Math 301  Name: ______________________
Fall 2015
(Practice) Midterm Exam 1
10/15/2015
Time Limit: 1 hour and 15 minutes

- One side of each sheet is blank and may be used as scratch paper.
- Show your work clearly.

Grade Table (for instructor use only)

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A useful formula

Sum of first $n$ terms of an arithmetic progression with first term $a$ and constant increment $d$:

$$\frac{n}{2}[2a + (n - 1)d]$$
1. For each of the following propositions, determine whether or not it is satisfiable. If it is, give a satisfying assignment.

(a) (5 points) \((p \rightarrow q) \land (p \rightarrow \neg q) \land (\neg p \rightarrow q) \land (\neg p \rightarrow \neg q)\)

**Solution:**

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None of the assignments of \(p\) and \(q\) make all four of the terms in the conjunction true, so the proposition is unsatisfiable.

(b) (5 points) \((p \leftrightarrow (q \lor r)) \land (r \leftrightarrow \neg (p \land q))\)

**Solution:** The proposition is satisfied when \(p\) is true and \(q\) is false and \(r\) is true.

(c) (5 points) \((\neg p \land q) \rightarrow (\neg q \land p)\)

**Solution:** The proposition is satisfied when \(p\) is true and \(q\) is false.
2. Negate the following statements:
   
   (a) (5 points) If all left-handed people are happy, then all right-handed people are sad.

   \[\text{Solution:} \text{ All left-handed people are happy, and there is a right-handed person who is happy.}\]

   (b) (5 points) Sometimes, when I eat too many malasadas, I get a stomachache.

   \[\text{Solution:} \text{ I never get a stomachache from eating malasadas, no matter how many I eat.}\]

   (c) (5 points) Everyone owns at least one musical instrument that they cannot play well.

   \[\text{Solution:} \text{ There’s a person who can play all the musical instruments they own well.}\]
3. (5 points) List the elements of the power set of the power set of the empty set.

**Solution:** \( \mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} \)

4. (5 points) Give an example of a function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) that is surjective but not injective.

**Solution:** Let \( f(x) = x \) if \( x \) is odd and \( f(x) = x/2 \) if \( x \) is even. This is surjective, because \( f(2x) = x \) for all \( x \) in \( \mathbb{Z} \). However, it is not injective since \( f(2) = f(1) = 1 \).

5. (5 points) How many subsets are there of the set \( \{1, 3, 5, 6, 19, 20, 23\} \) that contain both 1 and 5?

**Solution:** The subsets that contain both 1 and 5 are exactly the ones of the form \( \{1, 5\} \cup B \) where \( B \) is a subset of \( \{3, 6, 19, 20, 23\} \), and from your homework, you know that the number of such \( B \) is \( 2^5 = 32 \).
6. Write down non-recurrent definitions of the following sequences (you may use piecewise definitions):

(a) (10 points) \( a_0 = 1 \), and for \( n > 1 \), \( a_n = a_{n-1}/2 \).

**Solution:** Write down a few terms:

\[
a_0 = 1, a_1 = 1/2, a_2 = 1/2^2, a_3 = 1/2^3, ...
\]

In general, \( a_n = 1/2^n \).

(b) (10 points) \( a_0 = -5 \), and for \( n > 1 \), \( a_n = 5 - a_{n-1} \).

**Solution:** We write down a few terms:

\[
a_0 = -5, a_1 = 10, a_2 = -5, a_3 = 10, a_4 = -5, ...
\]

In general, \( a_n = -5 \) if \( n \) is even, and 10 if \( n \) is odd.
7. (10 points) You’re training for an ultramarathon, which entails running for 100 miles continuously. On your first attempt, you’re really out of shape and can only manage 4 miles. But on the second attempt, you manage 8 miles, and each attempt thereafter, increase your distance by 4 miles. When you finish running an ultramarathon for the first time, how many miles will you have run in total since you started training?

**Solution:** We need to sum an arithmetic progression. The first term is 4, the constant increment is 4, and the number of terms to sum is 25, so the total distance is

\[
\frac{25}{2} [8 + 24(4)] = 52 \times 25 = 1300.
\]
8. (10 points) Show that if $a$ and $b$ are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

**Solution:** Suppose for a contradiction that this isn’t the case. Then there are $a$ and $b$ such that $a + b < 2\sqrt{ab}$. This means that $(a + b)^2 < 4ab$ (it’s important here that $a$ and $b$ are positive). But then $a^2 + 2ab + b^2 - 4ab = a^2 - 2ab + b^2 = (a - b)^2 < 0$, which is impossible, so the proposition must be true.
9. Recall that the Fibonacci sequence is given by \(a_0 = 0, a_1 = 1,\) and for \(n > 1, a_n = a_{n-1} + a_{n-2}:\)

\[
0, 1, 1, 2, 3, 5, 8, 13, 21, ...
\]

(a) (5 points) Show that infinitely many odd numbers occur in this sequence.

**Solution:** Suppose for a contradiction that only finitely many odd numbers occur, and that \(a_n\) is the last one. Then \(a_{n+1}\) is even. But that would mean \(a_{n+2}\) is odd. So it must be the case that infinitely many odd numbers occur.

(b) (5 points) Show that no three consecutive numbers in this sequence can be odd.

**Solution:** Suppose that \(a_n\) and \(a_{n+1}\) are both odd. This means that \(a_{n+2}\) has to be even.

(c) (5 points) Show that no two consecutive numbers in this sequence can be even.

**Solution:** Suppose that \(a_n\) and \(a_{n+1}\) are even. Then \(a_{n+2}\) must be even, which means that \(a_{n+3}\) must be even, which in turn implies that \(a_{n+4}\) must be even, and so on. But we know from part (a) that infinitely many odd numbers appear in the sequence, so it must be the case the no two consecutive numbers in the sequence can be even.