + a^2_2 w_2 + \cdots + a^2_n w_n
\]

\[
\vdots
\]

\[
T(v_m) = a^m_1 w_1 + a^m_2 w_2 + \cdots + a^m_n w_n.
\]

We call matrix of \( T \) with respect to the basis \( \alpha \) and \( \beta \), denoted \( [T]_{\alpha}^{\beta} \), the matrix

\[
[T]_{\alpha}^{\beta} = \begin{pmatrix}
  a^1_1 & a^2_1 & \cdots & a^m_1 \\
  a^1_2 & a^2_2 & \cdots & a^m_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  a^1_n & a^2_n & \cdots & a^m_n
\end{pmatrix}.
\]

Denote \( \alpha = \{v_1, \ldots, v_n\} \) and \( \beta = \{w_1, \ldots, w_n\} \) two basis for a vector space \( V \). Assume that

\[
w_1 = p^1_1 v_1 + p^1_2 v_2 + \cdots + p^1_n v_n
\]

\[
w_2 = p^2_1 v_1 + p^2_2 v_2 + \cdots + p^2_n v_n
\]

\[\vdots\]

\[
w_n = p^n_1 v_1 + p^n_2 v_2 + \cdots + p^n_n v_n.
\]

We call change of basis matrix from \( \alpha \) to \( \beta \) the matrix

\[
P = \begin{pmatrix}
p^1_1 & p^2_1 & \cdots & p^n_1 \\
p^1_2 & p^2_2 & \cdots & p^n_2 \\
\vdots & \vdots & \ddots & \vdots \\
p^1_n & p^2_n & \cdots & p^n_n
\end{pmatrix}.
\]

Denote \( I : V \rightarrow V \) the identity transformation from \( V \) to \( V \) defined as \( I(v) = v \) for all \( v \) in \( V \). Then the change of basis matrix \( P \) from \( \alpha \) to \( \beta \) is given by

\[
P = [I]_\beta^\alpha.
\]

If \( P \) is the change of basis matrix \( P \) from \( \alpha \) to \( \beta \) then the change of basis from \( \beta \) to \( \alpha \) is \( P^{-1} \). In other words,

\[
[I]_\beta^\alpha = ([I]_\alpha^\beta)^{-1} = P^{-1}.
\]

Suppose \( T : V \rightarrow W \) is a linear transformation. Denote \( \alpha \) and \( \alpha' \) two basis for \( V \) and \( \beta \) and \( \beta' \) two basis for \( W \). If \( P \) is the change of basis matrix from \( \alpha \) to \( \alpha' \) and \( Q \) is the change of basis matrix from \( \beta \) to \( \beta' \) then

\[
[T]_{\alpha'}^{\beta'} = Q^{-1}[T]_{\alpha}^{\beta} P
\]

\[\text{3}\]
– Suppose \( T : V \rightarrow V \) is a linear transformation. Denote \( \alpha \) and \( \beta \) two basis for \( V \) and \( P \) the change of basis matrix from \( \alpha \) to \( \beta \). Then

\[
[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P
\]

– Suppose \( T : V \rightarrow V \) is a linear transformation. Let \( \alpha \) be a basis for \( V \) and \( \beta \) be a basis for \( W \). For any \( v \) in \( V \), denote \([v]_\alpha\) the coordinate vector of \( v \) relative to \( \alpha \) and \([T(v)]_\beta\) the coordinate vector of \( T(v) \) relative to \( \beta \). Then we have

\[
[T(v)]_\beta = [T]_\beta^\alpha[v]_\alpha.
\]

– If \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a linear transformation and \( A \) is the matrix of \( T \) with respect to the canonical basis for \( \mathbb{R}^m \) and \( \mathbb{R}^n \), then, for any \( X \) in \( \mathbb{R}^m \)

\[
T(X) = AX.
\]

– If \( A \) is an \( n \times n \) matrix, an eigenvector of \( A \) is a non-zero vector of \( \mathbb{R}^n \) so that

\[
Av = \lambda v
\]

for some \( \lambda \). The scalar \( \lambda \) is called an eigenvalue of \( A \).

– If \( A \) is an \( n \times n \) matrix, a number \( \lambda \) is an eigenvalue of \( A \) if and only if

\[
\det(A - \lambda I_n) = 0.
\]

The degree-\( n \) polynomial \( P(\lambda) = \det(A - \lambda I_n) \) is called the characteristic polynomial of \( A \). A number \( \lambda \) is an eigenvalue of \( A \) if and only if it is a root of the characteristic polynomial of \( A \).

– If \( A \) is an \( n \times n \) matrix and \( \lambda \) is an eigenvalue of \( A \), then the nullspace of \( A - \lambda I_n \), denoted \( NS(A - \lambda I_n) \), is a subspace of \( \mathbb{R}^n \) called the Eigenspace of \( A \) associated with \( \lambda \). It is denoted \( E_\lambda \) and it is formed by all the eigenvectors of \( A \) associated with \( \lambda \). That is to say

\[
E_\lambda = \{ \text{eigenvectors of } A \text{ associated with } \lambda \} = \{ v \in \mathbb{R}^n \mid Av = \lambda v \} = NS(A - \lambda I_n).
\]

– An \( n \times n \) matrix \( A \) is said to be diagonalizable if there exist a diagonal \( n \times n \) matrix \( D \) and an invertible \( n \times n \) matrix \( P \) so that

\[
D = P^{-1}AP.
\]

– An \( n \times n \) matrix \( A \) is diagonalizable if and only if there exist a basis \( \alpha \) for \( \mathbb{R}^n \) consisting of eigenvectors of \( A \). In that case, if \( T \) is the matrix transformation

\[
T : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
X \rightarrow AX
\]

then the matrix of \( T \) with respect to the basis \( \alpha \), denoted \([T]_\alpha^\alpha\), is a diagonal matrix.
Suppose that $A$ is an $n \times n$ matrix with distinct eigenvalues $r_1, r_2, \ldots, r_k$. Then $A$ is diagonalizable if and only if
\[ \dim(E_1) + \cdots + \dim(E_k) = n. \]

Suppose that $A$ is an $n \times n$ matrix with distinct eigenvalues $r_1, r_2, \ldots, r_k$. Denote, respectively, $m_{r_1}, m_{r_2}, \ldots, m_{r_k}$ the multiplicities of $r_1, r_2, \ldots, r_k$ in the characteristic polynomial of $A$. Then $A$ is diagonalizable if and only if
\[ \dim(E_1) = m_{r_1}, \ \dim(E_2) = m_{r_2}, \ldots, \ \dim(E_k) = m_{r_k}. \]

Suppose $T : V \rightarrow V$ is a linear transformation. An eigenvector of $T$ is a non-zero vector of $V$ so that
\[ T(v) = \lambda v \]
for some $\lambda$. The scalar $\lambda$ is called an eigenvalue of $T$.

If $T : V \rightarrow V$ is a linear transformation and $\lambda$ is an eigenvalue of $T$, then the kernel of $T - \lambda I$, denoted $\text{Ker}(A - \lambda I)$, is a subspace of $V$ called the Eigenspace of $T$ associated with $\lambda$. It is denoted $E_\lambda$ and it is formed by all the eigenvectors of $T$ associated with $\lambda$. That is to say
\[ E_\lambda = \{ \text{eigenvectors of } T \text{ associated with } \lambda \} = \{ v \in V \mid T(v) = \lambda v \} = \text{Ker}(A - \lambda I). \]

Suppose $T : V \rightarrow V$ is a linear transformation. Let $\alpha$ be a basis for $V$ and $A$ be the matrix of $T$ with respect to $\alpha$ (that-is-to-say, $A = [T]_\alpha^\alpha$). Then $v$ is an eigenvector of $T$ associated with $\lambda$ if and only if the coordinate vector of $v$ relative to $\alpha$, denoted $[v]_\alpha$, is an eigenvector of $A$ associated with $\lambda$. In other words,
\[ T(v) = \lambda v \iff [T]_\alpha^\alpha[v]_\alpha = \lambda [v]_\alpha. \]

Suppose $T : V \rightarrow V$ is a linear transformation. We say that $T$ is diagonalizable if there exists a basis $\alpha$ for $V$ consisting of eigenvectors of $T$.

Suppose $T : V \rightarrow V$ is a linear transformation. Then $T$ is diagonalizable if and only if there exists a basis $\alpha$ for $V$ so that the matrix $[T]_\alpha^\alpha$ is diagonalizable.