(1) Let $V$ be a vector space over $F$, let $\alpha \in F$, and let $T_\alpha : V \rightarrow V$ be the linear transformation $v \mapsto \alpha v$. Prove the following properties we discussed in class.

(a) $T_0 = 0$
(b) $T_1 = 1$
(c) $T_{\alpha_1 + \alpha_2} = T_{\alpha_1} + T_{\alpha_2}$
(d) $T_{\alpha_1 \alpha_2} = T_{\alpha_1} T_{\alpha_2}$

(Remark: these show that $F$ can be thought of as a “subalgebra” of $\text{End}(V)$.)

(2) Let $T : V \rightarrow W$ be a linear transformation. Prove the following statements (without assuming $V$ and $W$ are finite-dimensional).

(a) If $\dim(V) < \dim(W)$, then $T$ cannot be surjective.
(b) If $\dim(V) > \dim(W)$, then $T$ cannot be injective.

(3) Let $V = \mathbb{R}[x]_{\leq 3}$ be the $\mathbb{R}$-vector space of polynomials over $\mathbb{R}$ of degree at most 3 and let $\mathcal{B} := \{1, x, (3x^2 - 1)/2, (5x^3 - 3x)/2\}$, so that $\mathcal{B}$ is a basis of $V$.

(a) Find the matrix $\mathcal{B}[I]_{\mathcal{B}}$ of the bilinear form

$$I(f(x), g(x)) = \int_{-1}^{1} f(x)g(x)dx.$$ 

(b) What is the matrix of $I$ with respect to the standard basis $\{1, x, x^2, x^3\}$ of $V$.

(4) For $a < b$ in $\mathbb{R}$, let $V = C([a, b], \mathbb{R})$ be the $\mathbb{R}$-vector space of continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$ and let $w(x)$ be a fixed function in $V$. Define the following function $I : V \times V \rightarrow \mathbb{R}$

$$I(f(x), g(x)) = \int_{a}^{b} f(x)g(x)w(x)dx.$$ 

(a) Show that $I(x)$ is a bilinear form.

(b) Take $a = 0$ and $b = \pi$, let $f_n(x) = \cos(nx)$ for $n = 0, 1, 2$, and let $W = \text{Span}(f_0, f_1, f_2) \leq V$. Consider $I$ as a bilinear form $W \times W \rightarrow \mathbb{R}$ and find the matrix of $I$ with respect to the basis $\{f_0, f_1, f_2\}$ of $W$. 

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