8. Distributive Lattices

Every dog must have his day.

In this chapter and the next we will look at the two most important lattice varieties: distributive and modular lattices. Let us set the context for our study of distributive lattices by considering varieties generated by a single finite lattice. A variety $\mathbf{V}$ is said to be *locally finite* if every finitely generated lattice in $\mathbf{V}$ is finite. Equivalently, $\mathbf{V}$ is locally finite if the relatively free lattice $\mathcal{F}_{\mathbf{V}}(n)$ is finite for every integer $n > 0$.

Theorem 8.1. *If* $\mathcal{L}$ *is a finite lattice and* $\mathbf{V} = \mathrm{HSP}(\mathcal{L})$*, then*

$$|\mathcal{F}_{\mathbf{V}}(n)| \leq |L|^{|L|^n}.$$

Hence $\mathrm{HSP}(\mathcal{L})$ *is locally finite.*

Proof. If $\mathbf{K}$ is any collection of lattices and $\mathbf{V} = \mathrm{HSP}(\mathbf{K})$, then $\mathcal{F}_{\mathbf{V}}(X) \cong \mathrm{FL}(X)/\theta$ where θ is the intersection of all homomorphism kernels $\ker f$ such that $f : \mathrm{FL}(X) \to \mathcal{L}$ for some $\mathcal{L} \in \mathbf{K}$. (This is the technical way of saying that $\mathrm{FL}(X)/\theta$ satisfies exactly the equations that hold in every member of $\mathbf{K}$.) When $\mathbf{K}$ consists of a single finite lattice $\{\mathcal{L}\}$ and $|X| = n$, then there are $|L|^n$ distinct mappings of X into L, and hence $|L|^n$ distinct homomorphisms $f_i : \mathrm{FL}(X) \to \mathcal{L}$ $(1 \leq i \leq |L|^n)$.[1] The range of each f_i is a sublattice of $\mathcal{L}$. Hence $\mathcal{F}_{\mathbf{V}}(X) \cong \mathrm{FL}(X)/\theta$ with $\theta = \bigcap \ker f_i$ means that $\mathcal{F}_{\mathbf{V}}(X)$ is a subdirect product of $|L|^n$ sublattices of $\mathcal{L}$, and so a sublattice of the direct product $\prod_{1 \leq i \leq |L|^n} \mathcal{L} = \mathcal{L}^{|L|^n}$, making its cardinality at most $|L|^{|L|^n}$. $\square$

It is sometimes useful to view this argument constructively: $\mathcal{F}_{\mathbf{V}}(X)$ is the sublattice of $\mathcal{L}^{|L|^n}$ generated by the vectors $\overline{x}$ $(x \in X)$ with $\overline{x}_i = f_i(x)$ for $1 \leq i \leq |L|^n$.

We should note that not every locally finite lattice variety is generated by a finite lattice.

Now it is clear that there is a unique minimum nontrivial lattice variety, *viz.*, the one generated by the two element lattice $\mathbf{2}$, which is isomorphic to a sublattice of any nontrivial lattice. We want to show that $\mathrm{HSP}(\mathbf{2})$ is the variety of all distributive lattices.

[1] The kernels of distinct homomorphisms need not be distinct, of course, but that is okay.

Lemma 8.2. *The following lattice equations are equivalent.*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$
(2) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$
(3) $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$

Thus each of these equations determines the variety $\mathbf{D}$ *of all distributive lattices.*

Proof. If (1) holds in a lattice $\mathcal{L}$, then for any x, y, $z \in L$ we have

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= x \vee (x \wedge z) \vee (y \wedge z) \\ &= x \vee (y \wedge z)\end{aligned}$$

whence (2) holds. Thus (1) implies (2), and dually (2) implies (1).

Similarly, applying (1) to the left hand side of (3) yields the right hand side, so (1) implies (3). Conversely, assume that (3) holds in a lattice $\mathcal{L}$. Making the substitution $y \mapsto x \wedge y$, we see that (3) implies that

$$x \wedge ((x \wedge y) \vee z) \approx (x \wedge y) \vee (x \wedge z)$$

which is the modular law, so $\mathcal{L}$ must be modular. Now for arbitrary x, y, z in $\mathcal{L}$, meet x with both sides of (3) and then use modularity to obtain

$$\begin{aligned}x \wedge (y \vee z) &= x \wedge [(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)] \\ &= (x \wedge y) \vee (x \wedge z) \vee (x \wedge y \wedge z) \\ &= (x \wedge y) \vee (x \wedge z)\end{aligned}$$

since $x \geq (x \wedge y) \vee (x \wedge z)$. Thus (3) implies (1).

(Note that we have shown that (3) is equivalent to (1). Since (3) is self-dual, it follows that (3) is equivalent to (2). The first argument, that (1) is equivalent to (2), is redundant!) $\square$

In the first Corollary of the next chapter, we will see that a lattice is distributive if and only if it contains neither $\mathcal{N}_5$ nor $\mathcal{M}_3$ as a sublattice. But before that, let us look at the wonderful representation theory of distributive lattices. A few moments reflection on the kernel of a homomorphism $h : \mathcal{L} \twoheadrightarrow \mathbf{2}$ should yield the following conclusions. By a *proper* ideal or filter, we mean one that is neither empty nor the whole lattice.

Lemma 8.3. *Let* $\mathcal{L}$ *be a lattice and* $h : \mathcal{L} \twoheadrightarrow \mathbf{2} = \{0, 1\}$ *a surjective homomorphism. Then* $h^{-1}(0)$ *is a proper ideal of* $\mathcal{L}$*, and* $h^{-1}(1)$ *is a proper filter, and* L *is the disjoint union of* $h^{-1}(0)$ *and* $h^{-1}(1)$*.*

Conversely, if I is a proper ideal of $\mathcal{L}$ and F a proper filter such that $L = I \dot{\cup} F$ (disjoint union), then the map $h : \mathcal{L} \to \mathbf{2}$ given by

$$h(x) = \begin{cases} 0 & \text{if } x \in I, \\ 1 & \text{if } x \in F. \end{cases}$$

is a surjective homomorphism.

This raises the question: *When is the complement $L - I$ of an ideal a filter?* The answer is easy. A proper ideal I of a lattice $\mathcal{L}$ is said to be *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$. Dually, a proper filter F is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$. It is straightforward that the complement of an ideal I is a filter iff I is a prime ideal iff $L - I$ is a prime filter.

This simple observation allows us to work with prime ideals or prime filters (interchangeably), rather than ideal/filter pairs, and we shall do so.

Theorem 8.4. *Let $\mathcal{D}$ be a distributive lattice, and let $a \not\leq b$ in $\mathcal{D}$. Then there exists a prime filter F with $a \in F$ and $b \notin F$.*

Proof. Now $\uparrow a$ is a filter of $\mathcal{D}$ containing a and not b, so by Zorn's Lemma there is a maximal such filter (with respect to set containment), say M. For any $x \notin M$, the filter generated by x and M must contain b, whence $b \geq x \wedge m$ for some $m \in M$. Suppose x, $y \notin M$, with say $b \geq x \wedge m$ and $b \geq y \wedge n$ where m, $n \in M$. Then by distributivity

$$b \geq (x \wedge m) \vee (y \wedge n) = (x \vee y) \wedge (x \vee n) \wedge (m \vee y) \wedge (m \vee n).$$

The last three terms are in M, so we must have $x \vee y \notin M$. Thus M is a prime filter. $\square$

Now let $\mathcal{D}$ be any distributive lattice, and let $T_{\mathcal{D}} = \{\varphi \in \mathbf{Con}\ \mathcal{D} : \mathcal{D}/\varphi \cong \mathbf{2}\}$. Theorem 8.4 says that if $a \neq b$ in $\mathcal{D}$, then there exists $\varphi \in T_{\mathcal{D}}$ with $(a, b) \notin \varphi$, whence $\bigcap T_{\mathcal{D}} = 0$ in $\mathbf{Con}\ \mathcal{D}$, i.e., $\mathcal{D}$ is a subdirect product of two element lattices.

Corollary. *The two element lattice $\mathbf{2}$ is the only subdirectly irreducible distributive lattice. Hence $\mathbf{D} = \mathrm{HSP}(\mathbf{2})$.*

Corollary. $\mathbf{D}$ *is locally finite.*

Another consequence of Theorem 8.4 is that every distributive lattice can be embedded into a lattice of subsets, with set union and intersection as the lattice operations.

Theorem 8.5. *Let $\mathcal{D}$ be a distributive lattice, and let S be the set of all prime filters of $\mathcal{D}$. Then the map $\phi : \mathcal{D} \to \mathfrak{P}(S)$ by*

$$\phi(x) = \{F \in S : x \in F\}$$

is a lattice embedding.

For finite distributive lattices, this representation takes on a particularly nice form. Recall that an element $p \in L$ is said to be *join prime* if it is nonzero and $p \le x \vee y$ implies $p \le x$ or $p \le y$. In a finite lattice, prime filters are necessarily of the form $\uparrow p$ where p is a join prime element.

Theorem 8.6. *Let $\mathcal{D}$ be a finite distributive lattice, and let $J(\mathcal{D})$ denote the ordered set of all nonzero join irreducible elements of $\mathcal{D}$. Then the following are true.*

(1) *Every element of $J(\mathcal{D})$ is join prime.*
(2) *$\mathcal{D}$ is isomorphic to the lattice of order ideals $\mathcal{O}(J(\mathcal{D}))$.*
(3) *Every element $a \in D$ has a unique irredundant join decomposition $a = \bigvee A$ with $A \subseteq J(\mathcal{D})$.*

Proof. In a distributive lattice, every join irreducible element is join prime, because $p \le x \vee y$ is the same as $p = p \wedge (x \vee y) = (p \wedge x) \vee (p \wedge y)$.

For any finite lattice, the map $\phi : \mathcal{L} \to \mathcal{O}(J(\mathcal{L}))$ given by $\phi(x) = \downarrow x \cap J(\mathcal{L})$ is order preserving (in fact, meet preserving) and one-to-one. To establish the isomorphism of (2), we need to know that for a distributive lattice it is onto. If $\mathcal{D}$ is distributive and I is an order ideal of $J(\mathcal{D})$, then for $p \in J(\mathcal{D})$ we have by (1) that $p \le \bigvee I$ iff $p \in I$, and hence $I = \phi(\bigvee I)$.

The join decomposition of (3) is then obtained by taking A to be the set of maximal elements of $\downarrow a \cap J(\mathcal{D})$. $\square$

It is clear that the same proof works if $\mathcal{D}$ is an algebraic distributive lattice whose compact elements satisfy the DCC, so that there are enough join irreducibles to separate elements. In Lemma 10.6 we will characterize those distributive lattices isomorphic to $\mathcal{O}(\mathcal{P})$ for some ordered set $\mathcal{P}$.

As an application, we can give a neat description of the free distributive lattice $\mathcal{F}_{\mathbf{D}}(n)$ for any finite n, which we already know to be a finite distributive lattice. Let $X = \{x_1, \ldots, x_n\}$. Now it is not hard to see that any element in a free distributive lattice can be written as a join of meets of generators, $w = \bigvee w_i$ with $w_i = x_{i_1} \wedge \ldots \wedge x_{i_k}$. Another easy argument shows that the meet of a nonempty proper subset of the generators is join prime in $\mathcal{F}_{\mathbf{D}}(X)$; note that $\bigwedge \emptyset = 1$ and $\bigwedge X = 0$ do not count. (See Exercise 3). Thus the set of join irreducible elements of $\mathcal{F}_{\mathbf{D}}(X)$ is isomorphic to the ordered set of nonempty, proper subsets of X, ordered by reverse set inclusion, and the free distributive lattice is isomorphic to the lattice of order ideals of that. As an example, $\mathcal{F}_{\mathbf{D}}(3)$ and its ordered set of join irreducibles are shown in Figure 8.1.

Dedekind [7] showed that $|\mathcal{F}_{\mathbf{D}}(3)| = 18$ and $|\mathcal{F}_{\mathbf{D}}(4)| = 166$. Several other small values are known exactly, and the rest can be obtained in principle, but they grow quickly (see Quackenbush [12]). While there exist more accurate expressions, the

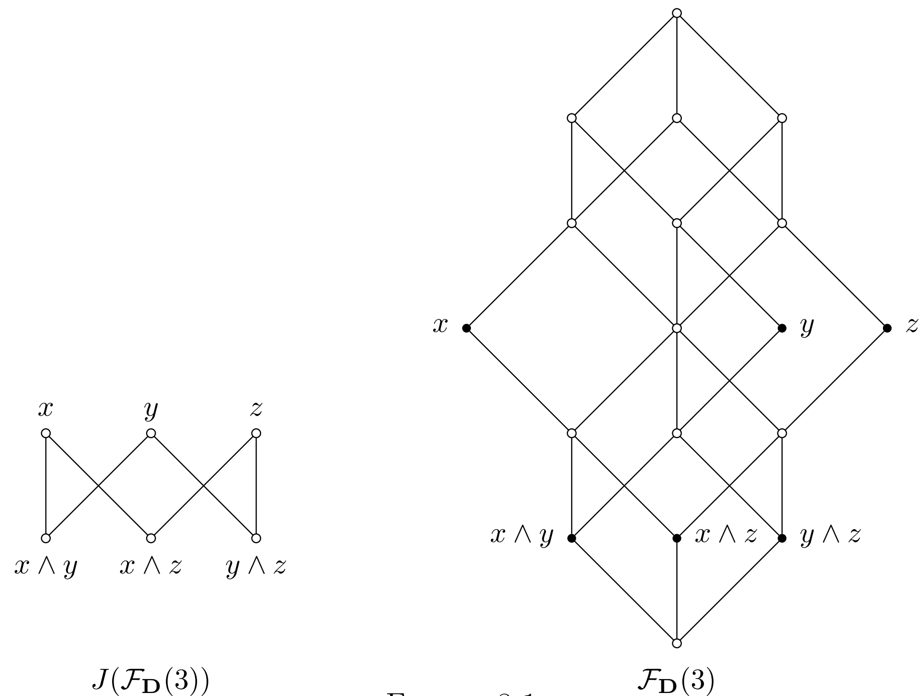

FIGURE 8.1

simplest estimate is an asymptotic formula due to D. J. Kleitman:

$$\log_2 |\mathcal{F}_{\mathbf{D}}(n)| \sim \binom{n}{\lfloor n/2 \rfloor}.$$

The representation by sets of Theorem 8.5 does not preserve infinite joins and meets. The corresponding characterization of complete distributive lattices that have a complete representation as a lattice of subsets is derived from work of Alfred Tarski and S. Papert [11], and was surely known to both of them. An element p of a complete lattice $\mathcal{L}$ is said to be *completely join prime* if $p \leq \bigvee X$ implies $p \leq x$ for some $x \in X$. It is not necessary to assume that $\mathcal{D}$ is distributive in the next theorem, though of course it will turn out to be so.

Theorem 8.7. *Let $\mathcal{D}$ be a complete lattice. There exists a complete lattice embedding $\phi : \mathcal{D} \to \mathcal{P}(X)$ for some set X if and only if $x \not\leq y$ in $\mathcal{D}$ implies there exists a completely join prime element p with $p \leq x$ and $p \not\leq y$.*

Thus, for example, the interval $[0, 1]$ in the real numbers is a complete distributive lattice that cannot be represented as a complete lattice of subsets of some set.

In a lattice with 0 and 1, the pair of elements a and b are said to be *complements* if $a \wedge b = 0$ and $a \vee b = 1$. A lattice is *complemented* if every element has at least one complement. For example, the lattice of subspaces of a vector space is a complemented modular lattice. In general, an element can have many complements, but it is not hard to see that each element in a distributive lattice can have at most one complement.

A *Boolean algebra* is a complemented distributive lattice. Of course, the lattice $\mathfrak{P}(X)$ of subsets of a set is a Boolean algebra. On the other hand, it is easy to see that $\mathcal{O}(\mathcal{P})$ is complemented if and only if $\mathcal{P}$ is an antichain, in which case $\mathcal{O}(\mathcal{P}) = \mathfrak{P}(\mathcal{P})$. Thus *every finite Boolean algebra is isomorphic to the lattice* $\mathfrak{P}(A)$ *of subsets of its atoms.*

For a very different example, the finite and cofinite subsets of an infinite set form a Boolean algebra.

If we regard Boolean algebras as algebras $\mathcal{B} = \langle B, \wedge, \vee, 0, 1, {}^c \rangle$, then they form a variety, and hence there is a *free Boolean algebra* $\mathrm{FBA}(X)$ generated by a set X. If X is finite, say $X = \{x_1, \ldots, x_n\}$, then $\mathrm{FBA}(X)$ has 2^n atoms, *viz.,* all meets $z_1 \wedge \ldots \wedge z_n$ where each z_i is either x_i or x_i^c. Thus in this case $\mathrm{FBA}(X) \cong \mathfrak{P}(A)$ where $|A| = 2^n$. On the other hand, if X is infinite then $\mathrm{FBA}(X)$ has no atoms; if $|X| = \aleph_0$, then $\mathrm{FBA}(X)$ is the unique (up to isomorphism) countable atomless Boolean algebra!

Another natural example is the Boolean algebra of all clopen (closed and open) subsets of a topological space. In fact, by adding a topology to the representation of Theorem 8.5, we obtain the celebrated Stone representation theorem for Boolean algebras [15]. Recall that a topological space is *totally disconnected* if for every pair of distinct points x, y there is a clopen set V with $x \in V$ and $y \notin V$.

Theorem 8.8. *Every Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of a compact totally disconnected (Hausdorff) space.*

Proof. Let $\mathcal{B}$ be a distributive lattice. (We will add the other properties to make $\mathcal{B}$ a Boolean algebra as we go along.) Let $\mathfrak{F}_p$ be the set of all prime filters of $\mathcal{B}$, and for $x \in B$ let

$$V_x = \{F \in \mathfrak{F}_p : x \in F\}.$$

The sets V_x will form a basis for the Stone topology on $\mathfrak{F}_p$.

With only trivial changes, the argument for Theorem 8.4 yields the following stronger version.

Sublemma A. *Let $\mathcal{B}$ be a distributive lattice, G a filter on $\mathcal{B}$ and $x \notin G$. Then there exists a prime filter $F \in \mathfrak{F}_p$ such that $G \subseteq F$ and $x \notin F$.*

Next we establish the basic properties of the sets V_x, all of which are easy to prove.

(1) $V_x \subseteq V_y$ iff $x \leq y$.
(2) $V_x \cap V_y = V_{x \wedge y}$.
(3) $V_x \cup V_y = V_{x \vee y}$.
(4) If $\mathcal{B}$ has a least element 0, then $V_0 = \emptyset$. Thus $V_x \cap V_y = \emptyset$ iff $x \wedge y = 0$.
(5) If $\mathcal{B}$ has a greatest element 1, then $V_1 = \mathfrak{F}_p$. Thus $V_x \cup V_y = \mathfrak{F}_p$ iff $x \vee y = 1$.

Property (3) is where we use the primality of the filters in the sets V_x. In particular, the family of sets V_x is closed under finite intersections, and of course $\bigcup_{x \in B} V_x = \mathfrak{F}_p$, so we can legitimately take $\{V_x : x \in B\}$ as a basis for a topology on $\mathfrak{F}_p$.

Now we would like to show that if $\mathcal{B}$ has a largest element 1, then $\mathfrak{F}_p$ is a compact space. It suffices to consider covers by basic open sets, so this follows from the next Sublemma.

Sublemma B. *If $\mathcal{B}$ has a greatest element* 1 *and $\bigcup_{x \in S} V_x = \mathfrak{F}_p$, then there exists a finite subset $T \subseteq S$ such that $\bigvee T = 1$, and hence $\bigcup_{x \in T} V_x = \mathfrak{F}_p$.*

Proof. Set $I_0 = \{\bigvee T : T \subseteq S,\ T \text{ finite}\}$. If $1 \notin I_0$, then I_0 generates an ideal I of $\mathcal{B}$ with $1 \notin I$. By the dual of Sublemma A, there exists a prime ideal H containing I and not 1. Its complement $B - H$ is a prime filter K. Then $K \notin \bigcup_{x \in S} V_x$, else $z \in K$ for some $z \in S$, whilst $z \in I_0 \subseteq B - K$. This contradicts our hypothesis, so we must have $1 \in I_0$, as claimed. □

The argument thus far has only required that $\mathcal{B}$ be a distributive lattice with 1. For the last two steps, we need $\mathcal{B}$ to be Boolean. Let x^c denote the complement of x in $\mathcal{B}$.

First, note that by properties (4) and (5) above, $V_x \cap V_{x^c} = \emptyset$ and $V_x \cup V_{x^c} = \mathfrak{F}_p$. Thus each set V_x $(x \in B)$ is clopen. On the other hand, let W be a clopen set. As it is open, $W = \bigcup_{x \in S} V_x$ for some set $S \subseteq B$. But W is also a closed subset of the compact space $\mathfrak{F}_p$, and hence compact. Thus $W = \bigcup_{x \in T} V_x = V_{\bigvee T}$ for some finite $T \subseteq S$. Therefore W *is a clopen subset of* $\mathfrak{F}_p$ *if and only if* $W = V_x$ *for some* $x \in B$.

It remains to show that $\mathfrak{F}_p$ is totally disconnected (which makes it Hausdorff). Let F and G be distinct prime filters on $\mathcal{B}$, with say $F \nsubseteq G$. Let $x \in F - G$. Then $F \in V_x$ and $G \notin V_x$, so that V_x is a clopen set containing F and not G. □

There are similar topological representation theorems for arbitrary distributive lattices, the most useful being that due to Hilary Priestley in terms of ordered topological spaces. A good introduction is in Davey and Priestley [6].

In 1880, C. S. Peirce proved that every lattice with the property that each element b has a unique complement b^*, with the additional property that $a \wedge b = 0$ implies $a \leq b^*$, must be distributive, and hence a Boolean algebra. After a good deal of confusion over the axioms of Boolean algebra, the proof was given in a 1904 paper of E. V. Huntington [10]. Huntington then asked whether *every* uniquely complemented lattice must be distributive. It turns out that if we assume almost any additional finiteness condition on a uniquely complemented lattice, then it must indeed be distributive. As an example, there is the following theorem of Garrett Birkhoff and Morgan Ward [5].

Theorem 8.9. *Every complete, atomic, uniquely complemented lattice is isomorphic to the Boolean algebra of all subsets of its atoms.*

Other finiteness restrictions which insure that a uniquely complemented lattice will be distributive include weak atomicity, due to Bandelt and Padmanabhan [4], and upper continuity, due independently to Bandelt [3] and Saliĭ [13], [14]. A monograph written by Saliĭ [16] gives an excellent survey of results of this type.

Nonetheless, Huntington's conjecture is very far from true. In 1945, R. P. Dilworth [8] proved that *every lattice can be embedded in a uniquely complemented lattice*. This result has likewise been strengthened in various ways. See the surveys of Mick Adams [1] and George Grätzer [9].

The standard book for distributive lattices is by R. Balbes and Ph. Dwinger [2]. Though somewhat dated, it contains much of interest.

Exercises for Chapter 8

1. Show that a lattice $\mathcal{L}$ is distributive if and only if $x \wedge (y \vee z) \leq y \vee (x \wedge z)$ for all x, y, $z \in L$. (J. Bowden)

2. (a) Prove that every maximal ideal of a distributive lattice is prime.

(b) Show that a distributive lattice $\mathcal{D}$ with 0 and 1 is complemented if and only if every prime ideal of $\mathcal{D}$ is maximal.

3. These are the details of the construction of the free distributive lattice given in the text. Let X be a finite set.

- (a) Let δ denote the kernel of the natural homomorphism from $\mathrm{FL}(X) \twoheadrightarrow \mathcal{F}_{\mathbf{D}}(X)$ with $x \mapsto x$. Thus $u \;\delta\; v$ iff $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ in all distributive lattices. Prove that for every $w \in \mathrm{FL}(X)$ there exists w' which is a join of meets of generators such that $w \;\delta\; w'$. (Show that the set of all such elements w is a sublattice of $\mathrm{FL}(X)$ containing the generators.)
- (b) Let $\mathcal{L}$ be any lattice generated by a set X, and let $\emptyset \subset Y \subset X$. Show that for all $w \in L$, either $w \geq \bigwedge Y$ or $w \leq \bigvee (X - Y)$.
- (c) Show that $\bigwedge Y \nleq \bigvee (X - Y)$ in $\mathcal{F}_{\mathbf{D}}(X)$ by exhibiting a homomorphism $h : \mathcal{F}_{\mathbf{D}}(X) \to \mathbf{2}$ with $h(\bigwedge Y) \nleq h(\bigvee (X - Y))$.
- (d) Generalize these results to the case when X is a finite ordered set (as in the next exercise).

4. Find the free distributive lattice generated by

- (a) $\{x_0, x_1, y_0, y_1\}$ with $x_0 < x_1$ and $y_0 < y_1$,
- (b) $\{x_0, x_1, x_2, y\}$ with $x_0 < x_1 < x_2$.

5. Let $\mathcal{P} = \mathcal{Q} \dot{\cup} \mathcal{R}$ be the disjoint union of two ordered sets, so that q and r are incomparable whenever $q \in Q$, $r \in R$. Show that $\mathcal{O}(\mathcal{P}) \cong \mathcal{O}(\mathcal{Q}) \times \mathcal{O}(\mathcal{R})$.

6. Let $\mathcal{D}$ be a distributive lattice with 0 and 1, and let x and y be complements in $\mathcal{D}$. Prove that $\mathcal{D} \cong \uparrow x \times \uparrow y$. (Dually, $\mathcal{D} \cong \downarrow x \times \downarrow y$; in fact, $\uparrow x \cong \downarrow y$ and $\uparrow y \cong \downarrow x$. This explains why $\mathbf{Con}\, \mathcal{L}_1 \times \mathcal{L}_2 \cong \mathbf{Con}\, \mathcal{L}_1 \times \mathbf{Con}\, \mathcal{L}_2$ (Exercise 5.6).)

7. Show that the following are true in a finite distributive lattice $\mathcal{D}$.

- (a) For each join irreducible element x of $\mathcal{D}$, let $\kappa(x) = \bigvee \{y \in D : y \ngeq x\}$. Then $\kappa(x)$ is meet irreducible and $\kappa(x) \ngeq x$.
- (b) For each $x \in J(\mathcal{D})$, $D = \uparrow x \,\dot{\cup} \downarrow \kappa(x)$.
- (c) The map $\kappa : J(\mathcal{D}) \to M(\mathcal{D})$ is an order isomorphism.

8. A join semilattice with 0 is *distributive* if $x \leq y \vee z$ implies there exist $y' \leq y$ and $z' \leq z$ such that $x = y' \vee z'$. Prove that an algebraic lattice is distributive if and only if its compact elements form a distributive semilattice.

9. Find an infinite distributive law that holds in every algebraic distributive lattice. Show that this may fail in a complete distributive lattice.

10. Prove Theorem 8.7.

11. Prove Peirce's theorem: If a lattice $\mathcal{L}$ with 0 and 1 has a complementation operation $*$ such that

(1) $b \wedge b^* = 0$ and $b \vee b^* = 1$,
(2) $a \wedge b = 0$ implies $a \leq b^*$,
(3) $b^{**} = b$,

then $\mathcal{L}$ is a Boolean algebra.

12. Prove Papert's characterization of lattices of closed sets of a topological space [11]: *Let $\mathcal{D}$ be a complete distributive lattice. There is a topological space $\mathcal{T}$ and an isomorphism ϕ mapping $\mathcal{D}$ onto the lattice of closed subsets of $\mathcal{T}$, preserving finite joins and infinite meets, if and only if $x \nleq y$ in $\mathcal{D}$ implies there exists a (finitely) join prime element p with $p \leq x$ and $p \nleq y$.*

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