Math 242 Final, Fall 2023

Name: ______________________________ **Section:** ______

Instructor: ____

TA: ____

Question	Points	Score
1	7	
2	12	
3	12	
4	20	
5	10	
6	12	
7	18	
8	12	
9	12	
10	10	
11	16	
12	9	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- Organize your work neatly and legibly in the spaces provided under each problem.
- You do not need to simplify your answers unless explicitly told to do so.
- Clearly cross-out scratch work.
- You have exactly 2:00 hours to complete this Exam.
- Good luck!

1. Let $f(x)$ be the function given by

$$f(x) = \sqrt{1+x^4}, \quad x \geq 0.$$

(a) (2 points) Find the value of x such that $f(x) = \sqrt{2}$.

(b) (5 points) Compute $(f^{-1})'(\sqrt{2})$.

2. Compute the derivatives of the following functions. **No need to simplify your answers.**

   (a) (6 points) $f(x) = 2^x \arcsin(2x)$.

   (b) (6 points) $f(x) = \ln\left(\dfrac{\sqrt[3]{x^2+1}}{(x-1)\sqrt{2x+1}}\right)$. (Hint: Use properties of logarithmic functions.)

3. Compute the following limits. You must justify your solution using algebraic manipulations and/or l'Hopitals rule for full credit.

   (a) (6 points) $\lim_{x \to 0} \dfrac{\cos(2x) - \cos(3x)}{x^2}$.

   (b) (6 points) $\lim_{n \to \infty} \sqrt[n]{n}$.

4. Evaluate the definite integral or show that it diverges.

   (a) (6 points) $\displaystyle\int_0^{\pi/2} \sin^3(x)\cos^7(x)dx.$

(b) (6 points) $\displaystyle\int_1^2 \frac{3}{x(x+4)}dx.$

(c) (8 points) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\, dx.$

5. (10 points) Evaluate the integral $\displaystyle\int \frac{5}{(x^2+25)^{3/2}}\,dx$.

6. **Calculate the sum** of each series below, if it is convergent. If it diverges, explain why.

(a) (6 points) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^2-1}$.

(b) (6 points) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n-2^n}{6^n}$.

7. Determine whether the series below is absolutely convergent, conditionally convergent or divergent. State which test(s) you use and justify the answer.

   (a) (6 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3}{6n^3+5n^2+1}$.

(b) (6 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2+1}{(n+1)!}$.

(c) (6 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\cos^n(\frac{2}{n})}{2^n}$.

8. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-5)^n (x+2)^n}{n+3}.$$

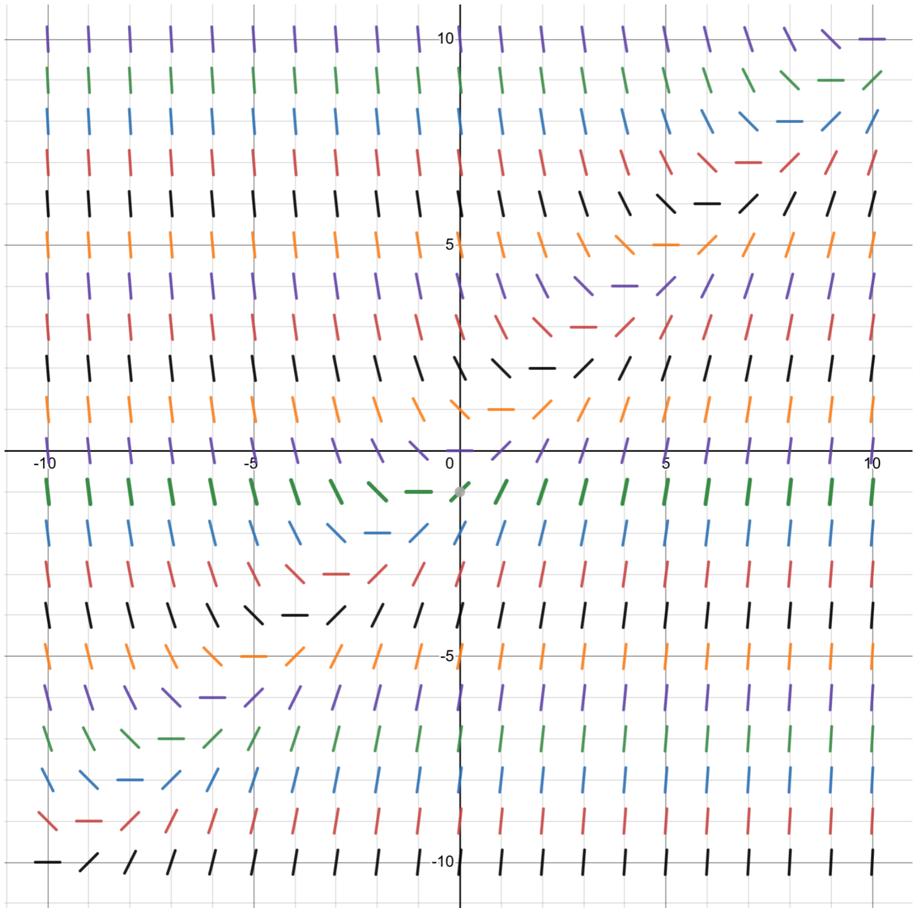
(a) (6 points) Find the radius of convergence of the series.

(b) (6 points) Find the interval of convergence for the above power series. Make sure to check the endpoints.

9. (a) (6 points) Find the Maclaurin series for the function $\cos(x^5)$.

(b) (6 points) Use part (a) to evaluate the integral $\int \cos(x^5)\, dx$ as a series.

10. Consider the direction field pictured below.



(a) (5 points) Which of the differential equations below matches this direction field?

$$\text{(a) } \frac{dy}{dx} = \frac{x}{y}, \qquad \text{(b) } \frac{dy}{dx} = x - y, \qquad \text{(c) } \frac{dy}{dx} = x + y, \qquad \text{(d) } \frac{dy}{dx} = x^2 + y^2.$$

(b) (5 points) Sketch the solution to this differential equation that satisfies $y(0) = 5$ on the direction field.

11. Solve the following differential equations. Either give the general solution, or solve for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.

(a) (8 points) $\dfrac{dy}{dx} = \dfrac{x\sin(x^2)}{y^2}$.

(b) (8 points) $y' + y = x, \quad y(0) = 5.$

12. Determine whether each of the following statements is true or false. **No need to justify your answer.**

(a) (3 points) $e^{\ln(x)} = x$, for all $x > 0$.

(b) (3 points) If the series $\sum_{n\geq 0}^{\infty} a_n$ and $\sum_{n\geq 0}^{\infty} b_n$ diverges, then the series $\sum_{n\geq 0}^{\infty}(a_n+b_n)$ also diverges.

(c) (3 points) $\lim_{n\to\infty} \frac{1}{n} = 0$, so the series $\sum_{n\geq 1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots$ converges, by the Test for Divergence.

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Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all x where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all x).

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1-\cos(2x))$$

- Addition formulas for sine and cosine (true for all x and y).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Trapezoidal Rule and Simpson's Rule:

$$T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error Bound for Trapezoidal Rule and Simpson's Rule:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \qquad \text{where } |f''(x)| \leq K \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \leq \frac{M(b-a)^5}{180n^4}, \qquad \text{where } |f^{(4)}(x)| \leq M \text{ for all } x \text{ in } [a,b]$$

- Standard power series expansions (centered at $a = 0$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{).}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{).}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{).}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{).}$$

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{).}$$

- Error estimate for approximations by Taylor polynomials.
  Say $f(x)$ is a function with derivatives of all orders on an interval $[b,c]$, and a is a point in $[b,c]$. Say $T_N(x)$ is the N^{th} Taylor polynomial for $f(x)$ centered at a, and $R_N(x) = f(x) - T_N(x)$ is the error when approximating $f(x)$ by $T_N(x)$. Then for all x in $[b,c]$

$$|R_N(x)| \leq \frac{M_{N+1}|x-a|^{N+1}}{(N+1)!},$$

  where M_{N+1} is the largest value taken by the $(N+1)^{\text{st}}$ derivative of $f(x)$ on $[b,c]$.