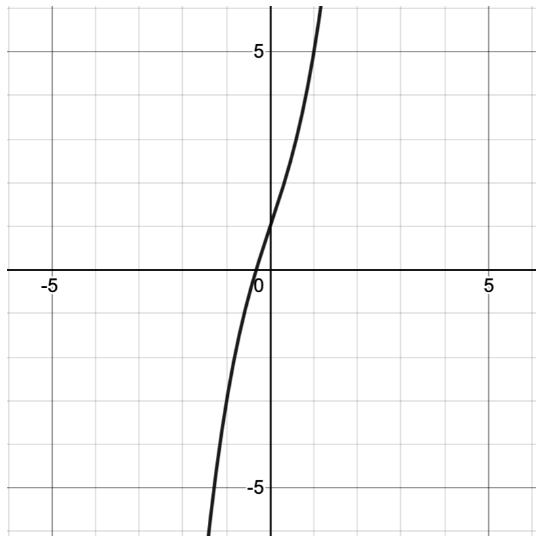
Math 242 Final

Name:

Question	Points	Score
1	12	
2	18	
3	14	
4	24	
5	14	
6	40	
7	15	
8	8	
9	5	
10	12	
Total:	162	

- You are allowed to use one page of your own notes together with the given formula sheets on the last two pages of the exam. Other than these resources, you may not use any additional notes, textbooks, or outside sources.
- You may not use calculators, phones, algebra software, or any other computational technology on the test.
- All work must be your own. You may not consult anyone else during the exam. You are not allowed to access the internet except for downloading and uploading the exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- As mentioned above, the last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. Let $f(x) = x^3 + 3x + 1$ with graph pictured below.



(a) (4 points) Does the function f have an inverse on the interval $[-5, 5]$? How do you know?

(b) (4 points) Determine $f^{-1}(5)$ and $(f^{-1})'(5)$.

(c) (4 points) The x-intercept for f is approximately $(-0.322, 0)$ and the y-intercept for f is $(0, 1)$. Sketch f^{-1} from $x = 1$ to $x = 5$ on the graph above or on another piece of paper, marking any intercepts clearly.

2. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 390.

   (a) (10 points) Find an expression for the number of bacteria after t hours.

   (b) (4 points) Find the number of bacteria after 3 hours. (You do not need to simplify).

   (c) (4 points) Find the rate of growth after 3 hours. (You do not need to simplify).

3. Compute the following limits. You must justify your solution using algebraic manipulations and / or l'Hôpital's rule for full credit. If you use l'Hôpital's rule, state the indeterminate form, for example, $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

   (a) (8 points) $\lim_{x \to 0} \frac{x \cdot 3^x}{3^x - 1}$.

   (b) (6 points) $\lim_{x \to 1^+} [\ln(x^2 - 1) - \ln(x - 1)]$.

4. Compute the following integrals, or say if they diverge. For full credit:

- If you use u-substitution, clearly state u, du, and dx.
- If you use integration by parts, clearly state u, v, du, and dv.
- If the integral is improper, make sure any limiting arguments that you use are explicit.

(a) (8 points) $\int_0^{\pi} \sin^5(x)\cos^2(x)dx$.

(b) (8 points) $\displaystyle\int \frac{5}{x^2 + x - 2} dx.$

(c) (8 points) $\int_{-\infty}^{0} xe^{2x}dx$

5. Compute the sum of each of the convergent series below. Simplify your answers.

(a) (7 points) The series $\sum_{n=0}^{\infty} a_n$ whose partial sums are given by

$$\sum_{k=0}^{n} a_k = \frac{2n}{3n+5}.$$

(b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$.

6. For each of the following series, say whether they converge or diverge. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

   (a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$.

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$.

(c) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$.

(d) (8 points) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n n^2}{n!}$.

(e) (8 points) $\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{\pi}{2} - \arctan n)$.

7. (15 points) Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

8. (a) (4 points) Find the degree three Taylor polynomial for $\sqrt[3]{x}$, centered at $a = 8$.

(b) (4 points) Approximate $\sqrt[3]{9}$ using your results from part (a). (You do not need to simplify).

9. (5 points) Find a power series representation centered at $x = 0$ of

$$\frac{1}{(1-x)^2}.$$

Please write your answer in summation notation.

10. Solve the following differential equations. Either give the general solution, or solve for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.

    (a) (6 points) $\dfrac{dy}{dx} = x^2 y$.

(b) (6 points) $x^2y' + xy = 1, \quad x > 0, \quad y(1) = 2.$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all x where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all x).

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

- Addition formulas for sine and cosine (true for all x and y).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Standard power series expansions (centered at $a = 0$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{)}.$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{)}.$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{)}.$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{)}.$$

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{)}.$$

- Error estimate for approximations by Taylor polynomials.
  Say $f(x)$ is a function with derivatives of all orders on an interval $[b, c]$, and a is a point in $[b, c]$. Say $T_N(x)$ is the N^{th} Taylor polynomial for $f(x)$ centered at a, and $R_N(x) = f(x) - T_N(x)$ is the error when approximating $f(x)$ by $T_N(x)$. Then for all x in $[b, c]$

$$|R_N(x)| \le \frac{M_{N+1}|x-a|^{N+1}}{(N+1)!},$$

  where M_{N+1} is the largest value taken by the $(N+1)^{\text{st}}$ derivative of $f(x)$ on $[b, c]$.