Math 242 Final, Spring 2020

Name:

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	12	
7	24	
8	10	
9	8	
10	5	
11	9	
12	10	
13	6	
14	6	
Total:	140	

- The exam is 2 hours long, plus an extra 15 minutes to upload your answers.
- You may not use any electronic devices or resources on the test.
- You may use the textbook, and your own personal notes, but no other notes.
- All work must be entirely your own. You cannot discuss the test with anyone else in any way.
- You must show all your work and make clear what your final solution is (for example, by drawing a box around it).
- You will get almost no credit for solutions that are not fully justified.
- You can write your answers on blank paper without printing out the test, as long as you make clear which question is which.
- Good luck!

1. Say f is a differentiable function that satisfies

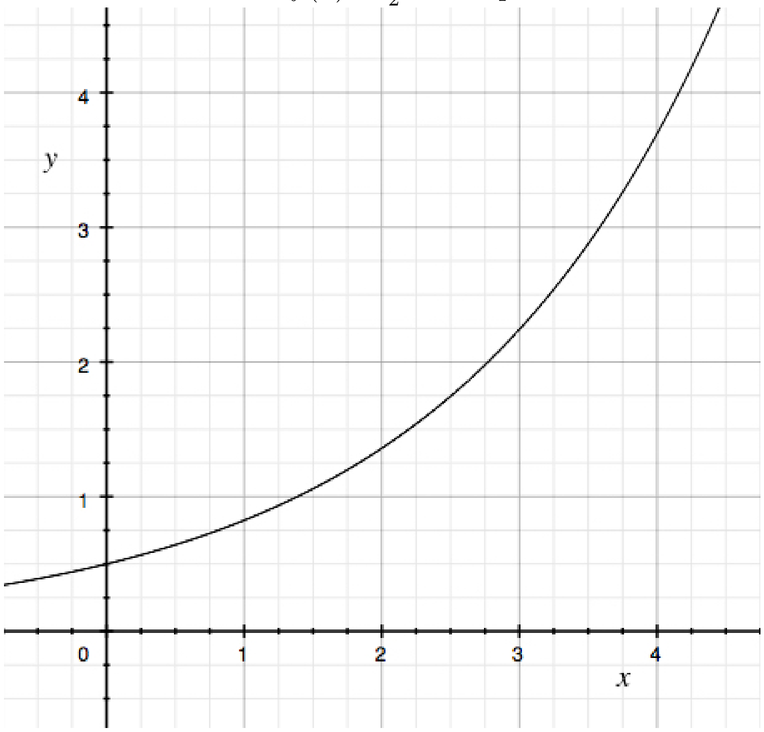
$$\lim_{x\to\infty} f(x) = 4, \quad \lim_{x\to\infty} f'(x) = -2, \quad \lim_{x\to 0} f(x) = 0, \quad \lim_{x\to 0} f'(x) = 5.$$

Find the following limits, making sure you explicitly state whether you are using L'Hospital's rule.

(a) (5 points) $\displaystyle\lim_{x\to 0} \frac{f(x)}{2^x - 1}$.

(b) (5 points) $\displaystyle\lim_{x\to\infty} \frac{f(x)}{\arctan(x)}$.

2. Consider the function $f(x) = \frac{1}{2}e^{x/2}$ as pictured below.



(a) (3 points) Draw on the graph the trapezoids you would get when approximating the integral $\int_1^4 \frac{1}{2}e^{x/2}dx$ using the trapezoid rule, and $n = 3$.

(b) (2 points) Would the estimate as in part (a) be an underestimate or overestimate of the actual integral $\int_1^4 \frac{1}{2}e^{x/2}$?

(c) (5 points) Estimate how large n should be if you want to estimate $\int_1^4 \frac{1}{2}e^{x/2}$ using the trapezoid rule with n-steps, and get accuracy to within 0.01.

3. (10 points) For the integral below, say whether it is proper, improper, or indefinite. Then compute the integral, or say if it diverges.

$$\int_4^\infty \frac{4}{x^2 - 4x + 3} dx.$$

4. (10 points) For the integral below, say whether it is proper, improper, or indefinite. Then compute the integral, or say if it diverges.

$$\int_2^\infty xe^{-3x} dx.$$

5. (a) (5 points) What trigonometric substitution of the form $x = f(\theta)$ would be a reasonable one to try for the integral below? Make this substitution to get an integral in terms only of θ.

$$\int \frac{x^3}{\sqrt{4+x^2}} dx$$

You do NOT have to compute the integral.

(b) (5 points) A student is given an indefinite integral in terms of x to do. They make the substitution $x = 5\sec(\theta)$, and get to the answer below in terms of θ, valid for $0 \leq \theta < \pi/2$. The problem is not finished at this stage: finish the problem!

$$\sin(\theta) + \theta\tan(\theta) + C$$

6. Compute the sums of the following convergent series.

(a) (6 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)}$.

(b) (6 points) $\displaystyle\sum_{n=0}^{\infty} (-1)^n e^{-n}$.

7. For each of the following series, say whether it converges or diverges. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

(a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}\cos(\pi n)}{\sqrt{4+n}}$.

(b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{4+7n}{n^3+1}$.

(c) (8 points) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$.

8. (10 points) Consider the series

$$\sum_{n=2}^{\infty} \frac{5}{\ln(n)4^n}(x-3)^n.$$

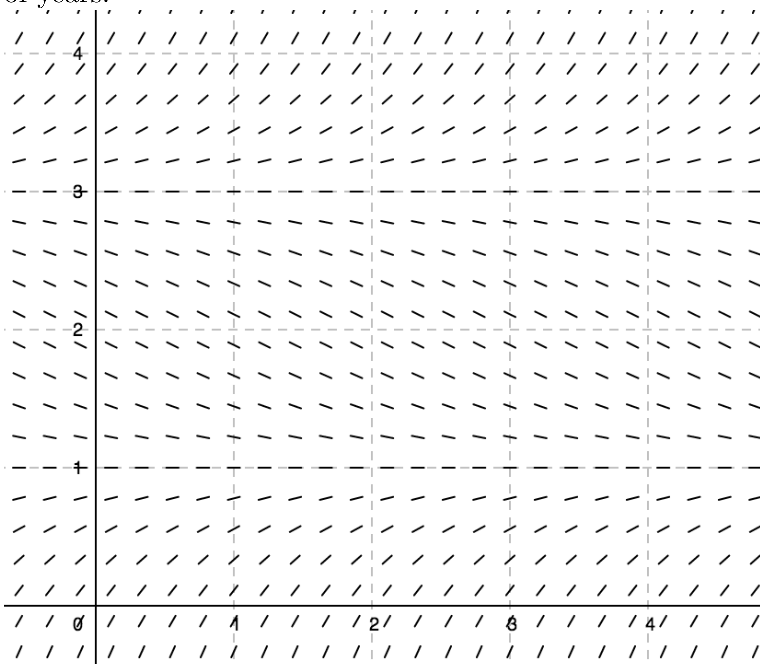
Find the values of x (if any) for which this series (i) converges absolutely; (ii) converges conditionally; (iii) diverges. Justify your answers.

9. (8 points) Compute the Maclaurin series for the integral

$$\int \ln(1+x^2)dx.$$

10. (5 points) It is known that a function f equals its Maclaurin series, and satisfies $|f^{(n)}(x)| \leq n$ on the interval $[0,1]$. Estimate how large n must be for the function to be within 0.1 of its n^{th} Taylor polynomial (centered at 0) on the interval $[0,1]$.

11. The direction field below is a model for the growth or decay of a certain type of star. The units on the y axis are the radius of the star millions of miles, and the units on the x axis are billions of years.



(a) (4 points) On the direction field, sketch the solution when the star initially has radius 2.5 million miles, and use this to estimate the radius of the star after 2 billion years.

(b) (2 points) What does the model predict happens to the star if its initial radius is 4 million miles?

(c) (3 points) Which of these ODEs best matches the direction field?

$$y' = y(y-1)(y-3), \quad y' = y(y-1)(3-y), \quad y' = (y-1)(y-3), \quad y' = (y-1)(3-y).$$

12. (10 points) Solve the following differential equation.

$$\frac{dy}{dx} = y^2 \cos(x) \sin(2x).$$

13. (6 points) Consider the differential equation:

$$\frac{dy}{dx} = \cos^2(x)\sin(x)y + x.$$

Find an integrating factor for the corresponding linear equation, and rearrange into the form

$$\frac{d}{dx}\Big(f(x)y\Big) = g(x).$$

You **DO NOT** need to solve the equation.

14. (6 points) A 10 gallon tank tank contains 4 pounds of dissolved potato starch. Water containing 1/2 pound per gallon of dissolved potato starch pours into the tank at a rate of 3 gallons per minute, and well-mixed liquid leaves the tank at the same rate.

Let p be the amount of potato starch in the tank. Set up a differential equation and initial condition that models the situation described. You **DO NOT** have to solve the resulting initial value problem.

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all x where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all x).

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

- Addition formulas for sine and cosine (true for all x and y).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Estimate for the trapezoid rule with n-steps
$$\frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)),$$
where $x_0, x_1, ..., x_n$ are the endpoints of the intervals involved, and Δx is the width of each interval.

- Estimate for the midpoint rule with n-steps
$$\Delta x(f(\overline{x_1})+f(\overline{x_2})+\cdots+f(\overline{x_n})),$$
where $x_0, x_1, ..., x_n$ are the endpoints of the intervals involved, $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$, and Δx is the width of each interval.

- Error formulas for the midpoint and trapezoid rules on an interval $[a, b]$:
$$|E_m| \leq \frac{M(b-a)^3}{24n^2}, \qquad |E_t| \leq \frac{M(b-a)^3}{12n^2},$$
where $|f''(x)| \leq M$ for all x in $[a, b]$.

- Standard power series expansions (centered at $a = 0$).
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{)}.$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{)}.$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{)}.$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{)}.$$
$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{)}.$$

- Error estimate for Taylor remainders. Say $f(x)$ is a function with derivatives of all orders on an interval $[b, c]$, and a is a point in $[b, c]$. Then for all x in $[b, c]$, the difference $R_N(x)$ between $f(x)$ and the N^{th} Taylor polynomial centered at a satisfies
$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!},$$
where $|f^{(N+1)}(x)| \leq M$ for all x in $[b, c]$.