Vectors (cont)

**Definition:** Dot Product (or scalar product):

Let \( u = \langle u_1, u_2, \cdots, u_n \rangle \), \( v = \langle v_1, v_2, \cdots, v_n \rangle \)

\[ u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \]

**Remark:** Odd kind of multiplication

**Examples:** (Class)

**Important note:**

\[ u \cdot u = \|u\|^2 \]
Algebraic properties of the dot product

**Theorem:** The dot product is commutative,
\[ \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \]
distributes with vector addition,
\[ (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) \]
associates with scalar multiplication,
\[ c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v}) \]
Moreover,
\[ \mathbf{u} \cdot \mathbf{u} \geq 0 \]
with equality if and only if \( \mathbf{u} \) is the zero vector

**Proof:** text

**Note:** Which operations make sense?
Since $\|u\|^2 = u \cdot u$, some of the above properties can be written as **basic norm properties**:

**Theorem:** Let $u$ be a vector in $\mathbb{R}^n$, $c \in \mathbb{R}$ a scalar. Then:

$$\|u\|^2 > 0 \text{ if } u \neq 0, \text{ and } \|0\|^2 = 0$$

$$\|u\| = \sqrt{u \cdot u}$$

$$\|u\| > 0 \text{ if } u \neq 0, \text{ and } \|0\| = 0$$

$$\|cu\| = |c|\|u\|$$

$$\|\frac{u}{\|u\|}\| = 1 \text{ if } u \neq 0 \text{ (normalization of } u)$$

**Proof:** Class

**Examples** Class
If \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), can interpret them as geometric rays with a common base point (picture; class)

There is a well-defined angle \( \theta \) between them, \( 0 \leq \theta \leq \pi \). This extends to \( \mathbb{R}^n \).

**Theorem:** If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^n \), and \( \theta \) is the angle between them, then

\[
\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta
\]

**Proof:** Class; text

Some geometric properties of dot product:

**Corollary:** Let \( \mathbf{u} \) and \( \mathbf{v} \) be nonzero vectors in \( \mathbb{R}^n \), \( \theta \) the angle between them. Then:

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}
\]

\( \mathbf{u} \perp \mathbf{v} \) if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \)

\[
|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\| \text{ (Cauchy-Schwarz Inequality – not in text)}
\]

\[
\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \text{ (Triangle Inequality – not in text)}
\]

**Proofs:** Text, class
Direction Angles/Cosines: If $\mathbf{u} = (u, v)$ is a vector in $\mathbb{R}^2$, the direction angles are the angles which $\mathbf{u}$ makes with $\mathbf{i}$ and $\mathbf{j}$ (that is, with the x- and y-axes), and the direction cosines are the cosines of these angles.

Similarly, if $\mathbf{u}$ is a vector in $\mathbb{R}^3$, the direction angles are the angles which $\mathbf{u}$ makes with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (that is, with the x-, y-, and z-axes), and the direction cosines are the cosines of these angles.

**Theorem:** If $\mathbf{u}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ then the direction cosines are the components of $\frac{\mathbf{u}}{\|\mathbf{u}\|}$.

**Proof:** Text, class

**Example:** Class
**Projections:** If \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors in \( \mathbb{R}^n \), the vector \( \mathbf{v} \) can be decomposed uniquely as the sum of two orthogonal vectors,

\[ \text{proj}_\mathbf{u} \mathbf{v} \text{ (the projection of } \mathbf{v} \text{ onto } \mathbf{u}) \]
\[ \text{proj}_{\mathbf{u}^\perp} \mathbf{v} \text{ (the projection of } \mathbf{v} \text{ orthogonal to } \mathbf{u}) \]

where \( \text{proj}_\mathbf{u} \mathbf{v} \parallel \mathbf{u} \) (ie, has the same direction as \( \mathbf{u} \))

**Theorem:** If \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors in \( \mathbb{R}^n \), then

\[ \text{proj}_\mathbf{u} \mathbf{v} = (\text{comp}_\mathbf{u} \mathbf{v}) \frac{\mathbf{u}}{||\mathbf{u}||} = (\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2})\mathbf{u} \]
\[ \text{proj}_{\mathbf{u}^\perp} \mathbf{v} = \mathbf{v} - \text{proj}_\mathbf{u} \mathbf{v} \]

where \( \text{comp}_\mathbf{u} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||} \) is the length of the projection of \( \mathbf{v} \) onto \( \mathbf{u} \)

**Derivation and examples:** class